Vol. 66, No. 2 DUKE MATHEMATICAL JOURNAL © May 1992

L^2-INDEX FOR CERTAIN DIRAC-SCHRÖDINGER OPERATORS

JOCHEN BRÜNING AND HENRI MOSCOVICI

1. Introduction. In [B2] a formula is given for evaluating the L^2-index of a Dirac-type operator D on a certain class of (noncompact) complete Riemannian manifolds. Although in principle computable, especially in the Fredholm case, this formula contains terms reflecting the contribution of the small eigenvalues, which are difficult to evaluate. We show in this paper that the addition of a skew-adjoint potential V, satisfying reasonable assumptions at infinity, has the effect of eventually overcoming the influence of the small eigenvalues of D. Thus, the L^2-index of the "Dirac-Schrödinger operator" $D + \lambda V$, for λ sufficiently large, is given by an "adiabatic limit" of η-invariants and is therefore local at infinity. (See Theorem 3.2 below.) This generalizes and at the same time explains index formulae of Callias type. (See [C], [A].)

Due in part to the nature of the problem, but mainly because of the limitations of the method we employ, the manifolds we are considering are subject to a number of constraints at infinity. Some of these conditions have a clear geometric meaning, but others do not. Thus, the class of manifolds to which our results apply is not easy to quantify. It seems possible to enlarge it to encompass all complete manifolds of strictly negative sectional curvature and finite volume; Theorem 3.5 below constitutes an important step in this direction.

The L^2-index theorem we prove in this paper can be used in conjunction with vanishing type arguments, much in the same way as the standard index theorem for Dirac operators and its relative version are employed in [GL], to gain information about the scalar curvature. To illustrate this, we discuss in Section 4 a "version with boundary" of the "conservation principle" for the scalar curvature of perturbations of the standard metric on the n-sphere, suggested by Gromov [G] and proved in [L]. We wish to thank Maung Min-Oo for making us aware of these references and for substantial help with the calculations in Section 4.

2. An abstract index theorem. In this section we recall the main facts from [B2], adapted to the following situation. Let M be a complete noncompact Riemannian manifold, of odd dimension $m = 2k + 1$, and let D be a generalized Dirac operator acting on the smooth compactly supported sections of a Clifford bundle E, equipped

Received 22 April 1991. Revision received 16 October 1991.

J. B. gratefully acknowledges partial support by Volkswagen-Stiftung and Deutsche Forschungsgemeinschaft and the hospitality of The Ohio State University.

H. M. acknowledges partial support from the National Science Foundation and thanks the University of Augsburg, where this work was begun, for its hospitality.

with a Hermitean structure and a metric connection ∇ compatible with the Clifford multiplication (to be denoted by $\cdot$). Thus, for any local orthonormal frame $\{F_1, \ldots, F_m\}$, one has

$$D\sigma = \sum_{i=1}^{m} F_i \cdot \nabla_{F_i} \sigma, \qquad \sigma \in C_0^\infty(E).$$

We also consider a skew-adjoint potential (i.e., a differential operator of order zero), $V = -V' \colon C_0^\infty(E) \to C_0^\infty(E)$, and form the one-parameter family of "Dirac-Schrödinger operators"

$$D_\lambda := D + \lambda V, \qquad \lambda \in \mathbb{R}.$$

For simplicity we assume that

$$\text{each } D_\lambda \text{ has a unique closed extension in } L^2(E). \tag{2.1}$$

Note that this is automatically satisfied if $\lambda = 0$ and the curvature of the bundle E induces a bounded operator in $L^2(E)$. (See [GL, Theorem 2.8].) More generally, if V and the commutator $[D, V]$ also induce bounded operators in $L^2(E)$, then (2.1) holds; moreover, the domain $\mathscr{D}(D_\lambda)$ is independent of λ and in fact coincides with $\mathscr{D}(D)$.

We will give sufficient conditions for the finiteness of dim ker D_λ and dim ker D_λ^*, and then compute the corresponding L^2-index

$$L^2\text{-ind } D_\lambda = \dim\ker D_\lambda - \dim\ker D_\lambda^*.$$

In what follows, the unique closed extension according to (2.1) will also be denoted by D_λ. Next, we assume that there is an open subset $U \subset M$ such that

$$\begin{aligned} &M_1 := M \setminus U \text{ is a compact manifold with boundary}, \\ &N := \partial M_1 \end{aligned} \tag{2.2}$$

and that, on U, D_λ has a nice representation in the following sense. There is a Hilbert space H and an isometry

$$\Phi \colon L^2(E|U) \to L^2(\mathbb{R}_+, H) \tag{2.3a}$$

which induces an isomorphism

$$H_0^1(E|U) \to H_0^1(\mathbb{R}_+, H) \cap L^2(\mathbb{R}_+, H_1) \tag{2.3b}$$

for some dense subspace H_1 of H. Moreover, there are smooth functions

$$\begin{aligned} \mathbb{R}_+ \ni y &\mapsto S(y) \in \mathscr{L}(H_1, H), \\ \mathbb{R}_+ \ni y &\mapsto W(y) \in \mathscr{L}(H), \end{aligned} \tag{2.4a}$$

such that for $u \in C_0^\infty(\mathbb{R}_+, H_1)$

$$\Phi D_\lambda \Phi^{-1} u(y) = -\partial_y u(y) + (S(y) + \lambda W(y))u(y). \tag{2.4b}$$

We also assume that

$$S(y) \text{ is self-adjoint in } H \text{ with domain } H_1 \text{ and discrete for all } y \geqslant 0 \tag{2.4c}$$

and that

$$W(y) \text{ is self-adjoint in } H. \tag{2.4d}$$

Then $(S + \lambda W)(y)$ is also self-adjoint in H with domain H_1. In all applications, $S + \lambda W$ can be realized as a first-order elliptic differential operator on the compact manifold N, which makes the following assumptions on ζ-functions (see [Gi, Lemma 1.10.1]) reasonable.

If $\zeta_\pm$ denotes the ζ-function of $(S + \lambda W)(0)_\pm$ (the positive and negative part of $(S + \lambda W)(0)$), then $\zeta_\pm$ is meromorphic in $\mathbf{C}$ and holomorphic in some right half-plane. (2.5)

It is convenient to work with the Hilbert space

$$\mathscr{H} := L^2(E|M_1) \oplus L^2(\mathbb{R}_+, H), \tag{2.6}$$

the elements of which we denote as (u_i, u_b), refering to "interior" and "boundary" parts.

We can conclude that D_λ has a finite L^2-index, in the sense that both dimensions in

$$L^2\text{-ind } D_\lambda = \dim \ker D_\lambda - \dim \ker D_\lambda^* \tag{2.7}$$

are finite but D_λ is not necessarily Fredholm, if the situation just described is "f-controlled". By this we mean the following: there is a positive function $f \in C_0^\infty(\mathbb{R}_+)$ with

$$\lim_{y \to \infty} f'(y) = a, \qquad 0 \leqslant a < 1, \tag{2.8}$$

such that the operator

$$\bar{S}_\lambda(y) := f(y)(S(y) + \lambda W(y)) =: \bar{S}(y) + \lambda \bar{W}(y) \tag{2.9}$$

has bounded variation and no small eigenvalues. More precisely, we assume that

$$|\bar{S}_\lambda(y)| \geqslant C_1 > 1/2, \qquad y \geqslant 0 \tag{2.10}$$

and that with

$$\alpha(y) := \|\bar{S}'_\lambda(y)(I + |\bar{S}_\lambda(y)|)^{-1}\|_H \tag{2.11a}$$

we have

$$\int_0^\infty f(y)\alpha(y)^2\, dy =: C_2 < \infty. \tag{2.11b}$$

If, finally, we assume that

$$(1 - y/\beta)\bar{S}_\lambda(y) = \bar{S}_\lambda(0) \qquad \text{for } y \text{ near } 0 \tag{2.12}$$

where β is a constant depending on f (given explicitly in [B2, (4.14)]), then we obtain from [B2, Theorem 4.4] the following lemma.

LEMMA 2.1. *Under the above assumptions, for λ sufficiently large we have*

$$L^2\text{-ind } D_\lambda = -\tfrac{1}{2}\eta(\bar{S}_\lambda(0)) \tag{2.13}$$

where $\eta(\bar{S}_\lambda(0))$ is the η-invariant of $\bar{S}_\lambda(0)$.

Proof. In the notation of [B2] we have $Q(y) = 0$, $y \geqslant 0$, from (2.10). Hence, the third and fourth contribution in [B2, (4.15)] vanishes. From [B2, Theorem 2.6 and (2.28)] we derive $h_0 = h_1 = 0$; hence, we obtain (2.13). Finally, the Atiyah-Singer integrand vanishes since m is odd [Gi, Lemma 1.7.4]. □

The main point of introducing the parameter λ in $\bar{S}_\lambda$ is that we obtain (2.10) and hence the "clean" index formula (2.13) under simple assumptions on $\bar{S}$ and $\bar{W}$, if λ is large.

We now consider an operator of the form (2.9) and try to reduce the assumptions (2.10), (2.11b) to more tractable conditions.

LEMMA 2.2. *Assume that*

$$0 < C_3^2 \leqslant \bar{W}(y)^2 \tag{2.14}$$

and that $(\overline{SW}+\overline{WS})(y)$ *is bounded in* H *with uniformly bounded norm*

$$\|(\overline{SW}+\overline{WS})(y)\|_H \leqslant C_4. \tag{2.15}$$

Then (2.10) *holds for* $\lambda \geqslant \lambda_0$, *with*

$$C_1 = \lambda C_3/2.$$

Proof. We compute

$$\begin{aligned}\bar{S}_\lambda(y)^2 &= \bar{S}(y)^2 + \lambda(\overline{SW}+\overline{WS})(y) + \lambda^2 \overline{W}(y)^2 \\ &\geqslant \lambda^2 C_3^2 - \lambda C_4 \geqslant \lambda^2 C_3^2/4\end{aligned}$$

if $\lambda \geqslant \lambda_0$. □

With any choice of f, conditions (2.14) and (2.15) impose restrictions on the choice of the potential, the geometry entering through $\bar{S}$. In order to clarify the nature of (2.11b) in this spirit, we make an assumption on S which will be justified in Section 3. Namely, we assume that for some smooth positive function h we have good control, not over $\bar{S}$, but over

$$\tilde{S}(y) := h(y)S(y). \tag{2.16}$$

Here, the idea is that h is "controlling the geometry of U" in the sense that, as it will be further explained below, we can estimate

$$\beta(y) := \|\tilde{S}'(y)(I + |\tilde{S}(y)|)^{-1}\|_H; \tag{2.17}$$

thus, h is a datum, whereas f can be chosen.

LEMMA 2.3. *Assume that for some* f *with* (2.8) *we have*

$$\begin{aligned}\int_0^\infty f(y)\|\overline{W}'(y)\|^2 + \int_0^\infty f(y)\left|\frac{f'}{f}-\frac{h'}{h}\right|^2(y)(1+\|\overline{W}(y)\|^2)\,dy \\ + \int_0^\infty f\beta^2(y)\left(1+\frac{f^2}{h^2}(y)+\|\overline{W}(y)\|^2\right)dy \leqslant C_5.\end{aligned} \tag{2.18}$$

Then (2.11b) *holds for* $\lambda \geqslant \lambda_1$ *with a constant* C_2 *depending only on* C_3 *and* C_5.

Proof. We have

$$\begin{aligned}\bar{S}_\lambda'(y) &= \frac{d}{dy}\left(\frac{f}{h}\tilde{S}(y) + \lambda\overline{W}(y)\right) \\ &= \left(\frac{f}{h}\right)'\tilde{S}(y) + \frac{f}{h}\tilde{S}'(y) + \lambda\overline{W}'(y).\end{aligned}$$

We estimate the right-hand side term by term. Using (2.17), we find

$$\begin{aligned}\left\|\left(\frac{f}{h}\right)'\tilde{S}(y)(I+|\bar{S}_\lambda(y)|)^{-1}\right\| &\leqslant \left|\left(\frac{f}{h}\right)'\frac{h}{f}\right|(y)[\|\bar{S}_\lambda(y)(I+|\bar{S}_\lambda(y)|)^{-1}\| \\ &\qquad + \|\lambda\overline{W}(y)(I+|\bar{S}_\lambda(y)|)^{-1}\|] \\ &\leqslant \left|\frac{f'}{f}-\frac{h'}{h}\right|(y)\left(1+\frac{2}{C_3}\|\overline{W}(y)\|\right).\end{aligned}$$

Similarly,

$$\|\lambda\overline{W}'(y)(I+|\bar{S}_\lambda(y)|)^{-1}\| \leqslant \frac{2}{C_3}\|\overline{W}'(y)\|.$$

Finally,

$$\begin{aligned}\frac{f}{h}(y)\|\tilde{S}'(y)(I+|\bar{S}_\lambda(y)|)^{-1}\| &\leqslant \beta(y)\frac{f}{y}(y)\|(I+|\tilde{S}(y)|)(I+|\bar{S}_\lambda(y)|)^{-1}\| \\ &\leqslant \beta\left[\frac{f}{h}(y)+1+\frac{2}{C_3}\|\overline{W}(y)\|\right].\end{aligned}$$

The assertion now follows. □

3. An index formula for Dirac-Schrödinger operators. We now consider a complete manifold M with decomposition $M = M_1 \cup U$, $N = \partial M_1$ compact, as in (2.2). In addition, we assume that U is "tame" in the sense that

$$U = (0, \infty) \times N \text{ with metric } g = dy^2 \oplus g_y, \text{ where } g_y \text{ is a smooth family of metrics on } N,\ y \geqslant 0. \tag{3.1}$$

Some of the assumptions in Lemma 2.1 already hold in this generality; they will be verified first. Then we impose more restrictions on the geometry of U, requiring it to be controlled by a simple function h, in a suitable sense. This will be further explained in the second half of this section.

We also assume that M is equipped with a spin structure, and we denote the corresponding spin bundle by S. Now S has a canonical Dirac connection induced by the Levi-Civita connection on M. Consider next a Hermitian vector bundle E over M with metric connection, and the associated Dirac operator, D_0, with coefficients in E. We consider D_0 as an operator in $L^2(S \otimes E)$ with domain $C_0^\infty(S \otimes E)$; then D_0 has a unique closed extension in $L^2(S \otimes E)$ which will also be denoted by D_0.

We introduce some convenient notation. $\pi_1\colon U \to (0, \infty)$ and $\pi_2\colon U \to N$ denote the natural projections associated with (3.1); for $y \in [0, \infty)$ we write $N_y := \{y\} \times N$, $N_0 =: N$, and i_y: $N \ni p \mapsto (y, p) \in N_y$ is the natural imbedding. Now y also denotes the global coordinate induced by π_1; we put $F_0 := \partial/\partial y$, the outward unit vector field normal to N_y. The second fundamental form of N_y will be denoted by Π_y,

$$\Pi_y(F) = -\nabla_F F_0, \qquad F \in C^\infty(TN_y), \tag{3.2}$$

and the mean curvature by H_y,

$$H_y = \operatorname{tr} \Pi_y. \tag{3.3}$$

Since π_1 is a Riemannian submersion, we have

$$\exp_{(y,p)} tF_0 = (y + t, p). \tag{3.4}$$

It follows that the vector fields defined by

$$\hat{E}(i_y(p)) := Ti_y(p)(E), \qquad E \in T_pN, \tag{3.5}$$

are Jacobi fields along the normal geodesic starting at p, with initial value E and initial velocity $-\Pi_0(E)$. The given orientation on M defines an orientation on each N_y; if ω_M and ω_y denote the respective volume forms, then we require that

$$\omega_M = F_0^\flat \wedge \omega_y. \tag{3.6}$$

Next, we recall the explicit form of the Dirac connection ∇ on $S \otimes E$: if ∇^M denotes the connection on S, induced by the Levi-Civita connection on TM, and ∇^E the given metric connection on E, then

$$\nabla = \nabla^M \otimes 1 + 1 \otimes \nabla^E. \tag{3.7a}$$

Moreover, if we choose a local orthonormal frame $(F_i)_{i \geq 0}$ for U (with $F_0 = \partial/\partial y$ as above) and denote by (S_α) the corresponding frame for S, then (see, e.g., [LM, Chap. II, Thm. 4.14]) for $F \in C^\infty(TU)$,

$$\nabla_F^M S_\alpha = \frac{1}{2} \sum_{j<k} < \nabla_F^M F_j, F_k > F_j \cdot F_k \cdot S_\alpha; \tag{3.7b}$$

here, $\cdot$ denotes the Clifford multiplication and also the left action of the Clifford bundle $\mathscr{C}\ell M$ on S.

Consider next $\Phi \in C^\infty(\operatorname{End} E)$ which is pointwise self-adjoint, $\Phi = \Phi^*$, and set

$$V := \sqrt{-1} I \otimes \Phi \in C^\infty(\operatorname{End} S \otimes E). \tag{3.8}$$

We call

$$D := D_0 + V\colon C_0^\infty(S \otimes E) \to C_0^\infty(S \otimes E) \tag{3.9}$$

the Dirac-Schrödinger operator with potential V.

The abstract machinery of Section 2 can be brought to bear on the present setting by separating variables. This will be carried out next, leading to the representation (2.4) for Dirac-Schrödinger operators.

Note that, since m is odd, $S^y := S|N_y$ is a spin bundle over N_y with grading given by $\sqrt{-1}F_0 =: \tau$. We compute with local orthonormal frames (F_i), (S_α) for TM and S (as above) and (E_β) for E, all assumed parallel along normal geodesics:

$$\begin{aligned}
D_0(S_\alpha \otimes E_\beta) &= \sum_{i \geqslant 1} F_i \cdot \nabla_{F_i}(S_\alpha \otimes E_\beta) \\
&= \frac{1}{2} \sum_{\substack{i \geqslant 1 \\ j<k}} \langle \nabla^M_{F_i} F_j, F_k \rangle (F_i \cdot F_j \cdot F_k \cdot S_\alpha) \otimes E_\beta + \sum_{i \geqslant 1} (F_i \cdot S_\alpha) \otimes \nabla^E_{F_i} E_\beta \\
&=: D_0^y(S_\alpha \otimes E_\beta) + \frac{1}{2} \sum_{i,k \geqslant 1} \langle \nabla^M_{F_i} F_0, F_k \rangle (F_i \cdot F_0 \cdot F_k \cdot S_\alpha) \otimes E_\beta \qquad (3.10) \\
&= D_0^y(S_\alpha \otimes E_\beta) - \frac{1}{2} \sum_{i,k \geqslant 1} \langle \Pi_y(F_i), F_k \rangle (F_i \cdot F_0 \cdot F_k \cdot S_\alpha) \otimes E_\beta \\
&=: (D_0^y + A^y)(S_\alpha \otimes E_\beta).
\end{aligned}$$

If we assume that $(F_i)_{i \geqslant 1}$ diagonalizes Π_y at (y, p) with eigenvalues $(\lambda_i(y, p))_{i \geqslant 1}$, we obtain

$$A^y(S_\alpha \otimes E_\beta)(y, p) = -\frac{1}{2} \sum_{i \geqslant 1} \lambda_i(y, p) F_0 \cdot S_\alpha \otimes E_\beta(y, p)$$

or

$$A^y = -\tfrac{1}{2} H_y F_0 \cdot\,. \tag{3.11}$$

With $f \in C^\infty(U)$ one has

$$\begin{aligned}
D_0(f S_\alpha \otimes E_\beta) &= \operatorname{grad}_M f \cdot S_\alpha \otimes E_\beta + f D_0(S_\alpha \otimes E_\beta) \\
&= (F_0 f) F_0 \cdot S_\alpha \otimes E_\beta + \operatorname{grad}_{N_y} f \cdot S_\alpha \otimes E_\beta \\
&\quad + f D_0^y(S_\alpha \otimes E_\beta) - \tfrac{1}{2} f H_y F_0 \cdot S_\alpha \otimes E_\beta \\
&= (F_0 \cdot \nabla_{F_0} + D_0^y - \tfrac{1}{2} H_y F_0 \cdot)(f S_\alpha \otimes E_\beta),
\end{aligned}$$

arriving thus at

$$D_0 = F_0 \cdot \nabla_{F_0} + D_0^y - \tfrac{1}{2} H_y F_0 . \tag{3.12}$$

To apply the results of Section 2 we need to trivialize the Hilbert bundle $L^2(S \otimes E \mid U)$. To this end we introduce the function $a \in C^\infty(\overline{U}) \simeq C^\infty(\mathbb{R}_+, C^\infty(N))$ by

$$i_y^* \omega_y =: a(y)^{-2} \omega_0 . \tag{3.13}$$

Next, we denote by P_y the parallel transport in $S \otimes E$ from N to N_y along normal geodesics and set $E^y := E \mid N_y$. Then we define the trivialization by

$$\begin{aligned} \Psi &: L^2(\mathbb{R}_+, L^2(S^0 \otimes E^0)) \to L^2(S \otimes E \mid U), \\ \Psi\sigma \circ i_y &:= a(y) P_y \sigma(y). \end{aligned} \tag{3.14}$$

Proceeding as in [B2, Sec. 5], we now obtain the following lemma.

LEMMA 3.1. *The Dirac-Schrödinger operator $D = D_0 + V$ in* (3.3) *is, on U, equivalent in the sense of* (2.3) *to the operator*

$$\begin{aligned} &-\partial_y + (D_0(y) + A(y)) + \tau W(y) \\ &=: -\partial_y + S(y) + \tau W(y) \end{aligned} \tag{3.15}$$

acting in $L^2(\mathbb{R}_+, L^2(S^0 \otimes E^0))$ with domain $C_0^\infty((0, \infty), C^\infty(S^0 \otimes E^0))$, where $\tau := \sqrt{-1} F_0 \cdot$ is the grading of $S^0 \otimes E^0$.

The operator functions S and W satisfy (2.4) *and* (2.5). *They are explicitly given by*

$$D_0(y) = \sum_{j \geq 1} F_0 \cdot F_j \cdot \tilde{\nabla}^y_{F_j(y)} \tag{3.16a}$$

where $Ti_y(F_j(y)(p)) := P_y(F_j(p))$ and $\tilde{\nabla}^y := \Psi^* \nabla^y$, ∇^y the canonical connection on $S^y \otimes E^y$;

$$A(y) = \sum_{j \geq 1} \langle F_j(y), \nabla^N \log a(y) \rangle F_0 \cdot F_j \cdot ; \tag{3.16b}$$

$$W := \Psi^{-1} (I \otimes \Phi) \Psi . \tag{3.16c}$$

Proof. The proof coincides with the proof of Theorem 5.3 in [B2], mutatis mutandis. For the convenience of the reader, we repeat the main steps. Observe first that multiplication by F_0 is unitary and that by (3.12)

$$F_0 \cdot D_0 = -\nabla_{F_0} + F_0 \cdot D_0^y + \tfrac{1}{2} H_y . \tag{3.17}$$

Under Ψ the various terms in (3.17) transform as follows. From [B2, (5.20a)] we obtain

$$-\Psi^{-1}\nabla_{F_0}\Psi = -\partial_y - \tfrac{1}{2}H_y.$$

The operator D_0^y is just the canonical Dirac operator on $S^y \otimes E^y$; so we obtain from the analogue of [B2, (5.20d,e)]

$$\Psi^{-1}F_0 \cdot D_0^y\Psi = \sum_{j\geq 1} [F_0 \cdot F_j \cdot \tilde{\nabla}^y_{F_j(y)} + \langle F_j(y), \nabla^N \log a(y)\rangle F_0 \cdot F_j \cdot].$$

This completes the proof. □

Note that the map ρ: $TN \to \operatorname{End} S \otimes E$, $\rho(F)(s \otimes e) := (F_0 \cdot F \cdot s) \otimes e$, induces a representation of $\mathscr{C}\ell N$ on $S \otimes E$. Thus, $D_0(y)$ is a Dirac-type operator; the connection $\tilde{\nabla}^y$ is, however, not a Dirac connection, and the vector fields $(F_j(y))_{j\geq 1}$ do not form an orthonormal frame for TN in general.

It follows from [B2, (5.39)] that for $y = 0$ we have $\tilde{\nabla}^0 = \nabla^N \otimes 1 + 1 \otimes \nabla^E$, $F_j(0) = F_j$, $a(0) = 1$; hence,

$$S(0) = \sum_{j\geq 1} F_0 \cdot F_j \cdot \tilde{\nabla}^0_{F_j} \tag{3.18}$$

is the Dirac operator on N with coefficients in E.

For simplicity we will assume in what follows that

$$M \text{ has the product metric near } N, \tag{3.19a}$$

$$W \text{ is constant in a neighborhood of } y = 0. \tag{3.19b}$$

Then the condition (2.12) reduces to an assumption on f

$$(1 - y/\beta)f(y) = f(0) = 1 \text{ near } y = 0. \tag{3.20}$$

This will also be assumed from now on.

Returning to the assumptions of Lemma 2.1, we see that in the geometric setting (3.1) we only have to find smooth positive functions h and f such that

$$\int_0^\infty \left[f(y)\left|\frac{f'}{f} - \frac{h'}{h}\right|^2(y) + f\beta^2(y)\left(1 + \frac{f^2}{h^2}(y)\right)\right] dy < \infty. \tag{3.21}$$

If this holds, then the potential $\overline{W}$ has to satisfy the conditions (3.19b), (2.14), (2.15), and

$$\int_0^\infty f(y)\,\|\overline{W}'(y)\|^2\, dy < \infty; \tag{3.22}$$

for all such potentials we then obtain the index formula (2.13). However, this formula can be made more precise by evaluating the adiabatic limit in (2.13). To do so, we note that $\overline{W}(0)$ is invertible in view of (2.14); so E^0 splits as $E^0 = E^0_+ \oplus E^0_-$ into the eigenspaces of $\overline{W}(0)$ with positive and negative eigenvalues, respectively. We project the connection ∇^E to connections $\nabla^{\pm}$ on $E^0_{\pm}$ and obtain from $D_0(0)$ two Dirac operators on N:

$$D_0^{++}\colon C^\infty(S^0_+ \otimes E^0_+) \to C^\infty(S^0_- \otimes E^0_+)$$

and

$$D_0^{+-}\colon C^\infty(S^0_+ \otimes E^0_-) \to C^\infty(S^0_- \otimes E^0_-).$$

With this notation we can prove the following theorem.

THEOREM 3.2. *Assume* (3.1), (3.19a,b), *and the existence of smooth positive functions h and f satisfying* (3.20) *and* (3.21). *Then for any potential V satisfying* (3.19b), (2.14), (2.15), *and for λ_0 sufficiently large, we have*

$$\begin{aligned} L^2\text{-ind}(D_0 + \lambda_0 V) &= -\frac{1}{2}\lim_{\lambda\to\infty} \eta(S(0) + \lambda\tau W(0)) \qquad (3.23)\\ &= -\text{ind } D_0^{++}\\ &= -\int_N \hat{A}(N) \wedge \text{ch } E^0_+ . \end{aligned}$$

Proof. Combining Lemma 2.1 with [BC, (2.45)], we obtain

$$L^2\text{-ind}(D_0 + \lambda_0 V) = -\tfrac{1}{2}[\text{ind } D_0^{++} - \text{ind } D_0^{+-}].$$

Since N is spin cobordant to zero, we also find

$$0 = \text{ind } D_0^+ = \text{ind } D_0^{++} + \text{ind } D_0^{+-}.$$

The proof is completed using the Atiyah-Singer formula. □

We will show below and also in the next section that the assumptions of Theorem 3.2 are satisfied in cases of interest. We now proceed to define the notion of "h-control".

M is said to have *h-controlled geometry at infinity* for some positive function $h \in C^\infty(\mathbb{R}_+)$ if (3.1) holds and we can find a smooth positive function f with (2.8) such that (3.21) holds for every Dirac operator on M canonically associated with the metric. Note that the operator enters only through the function β defined in

(2.16). To make this work, it is necessary to obtain estimates of β in terms of h, which in turn must be well adapted to the geometry of M near infinity.

The simplest case arises for warped products. If $U = (0, \infty) \times_{\tilde{h}} N$ with metric $dy^2 \oplus \tilde{h}(y)^2 g$, then the separation of variables yields (see [B2, Sec. 5])

$$D \simeq -\partial_y + \frac{1}{\tilde{h}(y)}[S(0) + \tau\overline{W}(y)] \tag{3.24}$$

with $S(0)$ a Dirac operator on N, given by (3.18). Note that the representation (3.24) for the Dirac operator does not contain a term involving f' if, e.g., the coefficient bundle together with its metric and connection are pulled back from N via the natural projection. (This is false in the general case; see [B2, Sec. 5].) Hence, the choice of h is now very simple: taking

$$h(y) := \tilde{h}(y),$$

we find

$$\beta(y) \equiv 0.$$

Thus, (3.21) is certainly satisfied with

$$f(y) := h(y);$$

so we only have to assume that h satisfies (2.8).

To derive the result of Callias, we let

$$U := \{x \in \mathbb{R}^m \,|\, |x| > 1\}$$

with metric $dy^2 \oplus y^2 g_0$, where $g_0 =$ the standard metric on $N = S^{m-1}$. We choose $0 < a < 1$ and put

$$h(y) := ay =: f(y), \qquad y \geqslant 1,$$

such that

$$D \simeq -\partial_y + \frac{1}{ay}\left[a \sum_{i \geqslant 1} F_0 \cdot F_j \cdot \tilde{\nabla}^0_{F_i} + \tau\overline{W}(y)\right].$$

Clearly, the factor a does not affect the index formula (3.23).

Next, we examine the assumptions on

$$\overline{W}(y) = ayW(y).$$

Except for the initial condition (3.19b), we have to require

$$C_3^2 \leqslant a^2 y^2 W(y)^2, \tag{3.25a}$$

$$\|S(0)W(y) + W(y)S(0)\| \leqslant C_4/ay, \tag{3.25b}$$

$$\int_0^\infty y\|(yW(y))'\|\,dy < \infty. \tag{3.25c}$$

This means essentially that $W(y) \approx y^{-1}W(1)$, i.e., the potential has to decay at infinity.

Though this is an interesting condition, too, it does not give the original result of Callias. To obtain it, we have to observe that the choice of f in our setting is by no means unique. In fact, we could also choose, e.g.,

$$f(y) \equiv 1,$$

in which case (3.21) reduces to

$$\int_1^\infty \frac{dy}{y^2} < \infty.$$

Then the conditions on W become

$$C_3^2 \leqslant W(y)^2,$$

$$\|S(0)W(y) + W(y)S(0)\| \leqslant C_4,$$

$$\int_0^\infty \|W'(y)\|\,dy < \infty.$$

Keeping this in mind, we obtain the following generalization of the Callias theorem [C, Theorem 2].

COROLLARY 3.3. *Let M satisfy* (3.1) *with $U = \{x \in \mathbb{R}^m \mid |x| > 1\}$, m odd, equipped with the standard metric. If f is a smooth positive function on $[1, \infty)$ satisfying* (2.8) *and*

$$\int_1^\infty f(y)\left|\frac{f'}{f}(y) - \frac{1}{y}\right|^2 dy < \infty,$$

and if the potential V satisfies the conditions (2.14), (2.15) *(with $\overline{W}(y) = fW(y)$), then $D + V$ has a finite L^2-index given by*

$$L^2\text{-ind}(D + V) = -\int_{S^{n-1}} \mathrm{ch}\, E_+^0.$$

In view of (3.24) we can immediately obtain analogous results for manifolds with more general ends which are warped products.

COROLLARY 3.4. *Let M satisfy* (3.1) *with $U = (0, \infty) \times_{\tilde{h}} N$, m odd. If f is a smooth positive function on* $[0, \infty)$ *satisfying* (2.8) *and*

$$\int_1^\infty f(y)\left|\frac{f'}{f}(y) - \frac{\tilde{h}'}{\tilde{h}}(y)\right|^2 dy < \infty$$

and if the potential V satisfies the conditions (2.14) *and* (2.15) (*with* $\overline{W}(y) = fW(y)$), *then $D + V$ has a finite L^2-index given by*

$$L^2\text{-ind}(D + V) = -\int_N \hat{A}(N) \wedge \operatorname{ch} E_+^0 .$$

Remark. N. Anghel [A, Theorem 0.4] has proved a generalization of the Callias Theorem to warped ends as above but imposing the conditions

$$\tilde{h}(y) \to \infty, \qquad y \to \infty,$$

and (essentially) $\overline{W}(y) := W(y)$, i.e., $f(y) \equiv 1$. This does not follow directly from Corollary 3.4 as it stands, as can be seen from the example $\tilde{h}(y) = e^y$. However, the representation (3.24) does not involve f'; hence, the Fredholm results given in [A, Prop. 4.2] allow us to deform $\tilde{h}$ to the function $h(y) = y$ without changing the index. Thus, after this deformation, the above corollary applies.

If we consider more general metrics than warped products, e.g., asymptotically Euclidean metrics as in [B1, Sec. 2], then β may no longer vanish and (3.21) is not so easy to deal with. The following result will allow us, however, to extend the index calculation to a large class of complete noncompact manifolds, e.g., with finite volume and pinched negative curvature. This will be the object of a future publication.

THEOREM 3.5. *Let M satisfy* (3.1) *and the following conditions.*

(a) *The curvature transformation $\mathcal{R}$ of the Dirac bundle $S \otimes E$ has uniformly bounded norm*

$$\|\mathcal{R}_q\| \leqslant C_7, \qquad q \in U.$$

(b) *The second fundamental form of N_y is uniformly bounded in y,*

$$\|\Pi_y(i_y(p))\| \leqslant C_8, \qquad y \geqslant 0, p \in \mathbb{N}.$$

(c) *The function $a(y) \in C^\infty(N)$, introduced in* (3.13), *is constant on N for all $y \geqslant 0$.*

Then, with $h \equiv 1$ in (2.17), we have for β in (2.16) the estimate

$$\beta(y) \leqslant C_9. \tag{3.26}$$

Remark. The proof of Theorem 3.5 will be broken up into a sequence of lemmas. Before embarking on it, we want to mention the following application. Assume again (3.1) but require in addition the following special properties.

$$TN \text{ has a global orthonormal frame } (F_i)_{i\geqslant 1} \text{ such that } [F_i, F_j] = \sum_k c_{ij}^k F_k \tag{3.27a}$$
with c_{ij}^k constant on N;

the parallel translates $\tilde{F}_i$ in the y-direction satisfy

$$[\tilde{F}_i, \tilde{F}_j] = \widetilde{[F_i, F_j]} = \sum_k c_{ij}^k \tilde{F}_k \tag{3.27b}$$

and

$$[F_0, \tilde{F}_i] = \lambda_i \tilde{F}_i \qquad \text{with } \lambda_i \text{ constant on } U. \tag{3.27c}$$

This can be satisfied, for example, if N is a nilpotent Lie group with a left invariant metric, and $\lambda_i + \lambda_j = \lambda_k$ if $c_{ij}^k \neq 0$. The metric on U is then defined by requiring that $(e^{\lambda_i y}\hat{F}_i \circ i_y)_{i\geqslant 1}$ is an orthonormal frame for all $y \geqslant 0$ and $p \in N$; the assumptions (3.27) will imply that $\tilde{F}_i(i_y(p)) = e^{\lambda_i y}\hat{F}_i(i_y(p))$. It is then easy to see that the assumptions of Theorem 3.5 are satisfied for all Dirac operators canonically associated to the given metric. In particular, we can apply Theorem 3.5 to all locally symmetric spaces of rank one with finite volume.

We now start the proof of Theorem 3.5 with a simple geometric lemma.

LEMMA 3.6. *Let $A(y) := Ti_y^{-1} \circ P_y \in C^\infty(\mathbb{R}_+, C^\infty(\mathrm{End}\, TN))$. Then we have*

$$\frac{\partial}{\partial y} A(y) = A(y) P_y^{-1} \Pi_y P_y. \tag{3.28}$$

Proof. For $G \in C^\infty(\mathbb{R}_+, C^\infty(TN))$ observe that

$$\frac{\partial}{\partial y} G(y) = Ti_y^{-1}[F_0, Ti_y G(y)]. \tag{3.29}$$

This implies for $F \in C^\infty(TN)$ that

$$\begin{aligned}\frac{\partial}{\partial y} A(y)F &= Ti_y^{-1}[F_0, \tilde{F}] = Ti_y^{-1}\Pi_y P_y F \\ &= A(y) P_y^{-1} \Pi_y P_y(F). \qquad \square\end{aligned}$$

With the vector fields $F_i(y) = A(y)F_i$, associated to the local orthonormal frame $(F_i)_{i\geqslant 1}$ for TN, we now define a family of quadratic forms on $C^\infty(S^0 \otimes E^0)$ as follows:

$$Q_y(\sigma) := \sum_{i\geqslant 1} \int_N |\tilde{\nabla}^y_{F_i(y)}\sigma|^2 \operatorname{vol}_N. \tag{3.30}$$

This is clearly independent of the choice of frame. It follows from Lemma 3.6 that

$$\begin{aligned} F_i'(y) &:= \frac{\partial}{\partial y} F_i(y) = Ti_y^{-1}[F_0, \tilde{F}_i] \\ &= A(y)P_y^{-1}\Pi_y P_y(F_i); \end{aligned} \tag{3.31}$$

the corresponding family of quadratic forms will be denoted by Q_y':

$$Q_y'(\sigma) = \sum_{i\geqslant 1} \int_N |\tilde{\nabla}^y_{F_i'(y)}\sigma|^2 \operatorname{vol}_N. \tag{3.32}$$

Now recall from Lemma 3.1 that, in view of assumption (c) in Theorem 3.5, we have

$$S(y) = \sum_{i\geqslant 1} F_0 \cdot F_j \cdot \tilde{\nabla}^y_{F_i(y)}. \tag{3.33}$$

LEMMA 3.7. *Under the assumptions of Theorem* 3.5 *we have for* $\sigma \in C^\infty(S^0 \otimes E^0)$

$$\|S'(y)\sigma\|^2 \leqslant 2n^2(1 + C_7)(Q_y'(\sigma) + \|\sigma\|^2).$$

Proof. Let $\sigma_j \in C^\infty(S^0 \otimes E^0)$, $j = 1, 2$. Using Lemma 3.1 and denoting the parallel transport by the superscript $\sim$, we compute

$$\begin{aligned} \frac{\partial}{\partial y}\langle S(y)\sigma_1, \sigma_2\rangle &= \frac{\partial}{\partial y} \sum_{i\geqslant 1} \langle F_0 \cdot F_i \cdot \tilde{\nabla}^y_{F_i(y)}\sigma_1, \sigma_2\rangle \\ &= \frac{\partial}{\partial y} \sum_{j\geqslant 1} \langle F_0 \cdot \tilde{F}_i \cdot \nabla_{\tilde{F}_i}\tilde{\sigma}_1, \tilde{\sigma}_2\rangle \circ i_y \\ &= \sum_{i\geqslant 1} (F_0\langle F_0 \cdot \tilde{F}_i \cdot \nabla_{\tilde{F}_i}\tilde{\sigma}_1, \tilde{\sigma}_2\rangle) \circ i_y \\ &= \sum_{i\geqslant 1} \langle F_0 \cdot \tilde{F}_i \cdot \nabla_{F_0}\nabla_{\tilde{F}_i}\tilde{\sigma}_1, \tilde{\sigma}_2\rangle \circ i_y \\ &= \sum_{I\geqslant 1} [\langle F_0 \cdot \tilde{F}_i \cdot \mathscr{R}(F_0, \tilde{F}_i)\tilde{\sigma}_1, \tilde{\sigma}_2\rangle \circ i_y + \langle F_0 \cdot \tilde{F}_i \cdot \nabla_{[F_0, \tilde{F}_i]}\tilde{\sigma}_1, \tilde{\sigma}_2\rangle \circ i_y] \end{aligned}$$

$$= \sum_{i\geq 1} [\langle F_0 \cdot F_i \cdot P_y^{-1}\mathscr{R}(F_0, \tilde{F}_i)P_y\sigma_1, \sigma_2\rangle$$
$$+ \langle F_0 \cdot F_i \cdot \tilde{\nabla}^y_{Ti_y^{-1}[F_0,\tilde{F}_i]}\sigma_1, \sigma_2\rangle].$$

Hence, in view of (3.31), $S'(y)\sigma$ is of the form

$$S'(y)\sigma =: \sum_{i\geq 1} [F_0 \cdot F_i \cdot \tilde{\nabla}^y_{F_i'(y)}\sigma + F_0 \cdot F_i \cdot \bar{\mathscr{R}}_y(F_0 \cdot F_i)\sigma]. \tag{3.34}$$

The assertion follows. □

The next lemma allows us to compare $Q_y'(\sigma)$ and $Q_y(\sigma)$.

LEMMA 3.8. *Under the assumptions of Theorem 3.5 we have for $\sigma \in C^\infty(S^0 \otimes E^0)$*

$$Q_y'(\sigma) \leqslant C_8^2 Q_y(\sigma).$$

Proof. We write $B(y) := P_y^{-1}\Pi_y P_y$ and

$$A(y)F_i =: \sum_j a_{ij}(y)F_j, \qquad B(y)F_i =: \sum_k b_{ik}(y)F_k,$$

$$a(y) := (a_{ij}(y)), \qquad b(y) := (b_{ij}(y)).$$

Then we find with (3.31)

$$Q_y'(\sigma) = \sum_i \int_N |\tilde{\nabla}^y_{AB(y)F_i}\sigma|^2$$

$$= \sum_{i,j,j',k,k'} \int_N b_{ij'}a_{j'i}b_{ik'}a_{k'k}(y)\langle\tilde{\nabla}^y_{F_j}\sigma, \tilde{\nabla}^y_{F_k}\sigma\rangle;$$

hence with

$$c_{ij}(y) := \langle\tilde{\nabla}^y_{F_i}\sigma, \tilde{\nabla}^y_{F_j}\sigma\rangle, \qquad c(y) := (c_{ij}(y)),$$

$$Q_y'(\sigma) = \int_N \operatorname{tr}(baca^tb^t)(y).$$

Now c is a matrix of Gram type and thus $\geqslant 0$. Hence, we obtain

$$Q_y'(\sigma) = \int_N \operatorname{tr}(c^{1/2}a^tb^tbac^{1/2})(y)$$

$$\leqslant \|b(y)\|^2 \int_N \operatorname{tr} aca^t(y)$$

$$\leqslant C_8^2 Q_y(\sigma).$$ □

Finally, we have to estimate $Q_y(\sigma)$ by $\|S(y)\sigma\|^2$ which amounts to an argument of Weitzenböck type.

LEMMA 3.9. *Under the assumptions of Theorem* 3.5 *there is a constant* C_{10} *such that*

$$|\|S(y)\sigma\|^2 - Q_y(\sigma)| \leqslant C_{10}(Q_y(\sigma)^{1/2}\|\sigma\| + C_7\|\sigma\|^2).$$

Proof. We compute, for $\sigma \in C^\infty(S^0 \otimes E^0)$ and $p \in N$,

$$\begin{aligned}
|S(y)\sigma(p)|^2 &= \langle S(y)^2\sigma(p), \sigma(p)\rangle \\
&= \sum_{i,j} \langle F_0 \cdot F_i \cdot \tilde{\nabla}^y_{F_i(y)} F_0 \cdot F_j \cdot \tilde{\nabla}^y_{F_j(y)}\sigma, \sigma\rangle(p) \\
&= \sum_{i,j} \langle F_0 \cdot \tilde{F}_i \cdot \nabla_{\tilde{F}_i} F_0 \cdot \tilde{F}_j \cdot \nabla_{\tilde{F}_j}\tilde{\sigma}, \tilde{\sigma}\rangle \circ i_y(p) \\
&= \sum_{i,j} [-\langle F_0 \cdot \tilde{F}_i \cdot \Pi_y(\tilde{F}_i) \cdot \tilde{F}_j \cdot \nabla_{\tilde{F}_j}\tilde{\sigma}, \tilde{\sigma}\rangle \circ i_y(p) \\
&\quad + \langle F_0 \cdot \tilde{F}_i \cdot F_0 \cdot \nabla_{\tilde{F}_i}\tilde{F}_j \cdot \nabla_{\tilde{F}_j}\tilde{\sigma}, \tilde{\sigma}\rangle \circ i_y(p) \qquad (3.35)\\
&\quad + \langle F_0 \cdot \tilde{F}_i \cdot F_0 \cdot \tilde{F}_j \cdot \nabla_{\tilde{F}_i}\nabla_{\tilde{F}_j}\tilde{\sigma}, \tilde{\sigma}\rangle \circ i_y(p)] \\
&=: A_y(\sigma)(p) + B_y(\sigma)(p) - \sum_i \langle \nabla_{\tilde{F}_i}\nabla_{\tilde{F}_i}\tilde{\sigma}, \tilde{\sigma}\rangle \circ i_y(p) \\
&\quad + \sum_{i<j} [\langle \tilde{F}_i \cdot \tilde{F}_j \cdot \mathscr{R}(\tilde{F}_i, \tilde{F}_j)\tilde{\sigma}, \tilde{\sigma}\rangle \circ i_y(p) + \langle \tilde{F}_i \cdot \tilde{F}_j \cdot \nabla_{[\tilde{F}_i, \tilde{F}_j]}\tilde{\sigma}, \tilde{\sigma}\rangle \circ i_y(p)] \\
&=: A_y(\sigma)(p) + B_y(\sigma)(p) + \tilde{Q}_y(\sigma)(p) + C_y(\sigma)(p) + D_y(\sigma)(p).
\end{aligned}$$

We estimate the terms in (3.35) one by one.

Clearly, the assumptions of Theorem 3.5 imply

$$|A_y(\sigma)(p)| \leqslant C\|\sigma(p)\| \left(\sum_i \|\tilde{\nabla}^y_{F_i(y)}\sigma(p)\|^2\right)^{1/2}. \qquad (3.36a)$$

Moreover, since $S(y)$ is independent of the choice of local orthonormal frame, we may assume that

$$\nabla^N_{\tilde{F}_i}\tilde{F}_j(i_y(p)) = 0, \qquad 1 \leqslant i, j \leqslant n.$$

This implies

$$D_y(\sigma)(p) = 0 \tag{3.36b}$$

and

$$\begin{aligned}\nabla_{\tilde{F}_i}\tilde{F}_j(i_y(p)) &= \langle \nabla_{\tilde{F}_i}\tilde{F}_j, F_0\rangle F_0(i_y(p)) \\ &= \langle \tilde{F}_j, \Pi_y(\tilde{F}_i)\rangle F_0(i_y(p)).\end{aligned}$$

Hence, we also find

$$|B_y(\sigma)(p)| \leqslant C\|\sigma(p)\| \left(\sum_i \|\tilde{\nabla}^y_{F_i(y)}\sigma(p)\|^2\right)^{1/2}. \tag{3.36c}$$

Finally, we have

$$|C_y(\sigma)(p)| \leqslant C_7 \frac{n(n-1)}{2} \|\sigma(p)\|^2. \tag{3.36d}$$

This implies the assertion, but with $Q_y(\sigma)$ replaced by

$$\tilde{Q}_y(\sigma) := -\int_N \sum_i \langle \nabla_{\tilde{F}_i}\nabla_{\tilde{F}_i}\tilde{\sigma}, \tilde{\sigma}\rangle \circ i_y\omega_0.$$

To complete the proof, we need an integration by parts. Recall from (3.13) that the volume form ω_y on N_y was related to ω_0 by $i_y^*\omega_y = a(y)^{-2}\omega_0$, with $a(y)$ constant on N by assumption. Thus, we find

$$\begin{aligned}\tilde{Q}_y(\sigma) &= -\int_{N_y} \sum_i \langle \nabla_{\tilde{F}_i}\nabla_{\tilde{F}_i}\tilde{\sigma}, \tilde{\sigma}\rangle a(y)^2\omega_y \\ &= \int_{N_y} \sum_i [-\tilde{F}_i\langle \nabla_{\tilde{F}_i}\tilde{\sigma}, \tilde{\sigma}\rangle + \|\nabla_{\tilde{F}_i}\tilde{\sigma}\|^2] a(y)^2\omega_y \\ &=: E_y(\sigma) + Q_y(\sigma).\end{aligned}$$

To calculate $E_y(\sigma)$ we define a vector field $V \in C^\infty(TN_y)$ by

$$\langle V, W\rangle := \langle \nabla_W\tilde{\sigma}, \tilde{\sigma}\rangle, \qquad W \in C^\infty(TN_y).$$

Then it follows that

$$\begin{aligned} E_y(\sigma) &= \int_{N_y} a(y)^2 \operatorname{div} V \\ &= \int_{N_y} \operatorname{div}(a(y)^2 V) \\ &= 0. \end{aligned}$$

The proof is complete. □

Proof of Theorem 3.5. The assertion of the theorem amounts to showing that

$$\|S'(y)(I + |S(y)|)^{-1}\| \leqslant C_9 \tag{3.37}$$

for some constant C_9, independent of y. Combining Lemmas 3.7, 3.8, and 3.9, we obtain for $\sigma \in C^\infty(S^0 \otimes E^0)$

$$\|S'(y)\sigma\|^2 \leqslant C(\|S(y)\sigma\|^2 + \|\sigma\|^2) \tag{3.38}$$

with C independent of y and σ. Then (3.38) holds for all $\sigma \in \mathscr{D}(S(y))$ since $S(y)$ with domain $C^\infty(S^0 \otimes E^0)$ is essentially self-adjoint. Substituting $\sigma = (I + |S(y)|)^{-1}\eta$, $\eta \in L^2(S^0 \otimes E^0)$, we obtain (3.37). □

Of course, it is not difficult to estimate C_9 explicitly in terms of C_7 and C_8.

4. An example. We shall now illustrate our results in a simple but interesting case. Let M be $\mathbb{R}^{2k+1}$. We consider various metrics, notably the flat metric $\bar{g}$, and for each $\varepsilon \in (0, 1/2]$ a "comparison metric" g_ε, $g_\varepsilon := dy^2 \oplus h_\varepsilon(y)^2 g_s$; here, y is the Euclidean length function, g_s the standard metric on S^{2k}, and $h_\varepsilon \in C^\infty(\mathbb{R}_+)$ is positive on $(0, \infty)$ which satisfies

$$h_\varepsilon(y) = \begin{cases} \sin y & 0 \leqslant y \leqslant \pi/2, \\ 1 & \text{near } \infty; \end{cases} \tag{4.1}$$

as well as, for $y \geqslant \pi/2$

$$|h_\varepsilon(y) - 1| \leqslant \varepsilon, \tag{4.2a}$$

$$|h_\varepsilon'(y)| \leqslant \varepsilon, \tag{4.2b}$$

$$h_\varepsilon''(y) \leqslant \varepsilon. \tag{4.2c}$$

Thus, $(\mathbb{R}^{2k+1}, g_\varepsilon)$ looks like the standard hemisphere S_+^{2k+1} with a cylinder attached.

Now consider a metric g on $\mathbb{R}^{2k+1}$ which coincides with g_ε on $y \geqslant \pi/2$ and satisfies

$$g \geqslant g_\varepsilon \text{ everywhere;} \tag{4.3}$$

then our manifold will be $M := (\mathbb{R}^{2k+1}, g)$. To link this to the notation of Section 3 we put

$$U := (\pi/2, \infty) \times S^{2k}, \tag{4.4}$$

which satisfies (3.19a) by construction. Denote by $S = S_g$ the spin bundle on M, constructed with the metric g. As a coefficient bundle, we choose $E := S_{\bar{g}}$, the spin bundle on M constructed with the flat metric. These data define the Dirac operator D_0 with coefficients in E.

Finally, we define the potential V by

$$V(p)s \otimes e := s \otimes \bar{F}(p) \circ e, \qquad s \in S_p, \qquad e \in E_p, \tag{4.5}$$

where "$\circ$" denotes Clifford multiplication in the flat metric $\bar{g}$ (whereas "$\cdot$" is Clifford multiplication in the metric g) and $\bar{F}$ is the smooth vector field on M given by

$$\bar{F}(y, \omega) := p(y)\frac{\partial}{\partial y}. \tag{4.6}$$

Here, $p \in C^\infty(\mathbb{R}_+)$ is odd near zero (i.e., $p^{(2j)}(0) = 0$, $j \geqslant 0$) and $p(y) = 1$, $y \geqslant \pi/2$. It is a routine matter to check the assumptions of Theorem 3.2: we choose $h := \tilde{h}$ and f such that (3.20) holds and $f = \tilde{h}$ near infinity. In the representation (3.15) we easily find

$$S(y) = D_0(y) = \sum_{i \geqslant 1} F_i \cdot \nabla_{F_i}, \quad \text{the canonical Dirac operator on } S^{2k} \text{ with coefficients in } E|S^{2k} \tag{4.7a}$$

and

$$W(y)(\omega)(s \otimes e) = s \otimes \omega \circ e, \qquad \omega \in S^{2k}, \qquad s \in S^0_\omega, \qquad e \in E^0_\omega. \tag{4.7b}$$

Hence, the remaining conditions (3.19b), (2.14), (2.15), (3.21), and (3.22) are all satisfied.

Thus, Theorem 3.2 gives (see [Gi, Lemma 3.8.9]) the following lemma.

LEMMA 4.1. *For λ sufficiently large we have*

$$L^2\text{-ind}(D_0 + \lambda V) = 1. \tag{4.8}$$

It is fairly easy to discuss the Fredholm properties of this special Dirac-Schrödinger operator. We need the following lemma.

LEMMA 4.2. *On U we have the estimate*

$$\|D_\lambda \varphi\|_{L^2}^2 \geqslant \|\nabla \varphi\|_{L^2}^2 + [(\lambda - k)^2 - k/2 + O(\varepsilon)]\,\|\varphi\|_{L^2}^2, \qquad \varphi \in C_0^\infty(S \otimes E\,|\,U). \quad (4.9)$$

Proof. We have $D_\lambda^* D_\lambda = (D - \lambda V)(D + \lambda V) = D^2 + \lambda[D, V] - \lambda^2 V^2$. It follows from the Lichnerowicz Theorem (see, e.g., [LM, Thm. 8.8]) that

$$D^2 = \nabla^* \nabla + \kappa_g/4 \quad (4.10)$$

where κ_g denotes the scalar curvature of g. A well-known formula (see [LM, Chap. IV (6.16)]) gives, in view of (4.2),

$$\kappa_g(x) \geqslant 2k(2k-1) + O(\varepsilon), \qquad x \in U. \quad (4.11)$$

Moreover, from (4.6) we see that

$$V^2(x) = -id, \qquad x \in U. \quad (4.12)$$

Hence, it remains to compute the commutator. We find, with $\tilde{F}_0 := \partial/\partial y$ and $(F_i)_{i\geqslant 1}$ a local orthonormal frame on S^{2k}, with parallel translates $(\tilde{F}_i)_{i\geqslant 1}$ in the y-direction, for $\sigma \in C^\infty(S)$, $\theta \in C^\infty(E)$,

$$\begin{aligned}
[D, V]\sigma \otimes \theta &= \sum_{i\geqslant 0} \tilde{F}_i \cdot \sigma \otimes [\nabla^{\bar{g}}_{\tilde{F}_i}(\tilde{F}_0 \circ \theta) - \tilde{F}_0 \circ \nabla^{\bar{g}}_{\tilde{F}_i}\theta] \\
&= \sum_{i\geqslant 0} \tilde{F}_i \cdot \sigma \otimes (\nabla^{\bar{g}}_{\tilde{F}_i}\tilde{F}_0) \circ \theta \\
&= \sum_{i\geqslant 1} \tilde{F}_i \cdot \sigma \otimes y^{-1}\tilde{F}_i \circ \theta =: \sum_{i\geqslant 1} A_i(\sigma \otimes \theta).
\end{aligned}$$

For the last equality we have used the second fundamental form (see (3.2)) and $\nabla^{\bar{g}}_{\tilde{F}_0}\tilde{F}_0 = 0$. Now observe only that $A_i^2 = y^{-2}\|\tilde{F}_i\|_{\bar{g}}^2$, which implies $\|A_i\| = 1$. Hence, we find

$$\|[D, V]\| \leqslant 2ky^{-1}\|\tilde{F}_i\|_{\bar{g}} = 2k. \quad (4.13)$$

With (4.10), (4.11), and (4.12), we derive the assertion. □

We obtain now easily the following. (See [GL, Sec. 3].)

LEMMA 4.3. (1) *D_λ has a unique closed extension with domain independent of λ, $\mathscr{D}(D_\lambda) = \mathscr{D}(D_0)$.*

(2) *If $\lambda \notin [k - \sqrt{k/2} + O(\varepsilon), k + \sqrt{k/2} + O(\varepsilon)]$, then D_λ is a Fredholm operator, depending continuously on λ.*

Proof. (1) This assertion is obvious since D_0 has a unique closed extension [GL, Thm. 1.17] and V is bounded in $L^2(S \otimes E)$.

(2) The Fredholm property is easily derived from [H, 19.1.3]: if $(f_j)_{j \in \mathbb{N}}$ is bounded in $\mathscr{D}(D_0)$ with respect to the graph norm and $(D_\lambda f_j)_{j \in \mathbb{N}}$ is convergent in $L^2(S \otimes E)$, then (4.9) implies that $(f_j)_{j \in \mathbb{N}}$ converges in $L^2(S \otimes E|U)$, provided that $\lambda \notin [k - \sqrt{k/2} + O(\varepsilon), k + \sqrt{k/2} + O(\varepsilon)]$. The estimate in Lemma 4.2 then implies that $(f_j)_{j \in \mathbb{N}}$ is bounded in the Sobolev space $H^1(S \otimes E|M \backslash U)$. Thus, the Rellich theorem implies the existence of a subsequence convergent in $L^2(S \otimes E)$.

We conclude that both D_λ and $D_\lambda^* = D_{-\lambda}$ have a finite-dimensional kernel and closed range. The continuity in λ is obvious. □

We proceed to show that the innocent-looking operator D_λ in fact loses Fredholm-ness for some $\lambda \in [k - \sqrt{k/2} + O(\varepsilon), k + \sqrt{k/2} + O(\varepsilon)]$. We use the Bochner-Lichnerowicz technique to prove that ind $D_\lambda = 0$ for λ sufficiently small.

LEMMA 4.4. *In $M \backslash U$ we have the estimate*

$$\|[D, V]\|_{L^2(S \otimes E|M \backslash U)} \leqslant \tau_k := (2k+1)\frac{\pi}{2} \sup_{0 \leqslant y \leqslant \pi/2} \left(|p'(y)| + \left| \frac{p(y)}{y} - p'(y) \right| \right). \tag{4.14}$$

Thus, for $\lambda < \min\{\min_{M \backslash U} \kappa_g / 4\tau_k, k - \sqrt{k/2} + O(\varepsilon)\}$ we have ind $D_\lambda = 0$.

Proof. Consider a local orthonormal frame $(F_i)_{i=0}^{2k}$ for the metric g. We decompose

$$F_i = \left\langle F_i, \frac{\partial}{\partial_y} \right\rangle_{\bar{g}} \cdot \frac{\partial}{\partial y} + F_i^\perp$$

and compute

$$\begin{aligned}
[D, V](\sigma \otimes \theta) &= \sum_{i \geqslant 0} F_i \cdot \sigma \otimes (\nabla^{\bar{g}}_{F_i} \bar{F}) \circ \theta \\
&= \sum_{i \geqslant 0} F_i \cdot \sigma \otimes (\nabla^{\bar{g}}_{\langle F_i, \partial/\partial y \rangle_{\bar{g}} \partial/\partial y} + \nabla^{\bar{g}}_{F_i^\perp}) \left(p(y) \frac{\partial}{\partial y} \right) \circ \theta \\
&= p'(y) \sum_{i \geqslant 0} F_i \cdot \sigma \otimes \left\langle F_i, \frac{\partial}{\partial y} \right\rangle_{\bar{g}} \frac{\partial}{\partial y} \circ \theta \\
&\quad + \frac{p(y)}{y} \sum_{i \geqslant 0} F_i \cdot \sigma \otimes F_i^\perp \circ \theta \\
&= p'(y) \sum_{i \geqslant 0} F_i \cdot \sigma \otimes F_i \circ \theta + \left(\frac{p(y)}{y} - p'(y) \right) \sum_{i \geqslant 0} F_i \cdot \sigma \otimes F_i^\perp \circ \theta.
\end{aligned}$$

To complete the proof of (4.14) we observe that we have, for $y \in [0, \pi/2]$,

$$1 \leqslant y/\sin y \leqslant \pi/2.$$

Hence, since $(F_i)_{i \geqslant 0}$ is g-orthonormal,

$$|F_i|_{\tilde{g}}^2 = \left\langle F_i, \frac{\partial}{\partial y} \right\rangle_{\tilde{g}}^2 + |F_i^{\perp}|_{\tilde{g}}^2 = \left\langle F_i, \frac{\partial}{\partial y} \right\rangle_{g_\varepsilon}^2 + \frac{y^2}{\sin^2 y} |F_i^{\perp}|_{g_\varepsilon}^2 \leqslant \frac{y^2}{\sin^2 y} |F_i|_{g_\varepsilon}^2 \leqslant \frac{\pi^2}{4},$$

$$|F_i^{\perp}|_{\tilde{g}}^2 \leqslant |F_i|_{\tilde{g}}^2 \leqslant \frac{\pi^2}{4};$$

thus,

$$\|[D, V]\|_{L^2(S \otimes E | M \setminus U)} \leqslant \tau_k.$$

Now assume $\lambda < \min\{\min_{M \setminus U} \kappa_g / 4\tau_k, k - \sqrt{k/2} + O(\varepsilon)\}$. It follows from Lemma 4.2 and (4.14) that in this case

$$D_\lambda^* D_\lambda \geqslant \delta, \qquad D_\lambda D_\lambda^* \geqslant \delta,$$

for some $\delta > 0$, hence the assertion.

We can now derive some interesting geometric consequences.
First, if $\lambda > k + \sqrt{k/2} + O(\varepsilon)$, we know that the estimate

$$\kappa_g(x) \geqslant 4\tau_k \lambda > 4\tau_k(k + \sqrt{k/2} + O(\varepsilon))$$

cannot hold for all $x \in M \setminus U$. Hence, for at least one $x_0 \in M \setminus U$, we obtain

$$\kappa_g(x_0) < 4(k + \sqrt{k/2} + O(\varepsilon))(2k+1)\frac{\pi}{2} \sup_{0 \leqslant y \leqslant \pi/2} \left(|p'(y)| + \left| \frac{p(y)}{y} - p'(y) \right| \right). \tag{4.15}$$

To derive a good bound, we use our freedom in the choice of p: we may consider any $p \in C^1[0, \pi/2]$, which is odd at 0 and satisfies $p(\pi/2) = 1$, $p'(\pi/2) = 0$. Construct now $t = p(y)$ as follows. Let the graph consist of two pieces: a small circular arc tangent to $t \equiv 1$ at $(\pi/2, 1)$ and the tangent to that arc through $(0, 0)$. If the point of tangency has abscissa $y_0 \in (0, \pi/2)$, then we have

$$\begin{aligned} p'(y) &\equiv \frac{p(y_0)}{y_0} \equiv \frac{p(y)}{y}, \qquad && y \in [0, y_0], \\ 0 \leqslant p'(y) &\leqslant \frac{p(y)}{y} \leqslant \frac{p(y_0)}{y_0}, \qquad && y \in [y_0, \pi/2]. \end{aligned} \tag{4.16}$$

Since $p(y_0)/y_0$ can be made arbitrarily close to $2/\pi$, it follows that

$$\kappa_g(x_0) \leqslant 2(2k + \sqrt{2k})2k + O(\varepsilon).$$

PROPOSITION 4.5. *Let g_s be the standard metric on S_+^{2k+1}. For any metric g on S_+^{2k+1} such that*

(i) *g extends to a metric $\geqslant g_\varepsilon$ on $\mathbb{R}^{2k+1}$,*

(ii) *$g = g_s$ on ∂S_+^{2k+1},*

and

(iii) *$g \geqslant g_s$ on S_+^{2k+1},*

one has

$$\inf \kappa_g \leqslant 2(2k + \sqrt{2k})2k + O(\varepsilon).$$

Proof. This is an immediate consequence of the above discussion. □

Remark. M. Llarul [L] proved a sharp version of this result, namely,

$$\inf \kappa_g \leqslant \kappa_{g_s} = (2k+1)2k$$

but for metrics on S^{2k+1} satisfying the condition $g \geqslant g_s$ on the *whole* sphere.

Another consequence of the preceding discussion is the following "semiglobal" estimate.

PROPOSITION 4.6. *With the same assumptions as in Proposition 4.5, given $\delta > 0$ and $\lambda > k + \sqrt{k/2}$, either*

$$\inf_{p(x)\geqslant\delta} \kappa_g(x) < 4\lambda(\tau_k - \lambda\delta)$$

or

$$\inf_{p(x)\leqslant\delta} \kappa_g(x) < 4\lambda\tau_k.$$

In particular, choosing $\lambda = \lambda_\delta := \tau_k/\delta^2$, one obtains that, if

$$\inf_{p(x)\leqslant\delta} \kappa_g(x) \geqslant 4\delta^{-2}\tau_k^2,$$

then

$$\inf_{p(x)\geqslant\delta} \kappa_g(x) < 0.$$

Proof. Since ind $D_\lambda = 1$ if $\lambda > k + \sqrt{k/2}$, the statement follows from the following estimates for the Dirac-Laplacian.

(i) On U one has

$$D_\lambda^* D_\lambda \geqslant \nabla^* \nabla + (\lambda - k)^2 - \frac{k}{2} + O(\varepsilon);$$

(ii) on $M_{\delta^+} = \{x \in S_+^{2k+1} | p(x) \geqslant \delta\}$ one has

$$D_\lambda^* D_\lambda \geqslant \nabla^* \nabla + \tfrac{1}{4}\kappa_g - \tau_k \lambda + \delta^2 \lambda^2;$$

(iii) on $M_{\delta^-} = \{x \in S_+^{2k+1} | p(x) \leqslant \delta\}$ one has

$$D_\lambda^* D_\lambda \geqslant \nabla^* \nabla + \tfrac{1}{4}\kappa_g - \tau_k \lambda. \qquad \square$$

Remark. The above results can be extended to spin manifolds (M, g) with boundary ∂M admitting 1-contracting maps $f: (M, g) \to (S_+^{2k+1}, g_s)$ such that $f|_{\partial M}: \partial M \to S^{2k}$ has nonzero $\hat{A}$-degree. (Cf. [L, Thm. 4.1].) In particular, similar estimates hold for even-dimensional hemispheres.

References

[A] N. Anghel, *L^2-index formulae for perturbed Dirac operators*, Comm. Math. Phys. **128** (1990), 77–97.

[BC] J.-M. Bismut and J. Cheeger, *η-invariants and their adiabatic limits*, J. Amer. Math. Soc. **2** (1989), 33–70.

[B1] J. Brüning, *L^2-index theorems for certain complete manifolds*, J. Differential Geom. **32** (1990), 491–532.

[B2] ———, *L^2-index theorems for complete manifolds of rank-one type*, Duke Math. J. **66** (1992), 257–310.

[C] C. Callias, *Axial anomalies and index theorems on open spaces*, Comm. Math. Phys. **62** (1978), 213–234.

[G] M. Gromov, "Large Riemannian manifolds" in *Curvature and Topology of Riemannian Manifolds: Proceedings of the Seventeenth International Taniguchi Symposium, Katata, Japan, 1985*, Lecture Notes in Math. **1201**, Springer, Berlin, 1985, 108–121.

[Gi] P. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, Publish or Perish, Wilmington, 1984.

[GL] M. Gromov and H. B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Institut Hautes Études Sci. Publ. Math. **58** (1983), 295–408.

[H] L. Hörmander, *The Analysis of Linear Partial Differential Equations, Volume III*, Springer, Berlin, 1985.

[L] M. Llarul, *Sharp estimates and the Dirac operator*, preprint, Univ. of Penn.

[LM] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, Princeton, 1989.

Brüning: Institut für Mathematik, Universität Augsburg, Universitätsstraße 8, D-8900 Augsburg, Germany

Moscovici: Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210-1174