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KÄHLER–HODGE THEORY FOR CONFORMAL COMPLEX CONES

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1. Introduction

In a recent publication [BL] we have introduced the notion of "Hilbert complex". By this we mean a differential complex of vector spaces,

$$0 \longrightarrow \mathcal{D}_0 \xrightarrow{D_0} \mathcal{D}_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{N-1}} \mathcal{D}_N \longrightarrow 0 , \tag{1.1}$$

where $\mathcal{D}_i$ is dense in some Hilbert space H_i and $D_i : \mathcal{D}_i \to H_{i+1}$ is closed. This additional functional analytic structure proved to be quite useful, and we obtained a very general setting to deal with questions like weak and strong Hodge decomposition, L^2–cohomology and L^2–index, Poincaré duality, smooth cohomology, Künneth type formulas etc. Our main motivation for this study was the analysis of elliptic complexes on singular spaces; we hoped that the notion of Hilbert complex would provide a convenient framework to compare the many results that exist in special cases, and to guide future analysis. We have substantiated this hope somewhat in [BL, Sec. 4] where we studied the de Rham complex in various situations. Consider, e.g. an *arbitrary* Riemannian manifold, M, and the de Rham complex with compact support,

$$0 \to \Omega_0^0(M) \xrightarrow{d_0} \Omega_0^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{N-1}} \Omega_0^N(M) \to 0 . \tag{1.2}$$

Then we ask whether we can find closed extensions of the d_i (in the natural L^2–spaces defined by the metric) which combine into a Hilbert complex of the form (1.1); every such choice will then be christened an "ideal boundary condition" (ibc) for (1.2) (inspired by the pioneering work of Cheeger on conical singularities). It turns out that this can always be done; e.g. by choosing $D_i = d_{i,\min}$ (the closure) or $D_i = d_{i,\max}$ (the adjoint of the closure of d_i^t); these two choices will be referred to as the "relative ibc" and the "absolute ibc", respectively. There may be, however, infinitely many other ibc's, with more or less pleasant features. So one is looking for further criteria to select "good" ibc's, and, of course, the case where the relative and the absolute ibc coincide deserves particular interest; this we call the "case of uniqueness".

The present paper attempts at further progress in this direction. Its main motivation is to incorporate the so called "L^2–Kähler package" (as introduced in [CGM]) into the framework of Hilbert complexes. Thus, as one of the main results (Thms. 5.6 and 5.8) we state rather simple functional analytic properties of the de Rham complex on an arbitrary Kähler manifold in order for the L^2–Kähler package to hold. Notably, we require that we have uniqueness of ibc's, which might be true for all projective varieties equipped with the Fubini study metric. This fact is beyond our reach, however, for the time being. Instead we look at the class of "conformally conic Kähler manifolds" for which we have uniqueness and where we achieve the most satisfying result (Thm. 5.9). Among these manifolds we find at least all (singular) algebraic curves and all complex cones. Thus we achieve a considerable extension of Cheeger's work in [C2], where the L^2–Kähler package for *metrically* complex cones was obtained under the additional assumption that the complex structure J is *conical* [C2, p. 119]. Conformally conic Kähler manifolds in our sense are a considerably more general class than metrically conic Kähler manifolds, and we do not require any additional assumption on the complex structure.

For general Riemann surfaces we do not have uniqueness. Nevertheless, we construct an ibc for the de Rham complex which always satisfies the L^2–Kähler package (Sec. 5).

We remark that other aspects of Kähler geometry are of interest in this context, too. For a remarkable contribution one may consult Ohsawa's work on the so-called Cheeger–Goresky–MacPherson conjecture [O].

This paper is organized as follows. Sec. 2 introduces the analytic tools to deal with conformally conic manifolds. They are closely related to the methods developed in [BS]. The results also resemble those in the conic case; in particular, the case of uniqueness prevails only if the dimension is even (Thms. 3.7, 3.8).

In Sec. 3 we investigate the relationship between unique ibc's and the Friedrichs extension of the corresponding Laplacians. This is a decisive tool in the uniqueness proof.

Sec. 4 is devoted to new invariants associated to Riemannian manifolds, M, of dimension $4k+1$, namely the deficiency indices, $n_\pm(M)$, of $\sqrt{-1}*_{2k+1}d_{2k}$. We show that these are invariant under quasi-isometries (Cor. 4.1) and that they are nontrivial (Lemma 4.3). Particularly intriguing is the formula

$$n_+(M) - n_-(M) = -\operatorname{sign}(N) \tag{1.3}$$

if M is the infinite cone over the compact manifold N. We will discuss generalizations of (1.3) elsewhere.

Finally, Sec. 5 contains the results on Kähler manifolds sketched above.

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2. The Model Situation

Assume that M is a Riemannian manifold of $\dim M = m$, with an open subset $U \subset M$ such that

$$M_1 := M \setminus U \text{ is a complete manifold with compact boundary } N, \tag{2.1}$$

U is isometric to $(0,\varepsilon) \times N$, $\dim N = m - 1 =: n$, with metric $g = h(x)^2(dx^2 \oplus x^2 g_N(x))$, where $g_N(x)$ is a family of metrics on N, smooth in $(0,\varepsilon)$ and continuous in $[0,\varepsilon)$, (2.2)

and $h \in C^\infty((0,\varepsilon) \times N)$ satisfies

$$\sup_{p\in N} |(x\partial_x)^j (x^{-c}h(x,p) - 1)| = O(x^\delta) \text{ as } x \to 0\,,\ j = 0,1\,, \tag{2.3}$$

and

$$\sup_{p\in N} \|h(x,p)^{-1} d_N h(x,p)\|_{T_p^* N, g_N(x)} = O(x^\delta) \text{ as } x \to 0\,, \tag{2.4}$$

for some $\delta > 0$ and $c > -1$.

Thus, we do not assume that $g_N(x)$ is smooth up to 0. But just continuity is not enough so we have to introduce an assumption on the asymptotic behaviour near 0. As in [B, Sec. 5] let

$$g^0 := dx^2 \oplus x^2 g_N(0)\,,$$

$$g^1 := h^{-2} g = dx^2 \oplus x^2 g_N(x)\,,$$

and denote by ∇^0, ∇^1 the Levi-Civita connections for g^0, g^1 with connection forms ω^0, ω^1. Then our assumption is (cf. [B, (5.2)])

$$\sup_{p\in N} (|g^1 - g^0|^0_{(x,p)} + x|\omega^0 - \omega^1|^0_{(x,p)}) = O(x^\delta)\,, \qquad x \to 0\,, \tag{2.5}$$

where δ is as above and the superscript ^{0} refers to g^0.

If the Riemannian manifold M satisfies (2.1) through (2.5) then we will call it a *conformally conic* manifold.

Now denote by d the exterior derivative, by d^t its formal adjoint with respect to the metric g, and by $D_{GB} : \Omega_0(M) \to \Omega_0(M)$ the Gauß-Bonnet operator; we also consider $D_{GB}^{\mathrm{ev/odd}} : \Omega_0^{\mathrm{ev/odd}}(M) \to \Omega_0^{\mathrm{odd/ev}}(M)$. Of course, $D_{GB} = D_{GB}^{\mathrm{ev}} \oplus D_{GB}^{\mathrm{odd}}$ and $(D_{GB}^{\mathrm{ev}})^t = D_{GB}^{\mathrm{odd}}$. These are closable operators in $L^2(\Lambda^* M)$ with many closed extensions in general. The relationship between the closed extensions of D_{GB}, d, d^t has been discussed in detail in [BL]; some further aspects will be given in the next section. The following fact is important for understanding the next theorem. From the weak Hodge decomposition [BL, Lemma 2.1] one easily derives that

$$\mathcal{D}(D_{GB,\min}) = \left\{ \begin{array}{l} \omega \in L^2(\Lambda^* M) \text{ there is a sequence } (\phi_n) \subset \Omega_0(M) \\ \text{such that } \phi_n \to \omega \text{ and } (d\phi_n), (d^t\phi_n) \text{ are Cauchy} \\ \text{sequences in } L^2(\Lambda^* M) \end{array} \right\} \tag{2.6}$$

thus

$$\mathcal{D}(D_{GB,\min}) \subset \mathcal{D}(d_{\min}) \cap \mathcal{D}(d^t_{\min}) \tag{2.7}$$

which is a proper inclusion in general, as we will see below. The aim of this section is to prove the following theorem.

THEOREM 2.1. *Let M be conformally conic. Then we have*

$$\mathcal{D}(d_{k,\max}) \cap \mathcal{D}(d^t_{k-1,\max}) \subset \mathcal{D}(D_{GB,\min}) \tag{2.8}$$

for all k except possibly $m/2, (m \pm 1)/2$.

For $k = m/2$ or $k = (m \pm 1)/2$ it may happen that $\mathcal{D}(D_{GB,\min}) \cap L^2(\Lambda^k T^* M)$ is a proper subset of $\mathcal{D}(d_{k,\min}) \cap \mathcal{D}(d^t_{k-1,\min})$.

This theorem is the source of all assertions concerning ideal boundary conditions and Hodge theory in the rest of the paper. Among other things, we will easily recover Cheeger's results on the L^2-Stokes theorem (cf. [C1]). But Theorem 2.1 above is much stronger than that, since it has implications in the Kähler case.

Via the transformation

$$y(x) := \frac{1}{c+1} x^{c+1} \tag{2.9a}$$

the metric changes into

$$\tilde{g} = \tilde{h}^2(dy^2 \oplus y^2 \tilde{g}_N(y)) \tag{2.9b}$$

which satisfies (2.1) through (2.5) with c=0. Thus $\tilde{g}$ is quasi-isometric to a conic metric. But we want to point out here that the inclusion (2.8) is not invariant under quasi-isometries; this will become clear in the proof below. Because of (2.9a,b), from now on we will assume $c = 0$ and write again g, h, g_N instead of $\tilde{g}, \tilde{h}, \tilde{g}_N$. As in [BS, Sec. 5] one constructs linear maps

$$\psi_k : C_0^\infty((0,\varepsilon), \Omega^{k-1}(N) \oplus \Omega^k(N)) \to \Omega_0^k(U) \tag{2.10a}$$

which extend to unitary maps

$$\psi_k : L^2((0,\varepsilon), L^2(\Lambda^{k-1}T^*N \oplus \Lambda^k T^*N, g_N(0)) \to L^2(\Lambda^k T^*U) \tag{2.10b}$$

such that

$$\psi_{k+1}^* d_k \psi_k =: h^{-1}\left[\begin{pmatrix} 0 & (-1)^k\partial_x \\ 0 & 0 \end{pmatrix} + \left(S_{0,k}^d + S_{1,k}^d(x)\right)\right] \tag{2.11a}$$

where

$$S_{0,k}^d := \begin{pmatrix} d_{N,k-1} & (-1)^k(k-n/2) \\ 0 & d_{N,k} \end{pmatrix} \tag{2.12a}$$

and $S_{1,k}^d(x)$ is a family of first order differential operators on $\Omega(N)$, smooth in $(0,\epsilon)$, and satisfying

$$\|S_{1,k}^d(x)\|_{H^1\to L^2} = O(x^\delta)\ ,\ x \to 0\ . \tag{2.13}$$

Taking adjoints we find

$$\psi_k^* d_k^t \psi_{k+1} = h^{-1}\left[\begin{pmatrix} 0 & 0 \\ (-1)^{k+1}\partial_x & 0 \end{pmatrix} + x^{-1}\left(S_{0,k}^{d^t} + S_{1,k}^{d^t}(x)\right)\right] \tag{2.11b}$$

where

$$S_{0,k}^{d^t} := \begin{pmatrix} d_{N,k-1}^t & 0 \\ (-1)^k(k-n/2) & d_{N,k}^t \end{pmatrix} \tag{2.12b}$$

and $S_{1,k}^{d^t}$ satisfies (2.13), too. Here we have used (2.3).

The ψ_k induce unitary operators [BS, (5.9a,b)]

$$\psi_{\text{ev/odd}} : C_0^\infty((0,\varepsilon), \Omega(N)) \to \Omega_0^{\text{ev/odd}}(U)\ , \qquad \text{(2.12b}$$

and from (2.11a,b) we obtain immediately

$$\psi_{\text{odd}}^* D_{GB}^{\text{ev}} \psi_{\text{ev}} = h^{-1}\left[\partial_x + x^{-1}(S_0 + S_1(x))\right] \tag{2.11c}$$

with ([BS, (5.10)])

$$S_0 = \begin{pmatrix} c_0 & d^t_{N,0} & 0 & \dots & 0 \\ d_{N,0} & c_1 & d^t_{N,1} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & d_{N,n-2} & c_{n-1} & d^t_{N,n-1} \\ 0 & \dots & 0 & d_{N,n-1} & c_n \end{pmatrix}, \quad c_k := (-1)^k(k-n/2), \tag{2.12c}$$

and $S_1(x)$ is a family of first order differential operators, smooth in $(0,\epsilon)$, satisfying the estimate (2.13). Since S_0 is elliptic, D_{GB} is a regular singular operator in the sense of [B, Sec. 3]. The spectrum of S_0, which is essential for the investigation of the closed extensions of D_{GB}, has been determined in [BS]. Because of its significance for the proof, we restate the result. We present it in a slightly different way, however, because we want to specify explicitly those forms which correspond to eigenvalues between $-1/2$ and $1/2$. Since N is compact, d_N and d^t_N have unique closed extensions, which we denote by d_N and d^t_N, too. Put ([BS, p. 699])

$$\mathcal{H}^k_{\lambda,ccl}(N) := \{\omega \in \Omega^k(N) \mid \Delta_k\,\omega = \lambda\omega, d^t_{N,k-1}\,\omega = 0\}\ , \tag{2.14}$$

the space of coclosed eigenforms of Δ_k with eigenvalue λ. In particular, $\mathcal{H}^k_{0,ccl}(N) = \hat{\mathcal{H}}^k(N)$ is the space of harmonic k-forms. Then we have an orthogonal decomposition

$$L^2(\Lambda^*N) = \bigoplus_{k\geq 0} \hat{\mathcal{H}}^k(N) \oplus \bigoplus_{\substack{k\geq 0\\ \lambda>0}} \left[\mathcal{H}^k_{\lambda,ccl}(N) \oplus d\mathcal{H}^k_{\lambda,ccl}(N)\right]. \tag{2.15}$$

LEMMA 2.2. *Let N be a compact Riemannian manifold of dimension n, and consider with $c = (c_0, \cdots, c_n) \in \mathbb{C}^{n+1}$, $\rho = (\rho_1, \cdots, \rho_n) \in \mathbb{C}^n$ the operator $S : \Omega(N) \to \Omega(N)$ defined by*

$$\omega \mapsto \begin{pmatrix} c_0 & \rho_1 d^t_0 & 0 & \dots & 0 \\ \rho_1 d_0 & c_1 & \rho_2 d^t_1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \rho_{n-1} d_{n-2} & c_{n-1} & \rho_n d^t_{n-1} \\ 0 & \dots & 0 & \rho_n d_{n-1} & c_n \end{pmatrix} \begin{pmatrix} \omega_0 \\ \\ \vdots \\ \\ \omega_n \end{pmatrix}. \tag{2.16}$$

Then the decomposition (2.15) *reduces S in the following sense:*

$$S|\hat{\mathcal{H}}^k(N) = c_k Id\ ; \tag{2.17}$$

for $\eta \in \mathcal{H}^{k-1}_{\lambda,ccl}(N)\setminus\{0\}$ *consider the space* $V_\eta \subset L^2(\Lambda^* N)$ *with basis* $\{\eta, \lambda^{-1/2}d\eta\}$. *Then*

$$S|V_\eta = \begin{pmatrix} c_{k-1} & \rho_k\lambda^{1/2} \\ \rho_k\lambda^{1/2} & c_k \end{pmatrix} . \tag{2.18}$$

Thus, from (2.18) *we obtain eigenvalues*

$$\mu_\pm(\lambda) = (c_{k-1}+c_k)/2 \pm (1/2)\sqrt{(c_{k-1}-c_k)^2+4\rho_k^2\lambda}\ , \tag{2.19}$$

if $\lambda > 0$ *and* $\mathcal{H}^{k-1}_{\lambda,ccl}(N) \neq 0$. *In particular, if* $S = S_0$ *in* (2.12 c) *we get*

$$\mu_\pm(\lambda) = (-1)^k/2 \pm \sqrt{(k-m/2)^2+\lambda}\ . \tag{2.20}$$

Proof: The proof follows from a straightforward calculation. □

COROLLARY 2.3. 1. *If* $m = 2\nu$ *is* <u>*even*</u> *then* $\operatorname{spec}(S_0)\cap(-1/2,1/2)\neq\emptyset$ *if and only if*

$$\operatorname{spec}\Delta_{\nu-1,ccl}\cap(0,1)\neq\emptyset\ .$$

Eigenforms of S_0 *with eigenvalues in* $(-1/2,1/2)$ *are in* $d^t_{N,\nu-1}(\Omega^\nu(N)) \oplus d_{N,\nu-1}(\Omega^{\nu-1}(N))$.

2. *If* $m = 2\nu+1$ *is* <u>*odd*</u> *then* $\operatorname{spec}(S_0)\cap(-1/2,1/2)\neq\emptyset$ *if and only if*

$$(\operatorname{spec}\Delta_{\nu-1,ccl}\cap(0,3/4))\cup(\operatorname{spec}\Delta_{\nu,ccl}\cap[0,3/4))\neq\emptyset\ .$$

Eigenforms of S_0 *with eigenvalues in* $(-1/2,1/2)$ *are in* $d^t_{N,\nu-1}(\Omega^\nu(N)) \oplus \Omega^\nu(N)\oplus d_{N,\nu}(\Omega^\nu(N))$.

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1: We consider first the case $k = 2j$. Pick $\omega \in \mathcal{D}(d_{2j,\max})\cap\mathcal{D}(d^t_{2j-1,\max})$ and $\varphi \in C_0^\infty((-\varepsilon,\varepsilon))$ with $\varphi = 1$ near 0. φ can be viewed as a C^∞-function on M. Clearly, we have $\omega \in \mathcal{D}(D^{\mathrm{ev}}_{GB,\max})$, and since $1-\varphi \in C_0^\infty(M)$ also

$$(1-\varphi)\omega \in \mathcal{D}(D^{\mathrm{ev}}_{GB,\max})\ .$$

Put for $t\in(0,\varepsilon)$ $U_t := (0,t)\times N$. By assumption, $M\setminus U_t$ is complete and for t small enough $(1-\varphi)|M\setminus U_t$ is 0 near the boundary $\{t\}\times N$. Thus we conclude

$$(1-\varphi)\omega \in \mathcal{D}(D^{\mathrm{ev}}_{GB,\min})\ .$$

Indeed, applying [Ch, Theorem 2.2] to the duplication of $M \setminus U_t$, which is complete and without boundary, we reach the conclusion. It remains to prove that $\varphi\omega \in \mathcal{D}(D^{\mathrm{ev}}_{GB,\min})$. Put

$$\phi := (\phi_{2j-1}, \phi_{2j}) := \psi^*_{\mathrm{ev}}(\varphi\omega) \in L^2((0,\varepsilon), L^2(\Lambda^{2j-1}T^*N \oplus \Lambda^{2j}T^*N, g_N(0)))\,; \tag{2.21}$$

then by [B, Cor. 3.2] we have to show that

$$\|\phi(x)\| = o(x^{1/2}|\log x|^{1/2})\ , \quad x \to 0\ . \tag{2.22}$$

We would like to apply [B, Lemma 3.7]. But, unfortunately, we may have $\operatorname{spec} S_0 \cap (-1/2, 1/2) \neq \emptyset$. We put for $l \geq 0$

$$\begin{aligned} a_{2j+l} &:= (-1)^l(|c_{2j}| + l)\operatorname{sgn} c_{2j}\ ,\\ a_{2j-1-l} &:= (-1)^l(|c_{2j-1}| + l)\operatorname{sgn} c_{2j-1}\ , \end{aligned}$$

and consider

$$\tilde{D}^{a,\mathrm{ev}}_{GB} := h^{-1}\left(\partial_x + x^{-1}(S^a_0 + S_1(x))\right)$$

where S^a_0 is the operator obtained by replacing c by a in the definition of S_0 (2.12c). Since $a_{2j-1} = c_{2j-1}, a_{2j} = c_{2j}$ we have

$$\tilde{D}^{a,\mathrm{ev}}_{GB}(\phi) = \psi^*_{\mathrm{odd}} D^{\mathrm{ev}}_{GB} \psi_{\mathrm{ev}} \phi\ .$$

Using Lemma 2.2 it is now easy to check that

$$\operatorname{spec} S^a_0 \cap (-1/2, 1/2) = \emptyset\ ,$$

for $2j \neq m/2, (m \pm 1)/2$. Thus we deduce (2.22) from [B, Lemma 3.7 and Lemma 3.2] and we are done in this case.

If $k = 2j + 1$ is odd, we repeat the above argument with D^{odd}_{GB} in place of D^{ev}_{GB} and everything goes through.

We prove the second assertion only in case $m = 2\nu$ even; see also the remark after Proposition 3.9 below, which yields a proof for arbitrary m. We look at the conic metric

$$g := dx^2 \oplus x^2 g_N\ .$$

For any $c > 0$, g_c is quasi-isometric to

$$g_c := dx^2 \oplus x^2 c^2 g_N =: dx^2 \oplus x^2 g_{N,c}\ . \tag{2.23}$$

Denoting by Δ_N^c the Laplacian on $\Omega(N)$ with respect to $g_{N,c}$, we clearly have

$$\Delta_N^c = c^{-2}\Delta_N^1 \ . \tag{2.24}$$

Thus, for c large enough, we have

$$\operatorname{spec}\Delta_{\nu-1,ccl}\cap(0,1)\neq\emptyset \ ,$$

and by Corollary 2.3 there is an eigenform $\eta \in d^t_{N,\nu-1}(\Omega^\nu(N)) + d_{N,\nu-1}(\Omega^{\nu-1}(N))$ of S_0 with eigenvalue $\mu\in(-1/2,1/2)$. With φ as above we put

$$\omega := \psi_\nu(x^{-\mu}\varphi\eta) \ .$$

Then by [BS, Lemma 3.2] $\omega\in\mathcal{D}(D^{\mathrm{ev}}_{GB,\max})\setminus\mathcal{D}(D^{\mathrm{ev}}_{GB,\min})$ and since ω is of degree $\nu=m/2$ this implies (see also the proof of Theorem 3.8 below)

$$\omega\in\mathcal{D}(d_{\nu,\max})\cap\mathcal{D}(d^t_{\nu-1,\max})\,.$$

In Theorem 3.7 we will show that $d_{k,\max}=d_{k,\min}$ for all k in this case and we are done. □

3. Uniqueness of ideal boundary conditions

We now study the question of uniqueness of ibc's. First we collect some general results which will be applied to the de Rham complex on conformal cones in Theorems 3.7 and 3.8 below.

We start with a general result on differential operators. Let E,F be hermitian vector bundles over an arbitrary Riemannian manifold M, and $d: C_0^\infty(E)\to C_0^\infty(F)$ a differential operator. We denote by $d^td^{\mathcal{F}}$ the Friedrichs extension of the symmetric nonnegative operator d^td, with domain $C_0^\infty(E)$, in $L^2(E)$.

LEMMA 3.1. 1. $d^td^{\mathcal{F}}=d^t_{\max}d_{\min}$.

2. *The map associating to each closed extension, D, of d the self-adjoint extension D^*D of d^td is injective.*

Proof: 1. Since both operators are self-adjoint, it is enough to prove that $\mathcal{D}(d^td^{\mathcal{F}})\subset\mathcal{D}(d^t_{\max}d_{\min})=\{u\in\mathcal{D}(d_{\min})\,|\,du\in\mathcal{D}(d^t_{\max})\}$. if $u\in\mathcal{D}(d^td^{\mathcal{F}})$ then, by definition, $u\in\mathcal{D}(d_{\min})$ and $u\in\mathcal{D}((d^td)_{\max})$ which, obviously, implies $du\in\mathcal{D}(d^t_{\max})$.

2. It follows from the polar decomposition, $D=U(D^*D)^{\frac12}$ [K, p. 334], and from the fact that $\mathcal{D}(D^*D)$ is a core for $\mathcal{D}((D^*D)^{\frac12})=\mathcal{D}(D)$ [K, p. 281], that $D=d_{\max}|\mathcal{D}((D^*D)^{\frac12})$. This implies the assertion. □

Consider now an arbitrary elliptic complex on the Riemannian manifold M,

$$0 \to C_0^\infty(E_0) \xrightarrow{d_0} C_0^\infty(E_1) \xrightarrow{d_1} \cdots \xrightarrow{d_{N-1}} C_0^\infty(E_N) \to 0 \ , \tag{3.1}$$

which we denote by $(C_0^\infty(E), d)$, $E = \bigoplus_{k=0}^{N} E_k$. For the corresponding Laplacian on $C_0^\infty(E)$,

$$\Delta = (d + d^t)^2 =: \bigoplus_{k=0}^{N} \Delta_k \ , \tag{3.2}$$

we obtain from Lemma 3.1

$$\Delta^{\mathcal{F}} = (d + d^t)_{\max}(d + d^t)_{\min} \ , \tag{3.3a}$$

in particular

$$\mathcal{D}(\Delta_i^{\mathcal{F}}) \subset \mathcal{D}(d_{i,\min}) \cap \mathcal{D}(d_{i-1,\min}^t) \ . \tag{3.3b}$$

Various other self-adjoint extensions are derived from ibc's for $(C_0^\infty(E), d)$. For example, with $k \in \mathbf{Z}_+ \cup \{\infty\}$ we may define

$$D_i^k := \begin{cases} d_{i,\min} \ , & i \leq k-1, \\ d_{i,\max} \ , & i \geq k. \end{cases}$$

This defines an ibc, $(\mathcal{D}^k, D^k)$, for $(C_0^\infty(E), d)$ as in [BL, Lemma 3.1]; in particular, for $k = 0$ and $k = \infty$ we obtain the absolute and relative boundary condition, respectively. $(C_0^\infty(E), d)$ admits a unique ibc iff $d_{k,\min} = d_{k,\max}$ for all k. We now give some conditions which imply this equality.

LEMMA 3.2. $d_{k,\max} = d_{k,\min}$ *iff* $\Delta_k^k = \Delta_k^\infty$.

Proof: We have

$$\Delta_k^k = d_{k-1,\min} d_{k-1,\max}^t + d_{k,\min}^t d_{k,\max} \ , \tag{3.4a}$$

$$\Delta_k^\infty = d_{k-1,\min} d_{k-1,\max}^t + d_{k,\max}^t d_{k,\min} \ . \tag{3.4b}$$

Thus, $d_{k,\max} = d_{k,\min}$ implies $d_{k,\max}^t = d_{k,\min}^t$ hence $\Delta_k^k = \Delta_k^\infty$.

Conversely, if $\Delta_k^k = \Delta_k^\infty$ then from [BL, Lemma 2.11] we infer that $\mathcal{D}(\Delta_k^k)$ is a core for $d_{k,\max}$ whereas $\mathcal{D}(\Delta_k^k) = \mathcal{D}(\Delta_k^\infty) \subset \mathcal{D}(d_{k,\min})$. □

LEMMA 3.3. *Assume that, for some* $k \in \mathbf{Z}_+$,

$$\mathcal{D}(d_{k,\max}) \cap \mathcal{D}(d_{k-1,\max}^t) \subset \mathcal{D}((d + d^t)_{\min}) \ . \tag{3.5}$$

Then

$$d_{k,\max} = d_{k,\min} \ , \tag{3.6a}$$

$$d_{k-1,\max}^t = d_{k-1,\min}^t \ , \tag{3.6b}$$

$$\Delta_k^{\mathcal{F}} = \Delta_k^k = d_{k-1,\min} d_{k-1,\min}^t + d_{k,\min}^t d_{k,\min} \ . \tag{3.7}$$

Proof: We prove first the equality (3.7). It is clearly enough to show the inclusion $\mathcal{D}(\Delta_k^{\mathcal{F}}) \supset \mathcal{D}(\Delta_k^k)$: for $u \in \mathcal{D}(\Delta_k^k)$ we have, using (3.4),

$$u \in \mathcal{D}(d_{k,\max}) \cap \mathcal{D}(d^t_{k-1,\max}) \subset \mathcal{D}((d+d^t)_{\min}),$$

hence $u \in \mathcal{D}((d+d^t)_{\max}(d+d^t)_{\min}) = \mathcal{D}(\Delta^{\mathcal{F}})$, by Lemma 3.1. This proves the first equality in (3.7); the second follows from the first, (3.3b), and (3.4a).

Now if $(\mathcal{D}, D)$ is an arbitrary ibc we find

$$\mathcal{D}((D^*D+DD^*)_k) \supset \mathcal{D}(d^t_{k,\min}d_{k,\min} + d_{k-1,\min}d^t_{k-1,\min}) = \mathcal{D}(\Delta_k^{\mathcal{F}}),$$

hence again $(D^*D+DD^*)_k = \Delta_k^{\mathcal{F}}$. Here D is the operator obtained by "rolling up" the complex $(\mathcal{D}, D)$, cf. [BL, (2.12)]. Choosing in particular $D_i = D_i^k$ we obtain (3.6) from Lemma 3.2. □

We also have a partial converse to Lemma 3.3

LEMMA 3.4. *Let $(\mathcal{D}, D)$ be an ibc for $(C_0^\infty(E), d)$ such that*

$$(DD^*+D^*D)_j = \Delta_j^{\mathcal{F}}, \quad 0 \le j \le k. \tag{3.8}$$

Then we have

$$D_j = d_{j,\min}, \quad j \le k, \tag{3.9a}$$

$$D_j^* = d^t_{j,\min}, \quad j \le k-1, \tag{3.9b}$$

in particular,

$$d_{j,\min} = d_{j,\max}, \quad j \le k-1. \tag{3.10}$$

Moreover,

$$\mathcal{D}((d+d^t)_{\min}) \cap L^2(E_j) = \mathcal{D}(d_{j,\min}) \cap \mathcal{D}(d^t_{j-1,\min}), \quad j \le k, \tag{3.11}$$

$$\mathcal{D}((d+d^t)_{\max}) \cap L^2(E_j) = \mathcal{D}(d_{j,\min}) \cap \mathcal{D}(d^t_{j-1,\min}), \quad j \le k-1 \tag{3.12}$$

Proof: Since $\mathcal{D}((DD^*+D^*D)_j)$ is a core for D_j and D^*_{j-1}, we obtain (3.9a,b) from (3.3b). (3.10) follows from $d^t_{j,\max} = (d_{j,\min})^*$.

To prove (3.11) we observe that, by (3.9), for the closed operator $T = D + D^*$ we have

$$\mathcal{D}(T) \cap L^2(E_j) = \mathcal{D}(d_{j,\min}) \cap \mathcal{D}(d^t_{j-1,\min}) \supset \mathcal{D}((d+d^t)_{\min}) \cap L^2(E_j), \quad j \le k.$$

On the other hand, $T^2 = DD^* + D^*D$ and $\mathcal{D}((T^2)_j) = \mathcal{D}(\Delta_j^{\mathcal{F}})$ is a core for $\mathcal{D}(T) \cap L^2(E_j)$; thus the definition of the Friedrichs extension implies

$$\mathcal{D}(T) \cap L^2(E_j) \subset \mathcal{D}((d+d^t)_{\min}) \cap L^2(E_j),$$

hence (3.11). Using the dual complex we also obtain (3.12). □

We state an important consequence of this lemma.

COROLLARY 3.5. *There is an ibc $(\mathcal{D}, D)$ with $(DD^*+D^*D)_j = \Delta_j^{\mathcal{F}}$, for all j, iff $(d+d^t)$ is essentially self-adjoint.*

Proof: If $(DD^* + D^*D)_j = \Delta_j^{\mathcal{F}}$ for all j then (3.11) and (3.12) imply $(d+d^t)_{\max} = (d+d^t)_{\min}$. If, conversely, $(d+d^t)$ is essentially self-adjoint then we have, for an arbitrary ibc $(\mathcal{D}, D)$, $D+D^* = (d+d^t)_{\max} = (d+d^t)_{\min}$ and hence by Lemma 3.1

$$DD^* + D^*D = (d+d^t)_{\max}\,(d+d^t)_{\min} = \Delta^{\mathcal{F}} \ . \qquad \square$$

The identities (3.11) and (3.12) are not independent. In fact, denote by D^r and D^a the closed operators in $L^2(E)$ for the relative and absolute boundary condition, respectively (corresponding to $D \oplus D^*$ with D from [BL, (2.12)]), and by $p_j : L^2(E) \to L^2(E_j)$ the orthogonal projection. Then we have four closed extensions of $d+d^t$, namely D^r, D^a, $D_{\min} := (d+d^t)_{\min}$, and $D_{\max} := (d+d^t)_{\max}$. They are related as follows.

LEMMA 3.6. *There are inclusions*

$$p_j\mathcal{D}(D_{\min}) \subset \mathcal{D}(D^{r/a}) \cap L^2(E_j) \ , \qquad (3.13r/a)$$

$$p_j\mathcal{D}(D_{\max}) \supset \mathcal{D}(D^{r/a}) \cap L^2(E_j) \ . \qquad (3.14r/a)$$

Moreover, if equality holds in (3.13 r/a) *for* $j-1$ *and* $j+1$, *then also for* j *in* $(3.14r/a)$, *and vice versa.*

Proof: Note that

$$\mathcal{D}(D^{r/a}) \cap L^2(E_j) = p_j\mathcal{D}(D^{r/a}) = \mathcal{D}(d_{j,\min/\max}) \cap \mathcal{D}(d^t_{j-1,\max/\min}) \quad (3.15)$$

which implies $(3.13r/a)$ and $(3.14r/a)$.

Assume next equality in $(3.13r/a)$ for $j \pm 1$, and pick $\omega \in \mathcal{D}(D_{\max})$, $\eta_{j-1} \in \mathcal{D}(d_{j-1,\min/\max})$. Then the weak Hodge decomposition [BL, Lemma 1.2] for $(\mathcal{D}^{r/a}, D^{r/a})$ implies that we can write

$$\eta_{j-1} = \eta'_{j-1} + \eta''_{j-1} \ , \ \ \eta'_{j-1} \in \ker{}_{,j-1,\min/\max} \ , \ \ \eta''_{j-1} \in \ker d^t_{j-2,\max/\min} \ ,$$

hence

$$\eta''_{j-1} \in \mathcal{D}(d_{j-1,\min/\max}) \cap \mathcal{D}(d^t_{j-2,\max/\min}) \subset p_{j-1}\mathcal{D}(D_{\min}) \subset \mathcal{D}(D_{\min}) \ ,$$

and

$$d_{j-1,\min/\max}\ \eta''_{j-1} = D^{r/a}\eta''_{j-1} = D_{\min}\eta''_{j-1} \ .$$

This implies

$$\begin{aligned} |(d_{j-1,\min/\max}\eta_{j-1}, \omega_j)| &= |(D_{\min}\eta''_{j-1}, \omega)| \\ &= |(\eta''_{j-1}, D_{\max}\omega)| \\ &\le C_\omega \|\eta''_{j-1}\| \\ &\le C_\omega \|\eta_{j-1}\| \ , \end{aligned}$$

thus $\omega_j \in \mathcal{D}(d^t_{j-1,\max/\min})$. A similar argument gives $\omega_j \in \mathcal{D}(d_{j,\min/\max})$, hence $\omega_j \in \mathcal{D}(D^{r/a}) \cap L^2(E_j)$.

Conversely, if equality holds in (3.14r/a) we pick $\omega_j \in \mathcal{D}(d_{j,\min/\max}) \cap \mathcal{D}(d^t_{j-1,\max/\min})$. Then for $\eta \in \mathcal{D}(D_{\max})$ we find $\eta_{j\pm 1} \in \mathcal{D}(d_{j\pm 1,\min/\max}) \cap \mathcal{D}(d^t_{j\pm 1-1,\max/\min})$ by assumption, hence

$$\begin{aligned}|(D_{\max}\ \eta,\omega_j)| &= |(d_{j-1,\min/\max}\ \eta_{j-1} + d^t_{j,\max/\min}\ \eta_{j+1},\omega_j)|\\ &= |(\eta_{j-1}, d^t_{j-1,\max/\min}\ \omega_j) + (\eta_{j+1}, d_{j,\min/\max}\ \omega_j)|\\ &\le C_\omega\|\eta\| .\end{aligned}$$

Thus, $\omega_j \in \mathcal{D}(D_{\min}) \cap L^2(E_j) = p_j\mathcal{D}(D_{\min})$. □

Remark: For $\omega \in C_0^\infty(E)$ we have, writing $\omega_j := p_j\omega$,

$$\|D_{\min}\omega\|^2 = \sum_{j\ge 0}\Big(\|d_j\omega_j\|^2 + \|d^t_{j-1}\omega_j\|^2\Big) .$$

This implies easily that

$$p_j\mathcal{D}(D_{\min}) = \mathcal{D}(D_{\min}) \cap L^2(E_j) \subset \mathcal{D}(d_{j,\min}) \cap \mathcal{D}(d^t_{j-1,\min}) , \qquad (3.16)$$

where equality does not hold in general.

On the other hand, for $\omega_j \in L^2(E_j)$ and $\eta \in \mathcal{D}(D_{\min})$ we have by (3.16)

$$(\omega_j, D_{\min}\eta) = (\omega_j, d_{j-1,\min}\eta_{j-1} + d^t_{j,\min}\eta_{j+1}) .$$

This implies that

$$p_j\mathcal{D}(D_{\max}) \supset \mathcal{D}(D_{\max}) \cap L^2(E_j) = \mathcal{D}(d_{j,\max}) \cap \mathcal{D}(d^t_{j-1,\max}) , \qquad (3.17)$$

where, again, equality does not hold in general.

So far we have dealt with arbitrary manifolds and arbitrary elliptic complexes. Specializing to conformally conic manifolds and the de Rham complex we obtain the main results of this section. It is convenient to distinguish two cases according to whether $m = \dim M$ is even or odd; we put

$$\nu := [m/2] . \qquad (3.18)$$

THEOREM 3.7. *Let $M = M_1 \cup U$ be a conformally conic manifold of even dimension $m = 2\nu$. Then we obtain the following facts for the de Rham complex on M, $(\Omega_0(M), d)$.*

a) *$(\Omega_0(M), d)$ has a unique ibc, say $(\mathcal{D}, D)$, which is Fredholm.*

b) *For the associated Laplacians we have*

$$\Delta_k = D_k^* D_k + D_{k-1} D_{k-1}^* = \Delta_k^{\mathcal{F}} \text{ if } k \neq \nu .$$

c) *The closed extensions of $d + d^t$ are restricted by the relations*

$$\mathcal{D}((d+d^t)_{\min}) \cap L^2(\Lambda^k T^* M) = \mathcal{D}(d_{k,\min}) \cap \mathcal{D}(d^t_{k-1,\min}) , \quad k \neq \nu , \tag{3.19}$$

and

$$p_k \mathcal{D}((d+d^t)_{\max}) = \mathcal{D}(d_{k,\min}) \cap \mathcal{D}(d^t_{k-1,\min}) , \quad |k-\nu| \neq 1 . \tag{3.20}$$

d) *For the Gauß-Bonnet operator associated with $(\mathcal{D}, D)$,*

$$D_{GB}^{\mathrm{ev}} := \bigoplus_{j \geq 0} (D_{2j} + D^*_{2j-1}) , \tag{3.21}$$

we have

$$D_{GB}^{\mathrm{ev}} = \begin{cases} (d+d^t)^{\mathrm{ev}}_{\max} , & \nu \text{ even} , \\ (d+d^t)^{\mathrm{ev}}_{\min} , & \nu \text{ odd} . \end{cases} \tag{3.22}$$

D_{GB} is a Fredholm operator and

$$\chi(\mathcal{D}, D) = \begin{cases} \operatorname{ind} (d+d^t)^{\mathrm{ev}}_{\max} , & \nu \text{ even} , \\ \operatorname{ind} (d+d^t)^{\mathrm{ev}}_{\min} , & \nu \text{ odd} . \end{cases} \tag{3.23}$$

Proof: It follows from Theorem 2.1 that (3.5) holds for $k \neq \nu$. Hence we obtain from (3.6a,b)

$$d_{k,\max} = d_{k,\min} , \; d^t_{k-1,\max} = d^t_{k-1,\min} \text{ for } k \neq \nu ,$$

thus

$$d_{k,\max} = d_{k,\min} \text{ for all } k .$$

This implies a) except the Fredholm property, which follows from d) which we prove below. Assertion b) follows similarly from Theorem 2.1 and (3.7), and (3.19) follows from Lemma 3.4 applied to $(\Omega_0(M), d)$ and its dual complex. (3.20) is a consequence of a) and (3.19), in view of Lemma 3.6, (3.15), and (3.16).

For the proof of d) we observe that, by definition,

$$\mathcal{D}(D_{GB}^{\mathrm{ev}}) = \bigoplus_{j \geq 0} \mathcal{D}(d_{2j,\min}) \cap \mathcal{D}(d^t_{2j-1,\max}) .$$

Hence, (3.22) follows from a) and c). The Fredholm property follows from [B, Theorem 3.4] (cf. 2.11c) and (3.23) is a consequence of (3.22). □

THEOREM 3.8. *Under the assumptions of Theorem 3.7 but with*

$$m = \dim M = 2\nu + 1 ,$$

we have the following.

a) $d_{k,\min} = d_{k,\max}$ *if* $k \neq \nu$, *whereas*

$$\mathcal{D}(d_{\nu,\max})/\mathcal{D}(d_{\nu,\min}) \simeq H^{\nu}_{dR}(N) , \tag{3.24}$$

the ν*th de Rham-cohomology of the cross-section.*

b) *If* $(\mathcal{D}, D)$ *is any ibc for* $(\Omega_0(M), d)$ *then the corresponding Laplacians satisfy*

$$\Delta_k = \Delta_k^{\mathcal{F}} \ \text{if}\, k \neq \nu, \nu + 1 .$$

c) *The closed extensions of* $d + d^t$ *are restricted by the relations*

$$\mathcal{D}((d+d^t)_{\min}) \cap L^2(\Lambda^k T^* M) = \mathcal{D}(d_{k,\min}) \cap \mathcal{D}(d^t_{k-1,\min}) \ \text{if}\, k \neq \nu, \nu+1 , \tag{3.25}$$

and

$$p_k \mathcal{D}((d+d^t)_{\max}) = \mathcal{D}(d_{k,\min}) \cap \mathcal{D}(d^t_{k-1,\min}) \ \text{if}\, k \notin [\nu-1, \nu+2] . \tag{3.26}$$

Proof: It follows from Theorem 2.1 that (3.5) holds for $k \neq \nu, \nu + 1$. Hence we arrive at all assertions except (3.24) as in the proof of Theorem 3.7.

Now (3.24) is invariant under quasi-isometries. Hence, by (2.9) we may assume that the metric g in (2.2) has the form

$$g = dx^2 \oplus x^2 c^2 g_N(0) , \ \text{on}\ U = (0, \varepsilon) \times N ,$$

with c^2 as small as we please. Now let $\omega \in \mathcal{D}(d_{\nu,\max})$; the weak Hodge decomposition for the relative ibc implies a decomposition

$$\omega = \omega_1 + \omega_2 , \ \omega_1 \in \mathcal{D}(d_{\nu,\min}) ,$$

$$\omega_2 \in \mathcal{D}(d_{\nu,\max}) \cap \mathcal{D}(d^t_{\nu-1,\max}) \subset \mathcal{D}((d + d^t)_{\max}) ,$$

where we also used (3.17). Now we invoke (the easy adaption of) [BS, Lemma 3.2]. With $\psi \in C_0^\infty(-\varepsilon, \varepsilon), \psi = 1$ near 0, this gives for $\widehat{\psi\omega_2}$, the transformation of $\psi\omega$ under (2.11),

$$\widehat{\psi\omega_2} =: \widehat{\omega}_3 + \sum_{\substack{s \in \operatorname{spec} S_0 \\ |s| < \frac{1}{2}}} \psi c_s(\omega_2) x^{-s} \pi_2^*(e_s) ,$$

where (in the terminology of [BS, Section 5]) S_0 is given by (2.12c), e_s is an eigenform of S_0 with eigenvalue s (counted with multiplicity), c_s is a continuous linear functional on $\mathcal{D}((d+d^t)_{\max})$, and $\pi_2 : U \to N$ is the natural projection. Moreover, we have ω_3, $(1-\psi)\omega_2 \in p_\nu \mathcal{D}((d+d^t)_{\min}) \subset \mathcal{D}(d_{\nu,\min})$. Choosing c^2 small enough we may assume that $\operatorname{spec} S_0 \cap (-\frac{1}{2}, \frac{1}{2}) = \{0\}$. Hence we can write

$$\widehat{\psi\omega_2} =: \hat{\omega}_3 + \psi\pi_2^*\alpha(\omega)\,, \quad \alpha(\omega) \in \hat{\mathcal{H}}^\nu(N)\,,$$

where $\hat{\mathcal{H}}^\nu(N)$ denotes the space of harmonic ν-forms on N. Thus we obtain a map

$$\alpha : \mathcal{D}(d_{\nu,\max}) \to \hat{\mathcal{H}}^\nu(N)$$

which is easily seen to be linear and surjective. It is also clear by construction that $\ker\alpha \subset \mathcal{D}(d_{\nu,\min})$. But $\psi\pi_2^*\alpha(\omega) = \psi\pi_2^*\widehat{\alpha(\omega)} \in \mathcal{D}(d_{\nu,\min})$ only if $\alpha(\omega) = 0$ since (cf. (2.11) and (2.12))

$$(d_{\nu,\max}\psi\pi_2^*\alpha(\omega), *_M\psi\pi_2^*(*_N\alpha(\omega))) - (\psi\pi_2^*\alpha(\omega), d^t_{\nu,\max} *_M \psi\pi_2^*(*_N\alpha(\omega))) = -\|\alpha(\omega)\|^2_{\hat{\mathcal{H}}^\nu(N)}\,.$$

Thus $\ker\alpha = \mathcal{D}(d_{\nu,\min})$ completing the proof. □

PROPOSITION 3.9. *In the situation of Theorem* 3.7 *or Theorem* 3.8 *the equality in* (3.19) *for* $k = \nu$ *or in* (3.25) *for* ν *and* $\nu + 1$ *is equivalent to the essential self-adjointness of* $(d + d^t)$.

Proof: If $(d + d^t)$ is essentially self-adjoint, then by Corollary 3.5 we have $\Delta_k = \Delta_k^{\mathcal{F}}$ for all k and the assertion follows from (3.11). The converse follows immediately from Lemma 3.6. □

Remark: Now we can give a more general proof of the second statement in Theorem 2.1. We consider M as above with N such that $H^\nu_{dR}(N) = 0$ and with a conic metric

$$dx^2 + x^2c^2g_N$$

on U. If c is large enough, then S_0 has eigenvalues in $(-1/2, 1/2)$, thus $(d+d^t)$ is not essentially self-adjoint and hence we have that $p_k\mathcal{D}((d+d^t)_{\min})$ is a proper subset of $\mathcal{D}(d_{k,\min}) \cap \mathcal{D}(d^t_{k-1,\min})$ for some $k \in \{\nu, \nu+1\}$.

Also, even though we have a unique ibc for the de Rham complex, the corresponding Laplacian is not given by the Friedrichs extension.

4. Poincaré Duality

Consider an ibc $(\mathcal{D}, D)$ of the de Rham complex $(\Omega_0(M), d)$ where M is an arbitrary oriented Riemannian manifold. In [BL, Sec. 3 and Lemma 4.3] we have introduced the notion of *Poincaré duality* for $(\mathcal{D}, D)$: it holds if the maps $g_i := \sqrt{-1}^{\beta(i)} *_i$ induce a complex isomorphism $(\mathcal{D}, D) \to (\mathcal{D}^*, D^*)$; here $\beta(i)$ is chosen in such a way that

$$d^t_{m-i-1} \circ g_i = g_{i+1} \circ d_i \quad \text{on } \Omega^i_0(M) \ . \tag{4.1}$$

In [BL, Lemma 3.7] we have shown that such ibc's do always exist if m is even. If $m = 2\nu + 1$ is odd then, since the g_i above are unitary, the existence of an ibc with Poincaré duality is equivalent to the existence of self-adjoint extensions of the operator

$$t := \sqrt{-1}^{\beta(\nu)} *_{\nu+1} d_\nu \ , \ \beta(\nu) = \begin{cases} 2 \ , & m \equiv 3 \bmod 4 \ , \\ 1 \ , & m \equiv 1 \bmod 4 \ , \end{cases} \tag{4.2}$$

with domain $\Omega^\nu_0(M)$ in $L^2(\Lambda^\nu T^* M)$. This leads us to introduce the deficiency indices of t,

$$n_\pm(t) := \dim\ker(t^* \mp \sqrt{-1}) \in \mathbb{Z}_+ \cup \{\infty\} \ . \tag{4.3}$$

If $m \equiv 3 \bmod 4$, t is real and hence has equal deficiency indices, but if $m \equiv 1 \bmod 4$ we meet an obstruction. Thus, one might hope to find interesting invariants of general Riemannian manifolds. The following simple fact is the key to such results.

LEMMA 4.1. *Let M be a Riemannian manifold, $E \to M$ a hermitian vector bundle, and t a densely defined symmetric operator in $L^2(E)$.*

a) *$t_{\min} \pm \sqrt{-1}$ is a semi-Fredholm operator in $L^2(E)$ with*

$$\operatorname{ind}(t_{\min} \pm \sqrt{-1}) = -n_\pm(t) \ . \tag{4.4}$$

b) *Consider, for $s \in [0,1]$, a family $(\ ,\ )_s$ of continuous scalar products on $L^2(E)$, equivalent to one another, and a family t^s of densely defined operators, symmetric with respect to $(\ ,\ )_s$, and with domain independent of s, $\mathcal{D}(t^s) = \mathcal{D}(t^0)$.*

If we have an estimate

$$\|t^{s_1}u - t^{s_2}u\|_{s_2} \le \varepsilon(|s_1 - s_2|)\|t^{s_2}u\|_{s_2} \ , \ s_1, s_2 \in [0,1], \ u \in \mathcal{D}(t^0) \ , \tag{4.5}$$

with $\varepsilon \in C(\mathbb{R}_+)$, $\varepsilon(0) = 0$, then the deficiency indices are independent of $s \in [0,1]$,

$$n_\pm(t^s) = n_\pm(t^0) \ . \tag{4.6}$$

Proof: Write $\mathcal{D}_0 := \mathcal{D}(t^0) = \mathcal{D}(t^s)$, then we have for $s \in [0,1]$

$$\|(t^s \pm \sqrt{-1})u\|_s^2 \geq \|u\|_s^2, \ u \in \mathcal{D}_0 \ .$$

Since the norms $\|\cdot\|_s$ on $L^2(E)$ are equivalent by assumption we obtain for $T_\pm^s := t_{\min}^s \pm \sqrt{-1}$

$$\mathcal{D}(T_\pm^s) = \mathcal{D}(T_\pm^0) =: \mathcal{D} \ ,$$

and the estimate

$$\|T_\pm^s u\|_s^2 \geq \|u\|_s^2 \ , \ u \in \mathcal{D} \ . \tag{4.7}$$

It follows that $T_\pm^s$ is a semi–Fredholm operator with

$$\operatorname{ind} T_\pm^s = -n_\pm(t^s);$$

this proves a).

Observe next that $T_\pm^s$ may be considered as a closed operator in the Hilbert space $(L^2(E), (\cdot,\cdot)_{s_2})$ for all s, s_2, to be denoted by $T_\pm^{s,s_2}$. Then we obtain from (4.7)

$$\|T_\pm^{s,s_2} u\|_{s_2}^2 \geq C(s,s_2)\|u\|_{s_2}^2 \ , \ u \in \mathcal{D} \ . \tag{4.8}$$

Moreover, $v \in \ker (T_\pm^s)^*$ is equivalent to

$$0 = (T_\pm^s u, v)_s = (T_\pm^{s,s_2} u, B_{s,s_2} v)_{s_2} \ , \ u \in \mathcal{D} \ ,$$

with some bounded invertible operator $B_{s,s_2} : (L^2(E), (\ , \ )_s) \to (L^2(E), (\ , \ )_{s_2})$. Thus, $T_\pm^{s,s_2}$ is a semi–Fredholm operator, too, with

$$\operatorname{ind} T_\pm^{s,s_2} = \operatorname{ind} T_\pm^s \ . \tag{4.9}$$

Now the estimate (4.5) implies

$$\|(T_\pm^{s_1,s_2} - T_\pm^{s_2})u\|_{s_2} \leq \varepsilon(|s_1 - s_2|)\|T_\pm^{s_2} u\|_{s_2}.$$

Hence, assertion b) follows from (4.9) and [K, Ch. IV, Thm. 5.17 and Thm. 2.14], since ε is continuous with $\varepsilon(0) = 0$. □

If g_0 and g_1 are quasi-isometric metrics on M and $g_t := (1-t)g_0 + tg_1$, with Hodge–operator $*_t$, then we easily see that

$$\| *_{t_1} - *_{t_2} \|_{L^2(\Lambda^* M, g_{t_2})} \leq C|t_1 - t_2|.$$

Hence Lemma 4.1 implies the following interesting fact.

COROLLARY 4.2. *Let M be a Riemannian manifold of dimension $m \equiv 1 \bmod 4$. Then the deficiency indices of the operator $\sqrt{-1} *_{\nu+1} d_\nu$, with domain $\Omega_0^\nu(M)$ in $L^2(\Lambda^\nu T^*M)$, are invariant under quasi-isometries.*

We will now show that the deficiency indices of

$$t_\nu := \sqrt{-1} * d_\nu \tag{4.10}$$

– which we denote by $n_\pm(M)$ – on a Riemannian manifold M of dimension $m = 4k+1$ depend nontrivially on the metric; thus Corollary 4.2 provides interesting invariants of the quasi-isometry class for these dimensions. We look at $M = (a,b) \times_f N$ where $-\infty \le a < b \le \infty$, N is compact, $f \in C^\infty(a,b)$ is positive, and M is equipped with the warped product metric

$$g = dx^2 \oplus f(x)^2 g_N \ , \tag{4.11}$$

with some metric g_N on N. We assume N oriented with volume form ω_N; M will be oriented by the volume form $dx \wedge \pi_2^*\omega_N$, π_2 the projection onto the second factor.

LEMMA 4.3. a) *If $a = -\infty$, $b = \infty$ then $n_+(M) = n_-(M) = 0$.*

b) *If $-\infty < a < b < \infty$ and f is continuous and positive in $[a,b]$, then $n_+(M) = n_-(M) = \infty$.*

c) *If $a = 0$, $b = \infty$, and both ends are in the limit point case for the differential operator $L = -\partial_x^2 + \lambda/f(x)^2$, for $\lambda \ge \lambda_0$, then*

$$n_+(M) - n_-(M) = -\operatorname{sign} N \ .$$

In particular, this is the case for $f(x) = x$ with $\lambda_0 = 3/4$.

Proof: We use the separation of variables as introduced in [BS, Sec. 5], mutatis mutandis. It is based on the unitary map ($\nu = 2k$, $I := (a,b)$, $\tilde{c}_j := (j-\nu)$)

$$\begin{aligned} \psi_\nu : C_0^\infty(I, \Omega^{\nu-1}(N) \oplus \Omega^\nu(N)) &\to \Omega_0^\nu(M) \ , \\ (\phi_{\nu-1}, \phi_\nu) &\mapsto f(x)^{\tilde{c}_{\nu-1}} \pi_2^* \phi_{\nu-1}(x) \wedge dx + f(x)^{\tilde{c}_\nu} \pi_2^* \phi_\nu(x) \ . \end{aligned} \tag{4.12}$$

A straightforward calculation gives

$$\psi_\nu^{-1} \sqrt{-1} *_{\nu+1} d_\nu \psi_\nu = \sqrt{-1} \begin{pmatrix} 0 & f(x)^{-1} *_N d_N \\ f(x)^{-1} *_N d_N & *_N \frac{\partial}{\partial x} \end{pmatrix} \ ,$$

hence we have to look for solutions of the system

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & *_N \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} \mp I & -f(x)^{-1} d_N^t *_N \\ f(x)^{-1} d_N^t *_N & \mp I \end{pmatrix} \right] \begin{pmatrix} \eta_{\nu-1} \\ \eta_\nu \end{pmatrix} = 0$$

in $L^2((a,b),\ L^2(\Lambda^{\nu-1}T^*N \oplus \Lambda^\nu T^*N))$. Eliminating $\eta_{\nu-1}$ from the system, and decomposing $\bar{\eta_\nu} := *_N\eta_\nu =: \xi_1 + \xi_2 + \xi_3$ according to the Hodge decomposition

$$L^2(\Lambda^\nu T^*N) = \hat{\mathcal{H}}^\nu(N) \oplus \overline{d_N\Omega^{\nu-1}(N)} \oplus \overline{d_N^t\Omega^{\nu+1}(N)}\ , \tag{4.13}$$

we are left with the equations

$$\left(\frac{\partial}{\partial x} \mp *_\nu\right)\xi_1(x) = 0, \tag{4.14 a$\pm$}$$

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\partial}{\partial x} \mp \begin{pmatrix}0 & *_N\\ f(x)^{-2}d_N^t d_N *_N + *_N & 0\end{pmatrix}\right]\begin{pmatrix}\xi_2\\ \xi_3\end{pmatrix}(x) = 0. \tag{4.14b$\pm$}$$

Since the decomposition (4.13) reduces Δ_N, (4.14b) decomposes according to the decomposition

$$\overline{d_N\Omega^{\nu-1}(N)} \oplus \overline{d_N^t\Omega^{\nu+1}(N)} = \bigoplus_{\lambda\in\operatorname{spec}\Delta_{N,\nu}\setminus\{0\}} E_\lambda \oplus *_N E_\lambda,$$

where $E_\lambda = \ker(\Delta_{N,\nu} - \lambda) \cap \overline{d\Omega^{\nu-1}(N)}$. This leads finally to the system

$$\left[\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\frac{\partial}{\partial x} \mp \begin{pmatrix}0 & 1\\ 1+\lambda/f(x)^2 & 0\end{pmatrix}\right]\begin{pmatrix}u\\ v\end{pmatrix}(x) = 0, \tag{4.15$_\lambda,\pm$}$$

$\lambda > 0$. Now we study the cases listed above separately.

a) If $a = -\infty$, $b = \infty$ then M is complete. Since the symbol of $\sqrt{-1}*_{\nu+1}\, d_\nu$ is uniformly bounded on the cosphere bundle of M, the method of Chernoff [Ch] proves essential self-adjointness in this case (alternatively, analyze (4.14 a) and (4.15) as done below).

b) The second assertion is clear from standard theory applied to (4.15).

c) We split $\hat{\mathcal{H}}^\nu(N) =: \hat{\mathcal{H}}^\nu_+(N) \oplus \hat{\mathcal{H}}^\nu_-(N)$, according to the ± 1 eigenspaces of $*_{N,\nu}$. Denoting their dimensions by $b_\nu(N)_\pm$ we get

$$b_\nu(N) = b_\nu(N)_+ + b_\nu(N)_-,\ \ \operatorname{sign} N = b_\nu(N)_+ - b_\nu(N)_-.$$

Obviously, the L^2–solutions of (4.14 a $\pm$) form a space of dimension $b_\nu(N)_\mp$; thus, our assertion follows if we can show that the system $(4.15_\lambda, \pm)$ admits no L^2–solutions for any $\lambda > 0$, under our assumptions.

Let us consider the system $(4.15_\lambda, +)$; the other case is reduced to this one by the transformation $\binom{u}{v}(x) = \binom{\tilde{u}}{\tilde{v}}(-x)$. Assuming the existence of an L^2–solution $\binom{u}{v}$, we obtain

$$-u'' + (1 + \lambda/f(x)^2)u = 0\ ,$$

i.e.

$$L^* u = -u \ . \tag{4.16}$$

Scaling the metric as in Sec. 2 and using Corollary 4.2 we may assume that $\operatorname{spec}\Delta_{N,\nu} \cap [-\lambda_0, \lambda_0] \subset \{0\}$. Thus L is essentially self-adjoint and hence $L^* = \bar{L}$ is positive. Then (4.16), of course, has no L^2–solutions. □

Lemma 4.3 is only a very special case of a fairly general "odd index theorem" to which we will return in a subsequent publication. Among other things this index theorem will imply that Lemma 4.3, 2) and 3) hold for any complete manifold with compact boundary and any conformally conic manifold, respectively. It will also provide a powerful tool for further study of the invariants $n_\pm(M)$.

5. Kähler Manifolds

Now we consider the case of a Riemannian manifold of dimension $m = 2\nu$ which carries a complex structure on the (real) tangent bundle TM. It induces a decomposition of the complexified tangent bundle, $\mathbb{C} \otimes_{\mathbb{R}} TM = T_{\mathbb{C}}M = T'M \oplus T''M$, and corresponding decompositions

$$\begin{aligned}
&\mathbb{C} \otimes_{\mathbb{R}} \Lambda T^*M =: \Lambda T^*_{\mathbb{C}}M = \bigoplus_{r,s\geq 0} \Lambda^r T'^*M \otimes \Lambda^s T''^*M =: \bigoplus_{r,s\geq 0} \Lambda^{r,s}M \ , \\
&\mathbb{C} \otimes_{\mathbb{R}} \Omega_0(M) =: \Omega_{\mathbb{C},0}(M) =: \bigoplus_{r,s\geq 0} \Omega^{r,s}_{\mathbb{C},0}(M) \ .
\end{aligned}$$

We assume next that M is a complex manifold (of complex dimension ν), and that the Riemannian metric is induced from a hermitian metric on $T'M$, with positive definite (1,1)–form $\omega \in \Omega^{1,1}_{\mathbb{C}}(M)$. Then we can form the corresponding L^2-spaces, with decomposition

$$\begin{aligned}
L^2(\Lambda T^*_{\mathbb{C}}M) &= \bigoplus_{r,s\geq 0} L^2(\Lambda^r T'^*M \otimes \Lambda^s T''^*M) \\
&= \bigoplus_{r,s\geq 0} L^2(\Lambda^{r,s}M) \ .
\end{aligned} \tag{5.1}$$

If we denote by $\pi^{r,s} : L^2(\Lambda T^*_{\mathbb{C}}M) \to L^2(\Lambda^{r,s}M)$ the natural projections we can further decompose

$$\begin{aligned}
d_{r+s}|\Omega^{r,s}_{\mathbb{C}}(M) &= \pi^{r+1,s} \circ d_{r+s}|\Omega^{r,s}_{\mathbb{C}}(M) + \pi^{r,s+1} \circ d_{r+s}|\Omega^{r,s}_{\mathbb{C}}(M) \\
&=: (\partial_{r,s} + \bar{\partial}_{r,s})|\Omega^{r,s}_{\mathbb{C}}(M) \ .
\end{aligned}$$

Then the de Rham complex, $(\Omega_{\mathbb{C},0}(M), d)$, splits into the subcomplexes

$$(\Omega^{*,s}_{\mathbb{C},0}(M), \partial) \ \text{and} \ (\Omega^{r,*}_{\mathbb{C},0}(M), \bar{\partial}),$$

$0 \le r, s \le \nu$. Thus, on $\Omega^r_{\mathbf{C},0}(M)$ we can define three Laplace-type operators: Δ_d, Δ_∂, and $\Delta_{\bar\partial}$, the subscript indicating the defining complex. Now, in general these operators are quite unrelated, but a close relation exists for *Kähler manifolds* and, in fact, characterizes this class. So we assume from now on that M is Kähler in the sense that the positive (1,1)–form ω defining the metric is closed; ω is called the Kähler form. Then we have

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial} \ . \tag{5.2}$$

More precisely, with $\Delta^{r,s}_\partial := \Delta_\partial | \Omega^{r,s}_{\mathbf{C},0}(M)$ (resp. $\Delta_{\bar\partial} := \Delta_{\bar\partial}|\Omega^{r,s}_{\mathbf{C},0}(M)$) we have

$$\Delta^t_d := \Delta_d | \Omega^t_{\mathbf{C},0}(M) = 2 \bigoplus_{r+s=t} \Delta^{r,s}_\partial = 2 \bigoplus_{r+s=t} \Delta^{r,s}_{\bar\partial} \ . \tag{5.3}$$

Now we may ask whether these identities persist to hold for suitably chosen ibc's of $(\Omega_{\mathbf{C},0}(M), d)$.

DEFINITION 5.1: Let M be a Kähler manifold and let $(\mathcal{D}, D)$ be an ibc for $(\Omega_{\mathbf{C},0}(M), d)$, with Laplacian Δ_D. We say that the Kähler-Hodge Theorem (KHT) holds for $(\mathcal{D}, D)$ if the decomposition (5.1) reduces Δ_D, i.e.

$$\mathcal{D}(\Delta^t_D) = \bigoplus_{r+s=t} \mathcal{D}(\Delta^t_D) \cap L^2(\Lambda^{r,s}M) \ . \tag{5.4}$$

Of course, KHT holds for compact Kähler manifolds. In this case, however, various other important assertions are true which are collectively known as the *Kähler package* (cf. [CGM, p. 303]). To describe it we need some more structure. We write the Kähler form in terms of a local orthonormal frame, (φ_i), for $\Lambda^{1,0}M$,

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{\nu} \varphi_i \wedge \bar\varphi_i \ . \tag{5.5}$$

Hence $|\omega(p)|^2 = \nu/4$ for all $p \in M$ and

$$L := \quad \text{wedge multiplication by } \omega \tag{5.6}$$

extends to a bounded operator in $L^2(\Lambda T^*_{\mathbf{C}} M)$. Put

$$\Lambda := L^*, \quad H := [\Lambda, L] \ , \tag{5.7}$$

then

$$H = \sum_{t=0}^{m} (\nu - t)\pi^t \quad , \quad \pi^t := \bigoplus_{r+s=t} \pi^{r,s} \ , \tag{5.8}$$

$$[H, L] = -2L \quad , \quad [H, \Lambda] = 2\Lambda \ . \tag{5.9}$$

Hence we obtain a representation of $sl(2, \mathbb{C})$ on $\Lambda T_p^* M$, $p \in M$, hence also on $L^2(\Lambda T_{\mathbb{C}}^* M)$, from

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \Lambda, \ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto L, \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H \ . \tag{5.10}$$

This representation gives rise to the "Lefschetz decomposition" of $L^2(\Lambda T_{\mathbb{C}}^* M)$ and of the cohomology of a compact Kähler manifold. Finally, we introduce the unitary operator

$$J := \bigoplus_{r,s \geq 0} \sqrt{-1}^{\,r-s} \pi^{r,s} ; \tag{5.11a}$$

for a linear operator, T, in $L^2(\Lambda T_{\mathbb{C}}^* M)$ we then put

$$T_c := J^* T J, \tag{5.11b}$$

with $\mathcal{D}(T_c) = J^* \mathcal{D}(T)$.

DEFINITION 5.2: Let M be a Kähler manifold. We say that the L^2-Kähler package holds for M, if we can find ibc's $(\mathcal{D}, D)$ for $(\Omega_{\mathbb{C},0}(M), d)$ and $(\mathcal{D}^r, D^r)$ for $(\Omega_{\mathbb{C},0}^{r,*}, \bar\partial)$ such that the following assertions are true (for simplicity of notation, we suppress the dependence on the ibc's in the formulas below).

a) *Hodge decomposition*

$$\Delta_d^t = \bigoplus_{r+s=t} 2\Delta_{\bar\partial}^{r,s} \ , \tag{5.12a}$$

$$\hat{\mathcal{H}}_d^t := \ker \Delta_d^t = \bigoplus_{r+s=t} \ker \Delta_{\bar\partial}^{r,s} =: \bigoplus_{r+s=t} \hat{\mathcal{H}}_{\bar\partial}^{r,s} \ , \tag{5.12b}$$

$$\overline{\hat{\mathcal{H}}_{\bar\partial}^{r,s}} = \hat{\mathcal{H}}_{\bar\partial}^{s,r} \ . \tag{5.12c}$$

b) *Poincaré duality* If $*$ denotes the Hodge star operator, $*_t = \oplus_{r+s=t} *_{r,s}$ then $\oplus_t (-1)^t *_t$ is a complex isomorphism from $(\mathcal{D}, D)$ onto its dual complex, $(\mathcal{D}^*, D^*)$. In particular, we have isomorphisms

$$*_t : \hat{\mathcal{H}}_d^t \to \hat{\mathcal{H}}_d^{m-r} \ , \tag{5.13a}$$

$$*_{r,s} : \hat{\mathcal{H}}_{\bar\partial}^{r,s} \to \hat{\mathcal{H}}_{\bar\partial}^{\nu-r,\nu-s} . \tag{5.13b}$$

c) *Hard Lefschetz* If all ibc's involved are Fredholm then L induces isomorphisms

$$L^k : \hat{\mathcal{H}}_d^{\nu-k} \to \hat{\mathcal{H}}_d^{\nu+k} \ , \quad 0 \leq k \leq \nu \ . \tag{5.14}$$

d) *Hodge signature* If $\nu = 2k$ is even, if b) holds, and if all ibc's are Fredholm, then we have the L^2-signature theorem

$$L^2\text{-sign}\, M = \sum_{r+s \equiv 0(2)} (-1)^r \dim \hat{\mathcal{H}}_{\bar\partial}^{r,s} \ . \tag{5.15}$$

Remarks: One usually includes the Lefschetz hyperplane theorem into the Kähler package of which, however, no proof by L^2-methods is known as far as we know. Therefore, we refer to the properties listed above as the "L^2-Kähler package", with abbreviation L^2-KP.

The L^2-signature of M is defined as the index of the elliptic operator D_S, arising from restricting the canonical operator, D, associated to the ibc $(\mathcal{D}, D)$ for $(\Omega_{\mathbf{C},0}(M), d)$ to the $+1$-eigenspace of the involution

$$c := \bigoplus_t (\sqrt{-1})^{\nu+t(t-1)} *_t ,$$

which anticommutes with D in view of b). Note that (5.15) continues to hold if the ibc is only weakly Fredholm in the sense of [BL, (2.24)].

Note also that in the view of (5.12a) $(\mathcal{D}, D)$ is Fredholm iff all $(\mathcal{D}^r, D^r)$ are. In this case, (5.12b) implies the usual relation between the de Rham and the Dolbeault cohomology.

Now we want to establish some simple properties of ibc's for $(\Omega_{\mathbf{C},0}(M), d)$ which imply L^2-KP. We start with an observation involving KHT.

LEMMA 5.3. *Let M be a Kähler manifold and let $(\mathcal{D}, D)$ be an ibc for $(\Omega_{\mathbf{C},0}(M), d)$.*

a) *If KHT holds then*

$$\Delta_D^t = J^* \Delta_D^t J = \Delta_{D_c}^t \ , \quad 0 \le t \le m \ . \tag{5.16}$$

b) *If the ibc is unique then* (5.16) *is equivalent to KHT.*

Proof: a) Note first that $(J^*\mathcal{D}, D_c)$ is a Hilbert complex with Laplacian $\Delta_{D_c} = J^* \Delta_D J$; thus the second equality in (5.16) is always true.

To prove the first it is enough to show that $J\mathcal{D}(\Delta_D^t) \subset \mathcal{D}(\Delta_D^t)$ since Δ_D^t is self-adjoint and $[\Delta_D^t, J] = 0$ on a dense subspace. But this is obvious in view of KHT and (5.11a).

b) Since $[\Delta_D^t, \pi^{r,s}] = 0$ on $\Omega_{\mathbf{C},0}^t(M)$ it is enough to prove that

$$\pi^{r,s} \mathcal{D}((\Delta_D^t)^2) \subset \mathcal{D}(\Delta_D^t) \ . \tag{5.17}$$

To do so, we use the following easy principle: if $\omega = \sum_{r,s} \omega_{r,s}$ is such that $\omega_{r,s} \ne 0$ implies $\omega_{r\pm 1, s\mp 1} = 0$, then $\omega \in \mathcal{D}(d_{t,\max}) \cap \mathcal{D}(d_{t-1,\max}^t)$ implies $\omega_{r,s} \in \mathcal{D}(d_{t,\max}) \cap \mathcal{D}(d_{t-1,\max}^t)$ for all r, s.

Then we see that

$$\begin{aligned} \omega_{\mathrm{ev/odd}} &= \tfrac{1}{2}(\omega \pm \sqrt{-1}^t J\omega) \in \mathcal{D}((\Delta_D^t)^2) \ , \\ d_t \omega_{\mathrm{ev/odd}} &\in \mathcal{D}(\Delta_D^{t+1}) \subset \mathcal{D}(d_{t+1,\max}) \cap \mathcal{D}(d_{t,\max}^t) \ , \\ d_{t-1}^t \omega_{\mathrm{ev/odd}} &\in \mathcal{D}(\Delta_D^{t-1}) \subset \mathcal{D}(d_{t-1,\max}) \cap \mathcal{D}(d_{t-2,\max}^t) \ . \end{aligned}$$

Applying the principle to $\omega_{\mathrm{ev/odd}}$ we find, from the uniqueness of ibc's,

$$\omega_{r,s} \in \mathcal{D}(D_t) \cap \mathcal{D}(D_{t-1}^*) \; .$$

Next we apply the principle again to $d_t \omega_{\mathrm{ev/odd}} \pm \sqrt{-1}^{\,t+1} J d_t \omega_{\mathrm{ev/odd}}$ and $d_{t-1}^t \omega_{\mathrm{ev/odd}} \pm \sqrt{-1}^{\,t-1} J d_{t-1}^t \omega_{\mathrm{ev/odd}}$ to conclude

$$D_t \omega_{r,s} \in \mathcal{D}(D_{t+1}) \cap \mathcal{D}(D_t^*), D_{t-1}^* \omega_{r,s} \in \mathcal{D}(D_{t-1}) \cap \mathcal{D}(D_{t-2}^*) \; . \qquad \square$$

The next step on the road to L^2-KP is the definition of natural ibc's on all the complexes $(\Omega_{\mathbb{C},0}^{r,*}, \bar\partial)$. For any operator T we write

$$\mathcal{D}^\infty(T) := \bigcap_{k \geq 1} \mathcal{D}(T^k) \; .$$

LEMMA 5.4. *Let M be a Kähler manifold and $(\mathcal{D}, D)$ an ibc for the de Rham complex with KHT. Then, for each $r \in \mathbb{Z}_+$, the differential complex*

$$0 \to \mathcal{D}^\infty(\Delta_D^{r,0}) \xrightarrow{\bar\partial_D^{r,0}} \mathcal{D}^\infty(\Delta_D^{r,1}) \xrightarrow{\bar\partial_D^{r,1}} \cdots \xrightarrow{\bar\partial_D^{r,\nu-1}} \mathcal{D}^\infty(\Delta_D^{r,\nu}) \to 0 \; , \qquad (5.18)$$

with $\bar\partial_D^{r,s} := \pi^{r,s+1} \circ D_{r+s} | \mathcal{D}^\infty(\Delta_D^{r,s})$, has essentially self-adjoint Laplacian. This defines an ibc for $(\Omega_{\mathbb{C},0}^{r,*}, \bar\partial)$ such that the Laplacians satisfy the conditions (5.12).

Proof: It follows from [BL, Lemma 2.11] that

$$D_t \mathcal{D}^\infty(\Delta_D^t) \subset \mathcal{D}^\infty(\Delta_D^{t+1}), \;\; D_{t-1}^* \mathcal{D}^\infty(\Delta_D^t) \subset \mathcal{D}^\infty(\Delta_D^{t-1}) \; .$$

This together with KHT implies

$$\bar\partial_D \mathcal{D}^\infty(\Delta_D^{r,s}) \subset \mathcal{D}^\infty(\Delta_D^{r,s+1}) \; , \qquad (5.19a)$$

and since $(\bar\partial_D)^* | \mathcal{D}^\infty(\Delta_D^{r,s+1}) = \pi^{r,s} \circ D_{r+s}^* | \mathcal{D}^\infty(\Delta_D^{r,s+1})$ we also get

$$(\bar\partial_D)^* \mathcal{D}^\infty(\Delta_D^{r,s}) \subset \mathcal{D}^\infty(\Delta_D^{r,s-1}) \; . \qquad (5.19b)$$

(5.19a) says that the complex is well defined, and with (5.19b) we find

$$\Delta_D^{r,s} = 2\Delta_{\bar\partial_D}^{r,s} \quad \text{on } \mathcal{D}^\infty(\Delta_D^{r,s}) \; , \qquad (5.20)$$

since this holds on $\Omega_{\mathbb{C},0}^{r,s}(M)$. Hence $\Delta_{\bar\partial_D}^{r,s}$ is essentially self-adjoint since Δ_D is.

Now we can take the closure of the operators in (5.18) which defines a Hilbert complex and hence an ibc for $(\Omega_{\mathbb{C},0}^{r,*}, \bar\partial)$ (cf. [BL, Lemma 3.1]). In view of (5.20) the corresponding Laplacian is $\frac{1}{2}\Delta_D^{r,s}$, so we obtain (5.12a), and (5.12b) is an easy consequence. Finally, since for $\omega \in \Omega_{\mathbb{C},0}^{r,s}(M)$

$$\Delta_D^{s,r} \bar\omega = \overline{\Delta_D^{r,s} \omega},$$

we obtain this identity also for $\mathcal{D}(\Delta_D^{r,s})$; this implies (5.12c). $\square$

Before we proceed we note that (5.18) gives rise to the following abstract concept which seems useful enough to be singled out.

DEFINITION 5.5: A differential complex of vector spaces

$$0 \to \mathcal{E}_0 \xrightarrow{d_0} \cdots \xrightarrow{d_{N-1}} \mathcal{E}_N \to 0 \tag{5.21}$$

will be called a pre-Hilbert complex if the following is true:

$$\mathcal{E}_i \text{ is a dense subspace of a Hilbert space } H_i,\ 0 \le i \le N, \tag{5.22a}$$

$$\begin{aligned} &d_i^*, \text{ the Hilbert space adjoint of } d_i : \mathcal{E}_i \to H_{i+1}, \text{ satisfies} \\ &\mathcal{E}_{i+1} \subset \mathcal{D}(d_i^*) \text{ and } d_i^*(\mathcal{E}_{i+1}) \subset \mathcal{E}_i. \end{aligned} \tag{5.22b}$$

Thus every elliptic complex on compactly supported smooth sections is a pre-Hilbert complex. Note that the operators D_i in (5.21) are closable in view of (5.22b). Hence we can introduce ibc's for pre-Hilbert complexes, and we easily obtain abstract analogues of many results in [BL, Sec. 3].

Now we are ready to deal with the full Kähler package.

THEOREM 5.6. *Let M be a Kähler manifold and assume that*

$$\textit{the de Rham complex on } M \textit{ has a unique ibc, } (\mathcal{D}, D), \textit{ which is Fredholm,} \tag{5.23}$$

and that

$$[J, \Delta_D] = 0 \ . \tag{5.24}$$

Then L^2-KP holds for M.

Proof: Introduce the ibc's for $(\Omega^{r,*}_{\mathbf{C},0}, \bar\partial)$ defined in Lemma 5.4. We deal with the various parts of L^2-KP separately.

a) We know from Lemma 5.3, 2. that KHT holds for $(\mathcal{D}, D)$, hence the assertion follows from Lemma 5.4.

b) Since $d^t_{m-k-1} \circ (\varepsilon *)_k = (\varepsilon *)_{k+1} \circ d_k$ on $\Omega_{\mathbf{C},0}(M)$, where $\varepsilon | \Omega^k(M) = (-1)^{(k-1)k/2}$, we see that we must have

$$D^*_{m-k-1} = (\varepsilon *)_{k+1} \circ D_k \circ (\varepsilon *)_{m-k} \ ,$$

since the adjoint complex has an unique ibc, too. It follows that $(\varepsilon *)_k \circ \Delta^k_D = \Delta^{m-k}_D \circ (\varepsilon *)_k$ which implies Poincaré duality.

c) The Hard Lefschetz Theorem follows from the representation theory of $sl(2, \mathbf{C})$ once we know that Λ and L map $\oplus_t \hat{\mathcal{H}}^t_D$ into itself and that all $\hat{\mathcal{H}}^t_D$ are finite-dimensional, cf. [GH, p. 122]. Thus we want the relations

$$L(\mathcal{D}(\Delta_D)) \subset \mathcal{D}(\Delta_D), \ \ [L, \Delta_D] = 0 \ , \tag{5.25a}$$

and

$$\Lambda(\mathcal{D}(\Delta_D)) \subset \mathcal{D}(\Delta_D), \ \ [\Lambda, \Delta_D] = 0 \ , \tag{5.25b}$$

which hold on $\Omega_{\mathbf{C},0}(M)$. Since L is bounded and Δ_D is self-adjoint, (5.25a) implies (5.25b). Also, it is enough to prove only

$$L(\mathcal{D}((\Delta_D)^2)) \subset \mathcal{D}(\Delta_D) \ . \tag{5.25c}$$

Now on $\Omega_{\mathbf{C},0}(M)$ we have the Kähler identities $[d, L] = 0$ and $[d^t, L] = -d_c$ [W, p. 192]. It follows that

$$[d_{\max}, L] = 0, \ \ [d_{\min}, L] = 0 \ , \tag{5.26a}$$

hence by uniqueness of ibc's

$$[D, L] = 0 \ ; \tag{5.26b}$$

moreover, evaluating $(L\omega, D\eta)$ for $\omega \in \mathcal{D}(\Delta_D) = \mathcal{D}(\Delta_{D_c})$ and $\eta \in \Omega_{\mathbf{C},0}(M)$ gives

$$[D^*, L] = -D_c \ \text{ on } \ \mathcal{D}(\Delta_D) \ . \tag{5.26c}$$

Thus, for $\omega \in \mathcal{D}((\Delta_D^t)^2)$ we have $L\omega \in \mathcal{D}_{t+2} \cap \mathcal{D}^*_{t+1}$, by (5.26b) and (5.26c), and

$$D_{t+2} L\omega = L D_t \omega \in \mathcal{D}^*_{t+2},$$

by (5.26c), since $D_{t+2}\omega \in \mathcal{D}(\Delta_D^{t+2})$, as well as

$$D^*_{t+1} L\omega = L D^*_{t-1}\omega - D_c\omega \in \mathcal{D}_{t+1} \ ,$$

since $D^*_{t-1}\omega \in \mathcal{D}(\Delta_D)$ and $D_c\omega \in \mathcal{D}(\Delta_{D_c}) = \mathcal{D}(\Delta_D)$.

d) The Hodge signature theorem is a consequence of the representation theory of $sl(2, \mathbf{C})$, the Kähler geometry and a)–c) (since we have (5.25a,b)). The arguments in [W, Sec. V.6] or [GH, Sec. 0.7] carry over literally. One only has to be careful with domains; but (5.25a,b) guarantee that everything goes through. The proof is complete. □

Remarks: 1. We have in fact shown that (5.25a) holds for Δ_D^t if the ibc is unique and $\Delta_D^j = \Delta_{D_c}^j$ for $j = t, t+1$.

2. Note that (5.24) is a commutator of an unbounded operator with a bounded operator. Because of its significance for our considerations we emphasize that $[J, \Delta_D] = 0$ means

$$J\mathcal{D}(\Delta_D) \subset \mathcal{D}(\Delta_D)$$

and

$$J\Delta_D\omega = \Delta_D J\omega \ \text{ for } \ \omega \in \mathcal{D}(\Delta_D).$$

Since J and Δ commute on smooth forms with compact support, for proving $[J, \Delta_D] = 0$ it suffices to check the inclusion $J\mathcal{D}(\Delta_D) \subset \mathcal{D}(\Delta_D)$.

Let us observe that (5.23) and (5.24) are automatically satisfied if all $\Delta^t|\Omega^t_{\mathbb{C},0}(M)$ are essentially self-adjoint in $L^2(\Lambda^t T^*_{\mathbb{C}}M)$ and the unique ibc is Fredholm. The Fredholmness is equivalent to saying that 0 is not in the essential spectrum of any Δ^t_D, cf. [BL, Theorem 2.4]. Thus we find

COROLLARY 5.7. *If all $\Delta^t|\Omega^t_{\mathbb{C},0}(M)$ are essentially self-adjoint in $L^2(\Lambda^t T^*_{\mathbb{C}}M)$ and $0 \notin \cup_t \mathrm{spec}_e \Delta^t$ then L^2-KP holds on M.*

In particular, $\Delta^t|\Omega^t_{\mathbb{C},0}(M)$ is essentially self-adjoint for every complete Kähler manifold [Ch].

We now proceed to the main result of this section which will give the conclusion of Theorem 5.6 under a weaker condition than (5.24); this will be crucial for the applications we have in mind.

THEOREM 5.8. *Let M be a Kähler manifold with (5.23) and*

$$[J, \Delta^t_D] = 0 \quad \textit{for } \ t < \nu \ . \tag{5.24$'$}$$

Then L^2-KP holds for M.

Proof: Note first that Poincaré duality holds for M since this uses only (5.23). Then we have $(\varepsilon *)_k \Delta^k_D = \Delta^{m-k}_D (\varepsilon *)_k$ so (5.24′) implies

$$[J, \Delta^t_D] = 0 \quad \text{for } \ t \neq \nu \ . \tag{5.24$''$}$$

We want to show that (5.24″) already implies (5.24). To do so we proceed in two steps.

In the first step we start from the Lefschetz decomposition of $L^2(\Lambda^\nu T^*_{\mathbb{C}}M)$. Denote by $PM := \ker \Lambda$ the subbundle of $\Lambda T^*_{\mathbb{C}}M$ consisting of primitive elements, then

$$PM = \bigoplus_{t \geq 0} PM \cap \Lambda^t T^*_{\mathbb{C}}M =: \bigoplus_{t \geq 0} P^t M \ .$$

The representation theory gives

$$\Lambda^\nu T^*_{\mathbb{C}}M = [\bigoplus_{\substack{\nu - 2l \geq 0 \\ l > 0}} L^l P^{\nu - 2l} M] \oplus P^\nu M \ , \tag{5.27}$$

and the corresponding L^2-decomposition

$$L^2(\Lambda^\nu T^*M) = [\bigoplus_{\substack{\nu-2l\geq 0\\ l>0}} L^2(L^l P^{\nu-2l}M)] \oplus L^2(P^\nu M) \ . \tag{5.28}$$

Now we show that the decomposition (5.28) reduces both Δ_D^ν and $\Delta_{D_c}^\nu$. The key observation is that we have unitary isomorphisms (cf. [W, p. 182, (3.16)])

$$\begin{aligned} &U_l : P^{\nu-2l}M \to L^l P^{\nu-2l}M \ , \\ &U_l = \alpha_{\nu,l}\, L^l, \ \ U_l^* = \alpha_{\nu,l}\, \Lambda^l, \ \ \alpha_{\nu,l} > 0, \ \ l \geq 0 \ . \end{aligned} \tag{5.29}$$

From (5.25) we derive that the orthogonal projection onto $L^2(P^{\nu-2l}M)$ reduces $\Delta_D^{\nu-2l} = \Delta_{D_c}^{\nu-2l}$ if $l > 0$. Hence we find with (5.29) that also the orthogonal projection onto $L^2(L^l P^{\nu-2l}M)$ reduces Δ^ν, with intertwining relations

$$\Delta_{D_{(c)}}^\nu | L^2(L^l P^{\nu-2l}M) = U_l(\Delta_{D_{(c)}}^{\nu-2l} | L^2(P^{\nu-2l}M))U_l^* \ , \quad l > 0 \ ,$$

where we also use that $[J, L] = 0$. Since $\Delta_{D_c}^{\nu-2l} = \Delta_D^{\nu-2l}$ by assumption, we conclude in particular that

$$\Delta_D^\nu | L^2(L^l P^{\nu-2l}M) = \Delta_{D_c}^\nu | L^2\left(L^l P^{\nu-2l}M\right) \ , \quad l > 0 \ .$$

Thus we see that (5.28) reduces $\Delta_{D_{(c)}}^\nu$ and that $\Delta_D^\nu = \Delta_{D_c}^\nu$ coincide on the first summand of (5.28).

In the second step we show that $\Delta_{D_c}^\nu | L^2(P^\nu M) =: Q_c$ extends $\Delta_D^\nu | L^2(P^\nu M) =: Q$ implying equality since both operators are self-adjoint. It is enough to show that $\mathcal{D}^\infty(Q) \subset \mathcal{D}(Q_c)$ since $Q = Q_c$ on $C_0^\infty(P^\nu M)$. Thus pick $\omega \in \mathcal{D}^\infty(Q) \subset \mathcal{D}^\infty(\Delta_D^\nu)$ and observe that $H | L^2(\Lambda^\nu T_{\mathbb{C}}^* M) = 0$, implying

$$0 = H\omega = \Lambda L \omega \ ,$$

hence from [W, p. 181] also

$$0 = L\omega \ .$$

It follows, using more Kähler identities [W, p. 193] that

$$\begin{aligned} d_c\omega &= [L, d^t]\omega = L d^t \omega \in L^2(\Lambda^{\nu+1} T_{\mathbb{C}}^* M) \ , \\ d_c^t \omega &= [d, \Lambda]\omega = -\Lambda d\omega \in L^2(\Lambda^{\nu-1} T_{\mathbb{C}}^* M) \ , \end{aligned}$$

hence $\omega \in \mathcal{D}(D_{c,\nu}) \cap \mathcal{D}(D^*_{c,\nu-1})$ by uniqueness of ibc's. Moreover

$$\begin{aligned} d_c^t d_c \omega &= d_c^t L d^t \omega = [d_c^t, L] d^t \omega + L d_c^t d^t \omega \\ &= d d^t \omega + L d_c^t d^t \omega \in L^2(\Lambda^\nu T^*_{\mathbf{C}} M) \ , \end{aligned}$$

since $d^t\omega \in \mathcal{D}^\infty(\Delta_D^{\nu-1}) = \mathcal{D}^\infty(\Delta_{D_c}^{\nu-1})$. Finally,

$$\begin{aligned} d_c d_c^t \omega &= -d_c \Lambda d\omega = -[d_c, \Lambda] d\omega - \Lambda d_c d\omega \\ &= d^t d\omega - \Lambda d_c d\omega \in L^2(\Lambda^\nu T^*_{\mathbf{C}} M) \ , \end{aligned}$$

since $d\omega \in \mathcal{D}^\infty(\Delta_D^{\nu+1}) = \mathcal{D}^\infty(\Delta_{D_c}^{\nu+1})$. The theorem is proved. □

We are now in the position to establish the L^2-Kähler package for conformally conic Kähler manifolds.

THEOREM 5.9. *Let M be a conformally conic Kähler manifold. Then the L^2-Kähler package holds for M, and all ideal boundary conditions involved are Fredholm.*

Proof: The given Kähler metric on M is conformally conic in the sense of Sec. 2, by assumption. Hence we can apply Theorem 3.7, and it only remains to show that

$$[\Delta_D^t, J] = 0 \ \text{ if } \ t \neq \nu \ . \tag{5.30}$$

But $\Delta_D^t = (\Delta_d^t)^{\mathcal{F}}$ for $t \neq \nu$, so $\omega \in \mathcal{D}(\Delta_D^t)$ iff $\omega \in \mathcal{D}((\Delta_d^t | \Omega^t_{\mathbf{C},0}(M))^*)$ *and* we can find a sequence $(\omega_n)_{n\in\mathbf{N}} \subset \Omega^t_{\mathbf{C},0}(M)$ such that, with $\omega_{nm} := \omega_m - \omega_n$,

$$\omega_n \to \omega \ \text{ in } \ L^2(\Lambda^t T^*_{\mathbf{C}} M), \ \ n \to \infty,$$

$$(\Delta_d^t \omega_{nm}, \omega_{nm})_{L^2} + \|\omega_{nm}\|^2_{L^2} \to 0, \ \ m \geq n \to \infty \ .$$

Since $[\Delta_d^t, J] = 0$ on $\Omega^t_{\mathbf{C},0}(M)$ and J is unitary, it follows from a routine check that J maps $\mathcal{D}((\Delta_d^t)^{\mathcal{F}})$ into itself. But this implies (5.30) and completes the proof. □

We also have the following interesting consequence on uniqueness of ibc's for the Dolbeault complexes.

COROLLARY 5.10. *Under the assumptions of Theorem 5.9, the operators ∂ and $\bar\partial$ on $\Omega^{r,s}_{\mathbf{C},0}(M)$ have unique closed extensions in $L^2(\Lambda^{r,s} M)$ if $r + s \neq \nu - 1, \nu$.*

Proof: Denote again by $(\mathcal{D}, D)$ the unique ibc for the de Rham complex. Then $(\mathcal{D}, D)$ induces ibc's on the Dolbeault complexes by the prescription of Lemma 5.4. Now we know from Theorem 5.9 that KHT holds for $(\mathcal{D}, D)$, and Theorem 3.7 gives $\Delta_D^t = (\Delta_d^t)^{\mathcal{F}}$ for $t \neq \nu$. It then follows from the characterization of the Friedrichs extension above that $\pi^{r,s}$ maps $\mathcal{D}(\Delta_D^t)$ to itself for $t \neq \nu$, so the corresponding Dolbeault Laplacians also coincide with the Friedrichs extension if $r + s \neq \nu$. Now the assertion follows from Lemma 3.4 and Poincaré duality. □

To conclude this section we add a few remarks on the case of Riemann surfaces, i.e. $\nu = 1$. So let M be any Riemann surface; then every Riemannian metric on M is Kähler. If $M \subset \mathbb{C}P^N$ is e.g. an algebraic curve equipped with the Fubini-Study metric, then the assumptions of Theorem 5.8 are satisfied, as shown in [BPS]. However, $\bar{\partial} : \Omega^{0,0}_{\mathbb{C},0}(M) \to \Omega^{0,1}_{\mathbb{C},0}(M)$ in this case may have many closed extensions showing that Cor. 5.10 cannot be improved in general. We want to discuss now a special ibc which satisfies L^2-KP on *any* Riemann surface.

From the de Rham complex

$$0 \to \Omega^0_{\mathbb{C},0}(M) \xrightarrow{d_0} \Omega^1_{\mathbb{C},0}(M) \xrightarrow{d_1} \Omega^2_{\mathbb{C},0}(M) \to 0$$

we derive the ibc

$$0 \to \mathcal{D}_0 \xrightarrow{d_{0,\min}} \mathcal{D}_1 \xrightarrow{d_{1,\max}} \mathcal{D}_2 \to 0\ , \tag{5.31}$$

which obviously satisfies Poincaré duality. Here $\mathcal{D}_0 := \mathcal{D}(d_{0,\min})$, $\mathcal{D}_1 := \mathcal{D}(d_{1,\max})$, $\mathcal{D}_2 := L^2(\Lambda^2 T^*_{\mathbb{C}} M)$; the corresponding Laplacians will again be denoted by Δ_D^t, $0 \leq t \leq 2$.

From the corresponding Hodge diamond

$$\begin{tikzcd}
 & \Omega^{1,0}_{\mathbb{C},0}(M) \arrow[dr, "\bar{\partial}^{1,0}"] & \\
\Omega^{0,0}_{\mathbb{C},0}(M) \arrow[ur, "\partial^{0,0}"] \arrow[dr, "\bar{\partial}^{0,0}"'] & & \Omega^{1,1}_{\mathbb{C},0}(M) \\
 & \Omega^{0,1}_{\mathbb{C},0}(M) \arrow[ur, "\partial^{0,1}"'] &
\end{tikzcd}$$

we want to derive closed extensions for $\partial^{r,s}$ and $\bar{\partial}^{r,s}$ as possible candidates for a Hodge decomposition compatible with (5.31). In each step we will consider only $\partial^{r,s}_{\min/\max}$ or $\bar\partial^{r,s}_{\min/\max}$; the corresponding Laplacians will be denoted by $\Delta^{r,s}_{\partial_{\min/\max}}$ or $\Delta^{r,s}_{\bar\partial_{\min/\max}}$, $0 \le r, s \le 1$. We introduce

$$\mathcal{D}^\infty_t := \mathcal{D}^\infty(\Delta^t_D) = \bigcap_{k\ge 1} \mathcal{D}\left(\left(\Delta^t_D\right)^k\right), \; 0 \le t \le 2 ,$$

$$\overset{(-)}{\mathcal{D}}{}^\infty_{r,s} := \mathcal{D}^\infty(\Delta^{r,s}_{\overset{(-)}{\partial}_{\min}}) , \; 0 \le r, s \le 1 .$$

LEMMA 5.11. *The complex*

$$0 \to \mathcal{D}^\infty_0 \xrightarrow{d_{0,\min}} \mathcal{D}^\infty_{1,0} \oplus \overline{\mathcal{D}}^\infty_{0,1} \xrightarrow{d_{1,\max}} \mathcal{D}^\infty_2 \to 0 \tag{5.32}$$

is a pre-Hilbert complex with essentially self-adjoint Laplacians. In particular, the unique ibc of the de Rham complex generated by (5.32) *satisfies KHT.*

Proof: First we have to verify the conditions of Def. 5.5. Clearly, the various spaces are dense in the respective Hilbert spaces, so (5.22a) holds.

Next we verify that $\mathcal{D}^\infty_0 = \mathcal{D}^\infty_{0,0} = \overline{\mathcal{D}}^\infty_{0,0} \subset \mathcal{D}_0$ – which is obvious – and $d_{0,\min}(\mathcal{D}^\infty_0) \subset \mathcal{D}^\infty_{1,0} \oplus \overline{\mathcal{D}}^\infty_{0,1}$. Now on $\Omega^0_{\mathbf{C},0}(M)$ we have

$$d_0 = \partial^{0,0} \oplus \bar\partial^{0,0}$$

implying

$$\mathcal{D}_0 \subset \mathcal{D}(\partial^{0,0}_{\min}) \cap \mathcal{D}(\bar\partial^{0,0}_{\min}) .$$

Thus our assertion follows from [BL, (2.34)] applied to $\overset{(-)}{\partial}{}^{0,0}_{\min}$.

For the middle term, we have to show that

$$\mathcal{D}^\infty_{1,0} \oplus \bar{\mathcal{D}}^\infty_{0,1} \subset \mathcal{D}(d_{1,\max}) \cap \mathcal{D}(d^t_{0,\max}) , \tag{5.33a}$$

$$d_{1,\max}(\mathcal{D}^\infty_{1,0} \oplus \bar{\mathcal{D}}^\infty_{0,1}) \subset \mathcal{D}^\infty_2 , \tag{5.33b}$$

$$d^t_{0,\max}(\mathcal{D}^\infty_{1,0} \oplus \bar{\mathcal{D}}^\infty_{0,1}) \subset \mathcal{D}^\infty_0 . \tag{5.33c}$$

To see this, we use the following relations between the Laplacians:

$$\Delta^{1,0}_{\partial_{\min}} = \Delta^{1,0}_{\bar\partial_{\max}} , \; \Delta^{0,1}_{\bar\partial_{\min}} = \Delta^{0,1}_{\partial_{\max}} . \tag{5.34}$$

In fact, the Kähler form equals vol_M and we have $L = *\pi^0$, $\Lambda = *\pi^2$, hence the Kähler identities [W, p. 193] imply on $\Omega_{\mathbf{C},0}(M)$

$$(\partial^{0,0})^t = \sqrt{-1}[\Lambda, \bar\partial] = \sqrt{-1} *_2 \bar\partial^{1,0} , \tag{5.35a}$$

$$\partial^{0,0} = \sqrt{-1}[L, \bar{\partial}^t] = -\sqrt{-1}(\bar{\partial}^{1,0})^t *_0 \ . \tag{5.35b}$$

Thus, after taking closures

$$\Delta^{1,0}_{\partial_{\min}} = \partial^{0,0}_{\min}(\partial^{0,0}_{\min})^* = (\bar{\partial}^{1,0}_{\max})^* *_0 *_2 \bar{\partial}^{1,0}_{\max}$$
$$= \Delta^{1,0}_{\bar{\partial}_{\max}} \ .$$

The second relation in (5.34) follows by complex conjugation.

Now we can argue as before (using [BL, (2.34)]) to obtain (5.33). The same reasoning gives the result for the third term:

$$\mathcal{D}^\infty_2 \subset \mathcal{D}(d^t_{1,\min}),\ d^t_{1,\min}(\mathcal{D}^\infty_2) \subset \mathcal{D}^\infty_{1,0} \oplus \bar{\mathcal{D}}^\infty_{0,1} \ . \tag{5.36}$$

Turning to the Laplacians we observe that the Kähler structure implies that

$$\Delta^0_D = 2\Delta^{0,0}_{\partial_{\min}} = 2\Delta^{0,0}_{\bar{\partial}_{\min}} \quad \text{on } \mathcal{D}^\infty_0 \ , \tag{5.37a}$$

$$\Delta^1_D = 2\Delta^{1,0}_{\partial_{\min}} \oplus 2\Delta^{0,1}_{\bar{\partial}_{\min}} \quad \text{on } \mathcal{D}^\infty_{1,0} \oplus \bar{\mathcal{D}}^\infty_{0,1} \ , \tag{5.37b}$$

$$\Delta^2_D = 2\Delta^{1,1}_{\partial_{\min}} = 2\Delta^{1,1}_{\bar{\partial}_{\min}} \quad \text{on } \mathcal{D}^\infty_2 \ . \tag{5.37c}$$

Since the domain is, in each case, a core of the self-adjoint operator on the right hand side, the Laplacians are all essentially self-adjoint. Hence (5.37) also implies KHT. □

We note some corollaries.

COROLLARY 5.12. *We have*

$$\Delta^0_D = 2\Delta^{0,0}_{\partial_{\min}} = 2\Delta^{0,0}_{\bar{\partial}_{\min}} = (\Delta^0_d)^{\mathcal{F}} \ ,$$
$$\Delta^1_D = 2(\Delta^{1,0}_{\partial_{\min}} \oplus \Delta^{0,1}_{\bar{\partial}_{\min}}) \ ,$$
$$\Delta^2_D = 2\Delta^{1,1}_{\partial_{\min}} = 2\Delta^{1,1}_{\bar{\partial}_{\min}} \ .$$

COROLLARY 5.13. *Putting*

$$\hat{\mathcal{H}}^{1,0} := \hat{\mathcal{H}}^1 \cap L^2(\Lambda^{1,0}M),\ \hat{\mathcal{H}}^{0,1} := \hat{\mathcal{H}}^1 \cap L^2(\Lambda^{0,1}M) \ ,$$

we have

$$\hat{\mathcal{H}}^{1,0} = \ker(\partial^{0,0}_{\min})^* = \ker \bar{\partial}^{1,0}_{\max} =: \hat{\mathcal{H}}^{1,0}_{\bar{\partial}} \ , \tag{5.38}$$

$$\hat{\mathcal{H}}^{0,1} = \ker(\bar{\partial}^{0,0}_{\min})^* = \ker \partial^{0,1}_{\max} =: \hat{\mathcal{H}}^{0,1}_{\partial} \ , \tag{5.39}$$

$$\overline{\hat{\mathcal{H}}^{1,0}} = \hat{\mathcal{H}}^{0,1} \ , \tag{5.40}$$

$$\hat{\mathcal{H}}^1 = \hat{\mathcal{H}}^{1,0} \oplus \hat{\mathcal{H}}^{0,1} \ . \tag{5.41}$$

Moreover, if $d_{0,\min}$ has closed range then this is true for $\partial_{0,\min}$ and $\bar{\partial}_{0,\min}$, too, and

$$\mathcal{H}^1 \simeq \hat{\mathcal{H}}^1 \simeq L^2(\Lambda^{1,0}M)/\operatorname{im}\partial^{0,0}_{\min} \oplus \ker \partial^{0,1}_{\max} \ , \tag{5.42a}$$

$$\simeq L^2(\Lambda^{0,1}M)/\operatorname{im}\bar{\partial}^{0,0}_{\min} \oplus \ker \bar{\partial}^{1,0}_{\max} \ . \tag{5.42b}$$

Proof: In view of our previous results we only have to verify the assertion on domains; thus assume that $\operatorname{im} d_{0,\min}$ is closed, such that $d^t_{1,\min} = - *_2 d_{0,\min} *_0$ also has closed range. Then we have the strong Hodge decomposition:

$$L^2(\Lambda^{1,0}M) \oplus L^2(\Lambda^{0,1}M) \simeq \hat{\mathcal{H}}^1 \oplus \operatorname{im}\, d_{0,\min} \oplus \operatorname{im}\, d^t_{1,\min} \; .$$

Now pick $\omega \in L^2(\Lambda^{1,0}M) \cap \hat{\mathcal{H}}^{\perp}_{1,0}$; we can find $\xi \in \mathcal{D}(d_{0,\min}), \eta \in \mathcal{D}(d^t_{1,\min})$ with $\omega = d_{0,\min}\xi + d^t_{1,\min}\eta$. But $\xi \in \mathcal{D}(\partial^{0,0}_{\min})$ and $d_{0,\min}\xi = \partial^{0,0}_{\min}\xi$, and since $d_{1,\max}|\mathcal{D}(d_{1,\max}) \cap L^2(\Lambda^{1,0}M) = \bar{\partial}^{1,0}_{\max}$ we also get $d^t_{1,\min}\eta = (\bar{\partial}^{1,0}_{\max})^*\eta$. Using (5.35) we find $(\bar{\partial}^{1,0}_{\max})^* = \sqrt{-1}\partial^{0,0}_{\min} *_2$, hence

$$\omega = \partial^{0,0}_{\min}(\xi + \sqrt{-1} *_2 \eta) \; .$$

So $\operatorname{im} \partial^{0,0}_{\min}$ is closed, and by complex conjugation $\operatorname{im} \bar{\partial}^{0,0}_{\min}$ is also closed. □

COROLLARY 5.14. *$(\mathcal{D}, D)$ is a Fredholm complex if and only if $\bar{\partial}^{0,0}_{\min}$ is a Fredholm operator. In this case,*

$$\operatorname{ind}(\mathcal{D}, D) = 2 \operatorname{ind} \bar{\partial}_{0,\min}.$$

If $M \subset \mathbb{C}P^N$ is an algebraic curve with normalization $\pi : \tilde{M} \to M$ then $\operatorname{ind} \bar{\partial}_{0,\min} = \frac{1}{2}\chi(\tilde{M})$, $\chi(\tilde{M})$ the Euler characteristic of $\tilde{M}$, so

$$\operatorname{ind}(\mathcal{D}, D) = \chi(\tilde{M}) \; . \tag{5.43}$$

In this case, the L^2-cohomology coincides with the cohomology of $\tilde{M}$.

Proof: The Fredholm property is equivalent [BL, Thm. 2.4] to the fact that 0 is not in the essential spectrum of Δ_D. Thus the assertion follows from Cor. 5.13 observing that $\Delta^{1,0}_{\partial_{\min}}$ and $\Delta^{0,1}_{\bar{\partial}_{\min}}$ are unitarily equivalent. From Cor. 5.13 we also derive the index formula (5.43).

Finally, (5.44) follows from the main result in [BPS]. □

We remark that Nagase has obtained Corollary 5.12, Corollary 5.13 and Corollary 5.14 in the special case of algebraic curves, using heavily the conformally conic structure [N1], [N2]. Both proofs of Corollary 5.14 have the same source since his proof is an adaption of the method in [BPS] to the Gauß–Bonnet operator. We emphasize that in the case of Riemann surfaces the L^2-Kähler package always holds for our special ibc, and that the cohomology calculation reduces to the calculation of $\operatorname{ind} \bar{\partial}^{0,0}_{\min}$.

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