Géométrie différentielle/*Differential Geometry*

The spectral rigidity of curve singularities

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Abstract – In this Note, we announce results on spectral asymptotics of algebraic curves, equipped with any metric which can be induced from a hermitian metric on complex projective space, via some projective embedding. We prove that the ζ-function of the Laplacian is meromorphic, and that the singularities of the curve can be detected from spectral information.

La rigidité spectrale des singularités des courbes algébriques

Résumé – *Dans cette Note, nous annonçons des résultats sur l'asymptotique spectrale de courbes algébriques. Les courbes sont munies de la métrique induite par une métrique hermitienne sur l'espace projectif complexe par un plongement projectif. On démontre que la fonction zêta du laplacien est méromorphe, et que les singularités de la courbe sont détectées par le spectre du laplacien.*

Version française abrégée – Soient M, Σ les parties régulière et singulière d'une courbe algébrique $C \subset \mathbb{C}P^n$. Nous munissons M de la métrique induite par une métrique hermitienne sur l'espace projectif $\mathbb{C}P^n$. Dans ce travail, nous étudions les propriétés spectrales du laplacien.

Nous énonçons maintenant les principaux résultats de ce travail:

Théorème 1. – 1) *L'extension de Friedrich*, Δ_0, *du laplacien est un opérateur à spectre discret.*

2) *La fonction zêta de* Δ_0 *est méromorphe dans* $\mathbb{C}$ *et ses pôles sont au plus d'ordre* 2.

3) *On a le développement asymptotique*

$$\operatorname{tr}(e^{-t\Delta_0}) \underset{t\to 0+}{\sim} \sum_{j\geq 0} a_{2j}\, t^{j-1} + \sum_{j\geq 1} b_{2j}\, t^{j-1} \log t + \sum_{\substack{p\in\Sigma \\ 1\leq k\leq L(p)}} \sum_{j\geq 0} c_j(k, p)\, t^{j/(2N_k(p))}, \tag{1}$$

où $L(p)$ *est le nombre de composantes irréductibles de* C *au voisinage de* $p \in \Sigma$ *et* $N_k(p)$ *est la multiplicité de la k-ième composante,* $1 \leq k \leq L(p)$.

En ce qui concerne les coefficients apparaissant dans (1), on a les informations suivantes.

Théorème 2. – *On a les identités*

$$a_0 = \frac{\operatorname{vol} M}{4\pi}, \tag{2}$$

$$b_2 = 0. \tag{3}$$

De plus

$$\lim_{t\to 0+} (\operatorname{tr} e^{-t\Delta_0} - a_0\, t^{-1}) - \chi_{(2)}(M)/6 = \frac{1}{12} \sum_{\substack{p\in\Sigma \\ 1\leq k\leq L(p)}} (N_k(p) + N_k(p)^{-1} - 2), \tag{4}$$

où $\chi_{(2)}(M)$ *est la caractéristique d'Euler* L^2 *de* M. *En particulier,* M *possède des singularités autres que des points multiples si et seulement si le membre de droite de* (4) *est non nul.*

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Dans ce travail nous démontrons que le coefficient b_4 d'une certaine parabole généralisée est non nul. Néanmoins nous ne connaissons pas d'exemple de courbe algébrique possédant des coefficients $c_j(k, p)$ non nuls.

Dans la troisième partie nous esquissons la démonstration du théorème 1. Celle-ci repose essentiellement sur le "Singular Asymptotics Lemma" [3] et une analyse détaillée de certains opérateurs de Sturm-Liouville à coefficients opérateurs. Notre méthode de démonstration simplifie d'une manière considérable l'approche de [3], [4], [5].

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1. SPECTRAL DATA FOR ALGEBRAIC CURVES. – Let M be the regular part of an algebraic curve $C \subset \mathbb{CP}^n$. We equip M with a Riemannian metric induced from some hermitian metric on $\mathbb{CP}^n$; we denote by $\mathcal{M}$ the set of all such objects. We have $C = M \cup \Sigma$ where the singular set, Σ, is finite. Near a point $p \in \Sigma$, C decomposes in $L(p)$ irreducible components providing the multiplicities $N_k(p) \in \mathbb{N}$, $1 \leqq k \leqq L(p)$. If all $N_k(p)$ are one then p is just a multiple point which we do not regard as a singularity for the purpose of this study. If Σ is nonempty, the metric on M may be incomplete. The first analytic difficulty caused by this fact concerns the definition of "spectral data": the Laplacians derived from the de Rham complex,

$$0 \to \Omega_0^0(M) \xrightarrow{d_0} \Omega_0^1(M) \xrightarrow{d_1} \Omega_0^2(M) \to 0, \tag{1}$$

[where $\Omega_0^i(M)$ denotes smooth i-forms with compact support] may not be essentially self-adjoint in the respective Hilbert spaces. We have proved in [1], however, that we are in the *case of uniqueness* in the sense that

$$d_{i,\min} = d_{i,\max}, \qquad i = 0, 1. \tag{2}$$

Here, $d_{i,\min}$ denotes the closure of d_i, and $d_{i,\max}$ the adjoint of $d_i^t := -\star d_i \star$. Thus we obtain a Hilbert complex from the closed operators in (2) with self-adjoint Laplacians Δ_i. The following result is also contained in [1].

THEOREM 1. – *Each Δ_i is discrete, and Δ_0 equals the Friedrichs extension of its restriction to $\Omega_0^0(M)$.*

If we put $\beta_i := \dim\ker \Delta_i$, $0 \leqq i \leqq 2$, then a full set of spectral data is provided by spec Δ_0 *and*

$$\chi_{(2)}(M) := 2\beta_0 - \beta_1.$$

2. MAIN RESULT. – In view of theorem 1, it is enough to compute the L^2-Euler characteristic, $\chi_{(2)}(M)$, and the spectral asymptotics of Δ_0. $\chi_{(2)}(M)$ has been determined in [2] and [1]. Moreover, we have

THEOREM 2. – 1) *The ζ-function of Δ_0 is meromorphic in $\mathbb{C}$, with poles of at most second order.*

2) *We have the asymptotic expansion*

$$\operatorname{tr}(e^{-t\Delta_0}) \underset{t\to 0+}{\sim} \sum_{j\geqq 0} a_{2j}\, t^{j-1} + \sum_{j\geqq 1} b_{2j}\, t^{j-1} \log t + \sum_{\substack{p\in\Sigma\\ 1\leqq k\leqq L(p)}} \sum_{j\geqq 0} c_j(k, p)\, t^{j/(2N_k(p))}, \tag{3}$$

As usual, the existence result requires other methods of proof than the explicit computations necessary to exploit (3). We find the following.

THEOREM 3. – *In* (3), *we have*

$$a_0 = \frac{\operatorname{vol} M}{4\pi}, \tag{4}$$

$$b_2 = 0, \tag{5}$$

and

$$\lim_{t\to 0+} (\operatorname{tr} e^{-t\Delta_0} - a_0\, t^{-1}) - \chi_{(2)}(M)/6 = \frac{1}{12} \sum_{\substack{p\in\Sigma\\ 1\leqq k\leqq L(p)}} (N_k(p) + N_k(p)^{-1} - 2). \tag{6}$$

In particular, M *has singularities (other than multiple points) iff the left hand side of* (6) *is nonzero.*

The leading term (4) has also been determined by Yoshikawa [7]. [6] contains eigenvalue estimates via heat kernel comparison in arbitrary dimensions. The logarithmic terms in (3) are determined by algebraic expressions localized at the points of Σ. Thus, they can all be calculated, at least in principle. They detect rather subtle information, though, as the following family of examples (known as generalized parabolas) illustrates. Let, for $k, l \in \mathbb{N}$, $k, l \geqq 1$, $(k, l) = 1$,

$$C_{kl} := \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \,|\, z_0^{k+l} = z_1^l\, z_2^k\},$$

and denote by M_{kl} the regular part, equipped with the Fubini-Study metric. C_{kl} has singularities unless $l = k = 1$.

THEOREM 4. – *The coefficient b_4 in* (3) *vanishes if $k \geqq 2$, but is nonzero if $k = 1$ and $l \geqq 2$.*

The coefficients $c_j(k, p)$ in (3) are determined by expressions which involve analytic continuations, so cannot be computed by simple algorithms. So far, we do not know whether they do actually occur; this point deserves further study.

3. THE METHOD OF PROOF. – The well-known explicit parametrization of curve singularities (*cf.* e.g. [2], Sec. 2) allows a correspondingly explicit description of the metric near the points of Σ. In fact, near $p \in \Sigma$ an irreducible component of multiplicity N is found to be isometric to a punctured disc in $\mathbb{R}^2$, with metric

$$g(r, \varphi) = \alpha(r^{1/N}, \varphi)\, dr \otimes dr + r^2\, \gamma(r^{1/N}, \varphi)\, d\varphi \otimes d\varphi, \tag{7}$$

where $\alpha, \gamma \in C^\infty([0, \varepsilon] \times S^1)$, $\alpha(0, \gamma) = 1$, $\gamma(0, \varphi) = N^2$. Thus, (7) is a conic metric up to a perturbation of order $r^{1/N}$. This suggests that the problem at hand can be reduced to the Singular Asymptotics Lemma (SAL), proved in [3] and applied to conic singularities in [4]. Following this outline we have to study a regular singular model operator on the half-line, which corresponds to the metric (7). We write

$$\partial_x u(x) := \frac{\partial u}{\partial x}(x), \qquad Xu(x) := xu(x),$$

$$A_0 := \frac{1}{N^2}\Delta_{S^1} - \frac{1}{4},$$

and introduce the operator valued ordinary differential operator

$$\tau := -\partial_x^2 + X^{-2} A_0 + R_\varepsilon, \tag{8}$$

with domain $\mathcal{H}^\infty := \bigcap_{k\geq 1} C_0^\infty((0,\infty), \mathcal{D}(A_0^k))$, $\mathcal{H} := L^2(\mathbb{R}_+, H)$, where $H := L^2(S^1)$. The necessary assumptions on the perturbation are

$$(9)\qquad (\tau u, u) \geqq 0 \quad \text{for} \quad u \in \mathcal{H}^\infty,$$

and a representation of the form

$$(10)\qquad R_\varepsilon = \sum_{i,j=0}^{2} U_i^* C_\varepsilon^{ij} U_j.$$

Here the basic operators U_i are given by

$$(10\,a)\qquad \begin{cases} U_0 := I, & U_1 := \Omega^{1/N} X^{-1}(A_0 + I)^{1/2}, \\ U_2 := \Omega^{1/2}\,\partial_x; \end{cases}$$

here $\omega(x) := x/(1+x)$ and Ω is multiplication by ω. For the operators C_ε^{ij} we have

$$(10\,b)\qquad C_\varepsilon^{ij} u(x) = c_\varepsilon^{ij}(x^{1/N})\,u(x), \qquad c_\varepsilon^{ij} \in C^\infty([0,\infty), \mathcal{L}(H)),$$

and

$$(10\,c)\qquad \sum_{i,j=0}^{2} \sup_{x>0} \|c_\varepsilon^{ij}(x)\|_{\mathcal{L}(H)} \leqq \varepsilon,$$

where ε is fixed, but can be chosen as small as we please.

In view of (9) we can form the Friedrichs extension, T, of τ in $\mathcal{H}$, and the techniques of [4] then require that we apply the SAL to the expression

$$t_l(\varphi; z) := \int_0^\infty \varphi(x)\,\mathrm{tr}_H\, G^l(z; x, x)\,dx, \qquad \varphi \in C_0^\infty(\mathbb{R}),$$

where $G^l(z; x, y)$ is the operator kernel of $G^l(z) := (T + z^2)^{-l}$, which takes values in $\mathcal{L}(H)$. The conic scaling on $\mathcal{H}$,

$$U_t\,u(x) := t^{1/2}\,u(tx), \qquad t > 0,$$

gives, with $\tau_t := t^2\,U_t \tau U_t^* =: -\partial_x^2 + X^{-2} A_0 + R_{\varepsilon,t}$, that

$$T_t := t^2\,U_t T U_t^*$$

is the Friedrichs extension of τ_t. Thus, with $G_t^l(z) := (T_t + z^2)^{-l}$ we find

$$(11)\qquad t_l(\varphi; z) := \int_0^\infty \varphi(x)\,x^{2l-1}\,\mathrm{tr}_H\, G_x^l(xz; 1, 1)\,dx.$$

However, the integrand is not smooth in $x \geqq 0$ but only a smooth function of $x^{1/N}$. This lack of smoothness prevents the direct application of the SAL, but it also complicates considerably the approach of [4], to construct explicitly the resolvent using a Neumann series. The first difficulty is easily circumvented by substituting $x = y^N$ in (11), the second is more serious. To remove it we drop the Neumann series altogether and attack the necessary *a priori* estimates directly; these estimates are then used to prove the crucial "integrability condition" in the SAL precisely as it was done in [4], sec. 4. This leads to a simpler and at the same time more general approach than in [4].

4. *A PRIORI* ESTIMATES. – To formulate the crucial estimates (in theorem 7 below) we make use of

DEFINITION 5. – *A linear operator*, S, *in* $\mathcal{H}$ *will be called controlled by* τ *if the following is true.*

(1) $\mathcal{D}(S) = \mathcal{H}^\infty$,

(2) S *is closable,*

(3) *there is a constant*, $c(S)$, *such that for* $u \in \mathcal{H}^\infty$

$$\|Su\|^2 \leqq c(S)\,((\tau u, u) + \|u\|^2).$$

The set of all operators controlled by τ forms a linear space to be denoted by $\mathcal{S}$. We introduce a weight on $\mathcal{S}$ as follows: for $S \in \mathcal{S}$, we write $\sigma(S) = 0$ if S extends to a bounded operator on $\mathcal{H}$, and $\sigma(S) = 1$ otherwise. Clearly, $\mathcal{L}(\mathcal{H}) \subset \mathcal{S}$ so the question is whether $\mathcal{S}$ contains unbounded operators. We prove, with U_i the operators introduced in (10 *a*):

THEOREM 6. – $U_i \in \mathcal{S}$, $0 \leqq i \leqq 2$.

The proof is reduced to parametrix constructions for first order operators, as given in [5], sec. 2, by factorizing the unperturbed operator $\tau - R_\varepsilon$.

The structure of R_ε given in (10) and theorem 6 together are the essential points in proving the following estimate. We introduce the von Neumann-Schatten classes $C_p(\mathcal{H})$, $p > 0$, and put $C_\infty(\mathcal{H}) := \mathcal{K}(\mathcal{H})$ the compact operators equipped with the operator norm.

THEOREM 7. – *Assume that, for some* $p > 0$,

$$(12)\qquad (A_0 + I)^{-1} \in C_p(H).$$

Then, for $|\arg z| < \delta$, $|z| \geqq c_0$, *and* $S_i \in \mathcal{S}$, $i = 1, 2$, *we have*

$$(13)\qquad \|S_1 (I + X)^\nu\, G(z)\,(I + X)^\mu\, S_2^*\|_{C_q(\mathcal{H})} \leqq c_1(\delta, c_0, S_1, S_2, \mu, \nu, q).$$

Here

$$q = (2p + 1)\,(2 - \sigma(S_1) - \sigma(S_2))^{-1}$$

and we require that, for some $\alpha > 2$, $\nu + \mu \leqq -\alpha\,(2 - \sigma(S_1) - \sigma(S_2))$.

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