The local index theorem without smoothness

Jochen Brüning

Dedicated to Lars Hörmander

1. Introduction

The local index theorem has attracted much interest of analysts, geometers, and physicists over the last two decades, rendering its proof more and more perspicuous. All this work pertains to the smooth case whereas it is known that in the special case of the signature operator much less regularity is required, at least for an "almost local" signature theorem cf. [MoWu,CSuT]. Motivated by this, we study here the differentiability requirements necessary to formulate and prove the local index theorem for general Dirac operators. We will present a version for $C^{1,1}$ structures which seems close to the minimum requirement. Our proof relies on abstract heat kernel estimates and is perhaps general enough to extend to other singular situations.

To formulate the results, we begin by recalling the setting of the local version of the Atiyah-Singer index theorem for twisted Dirac operators. Let M be a smooth compact spin manifold of dimension $m = 2k$. We fix a spin structure, $P_{spin}M$, and a smooth Riemannian metric, g, on M. Let S be a spin bundle over M, equipped with the natural hermitian structure and unitary connection, ∇^S, defined by g. S is a left module over the bundle of complex Clifford algebras $\mathbb{C}\ell M$; the action of $\sigma \in \mathbb{C}\ell M$ will be denoted by $c(\sigma)$. Choose a hermitian bundle of coefficients, E, with unitary connection, ∇^E, and form the tensor product $S \otimes E$ which inherits naturally a hermitian structure and a unitary connection, $\nabla^{S\otimes E}$. The Dirac operator with coefficients in E, D^E, is then a first order elliptic differential operator on $C^\infty(S \otimes E)$, defined by

$$D^E f = \sum_{i=1}^{m} c(e_i)\, \nabla^{S\otimes E}_{e_i} f, \qquad f \in C^\infty(S \otimes E), \tag{1.1}$$

where $(e_i)_{i=1}^m$ is any (oriented) local orthonormal frame for TM. D^E is symmetric and essentially self-adjoint in $L^2(S \otimes E)$; the unique self-adjoint extension will also be denoted by D^E. To obtain a nontrivial index we bring in the involution on S defined by

$$c(\omega_{\mathbb{C}}) := \sqrt{-1}^{\,k} c(e_1) \circ \ldots \circ c(e_m) \tag{1.2}$$

which splits $S = S_+ \otimes S_-$. Then $\varrho := c(\omega_{\mathbb{C}}) \otimes I$ splits $S \otimes E = S_+ \otimes E \oplus S_- \otimes E$, and we obtain a decomposition

$$D^E = \begin{pmatrix} 0 & D_-^E \\ D_+^E & 0 \end{pmatrix} . \tag{1.3}$$

D_+^E is a closed Fredholm operator with adjoint D_-^E, and its index is given by the celebrated formula of Atiyah and Singer [AS]:

Theorem 1.1

$$\mathrm{ind}\ D_+^E = \int_M \hat{A}(M) \wedge \mathrm{ch}\ E\ . \tag{1.4}$$

Here $\hat{A}(M)$ and ch$E$ are certain characteristic differential forms which can be computed locally from the curvature of ∇^{TM}, the Levi-Civita connection defined by g, and ∇^E. This "locality" of the index can be made more precise if one brings in the heat kernel. As observed by McKean and Singer [McKS],

$$\mathrm{ind}\ D_+^E = \mathrm{tr}_{L^2(S\otimes E)}\ \left[\varrho\ e^{-t(D^E)^2}\right]\ , \tag{1.5}$$

for all $t > 0$. Since $(D^E)^2$ has discrete spectrum, $e^{-t(D^E)^2}$ is an operator with smooth kernel,

$$e^{-t(D^E)^2} f(p) = \int_M K_t(p,q)\ (f(q))\ \mathrm{vol}_M(q)\ , \tag{1.6}$$

where $\mathrm{vol}_M(q) = e_1^\flat \wedge \ldots \wedge e_m^\flat(q)$ (with $\flat : TM \to T^*M$ the 'musical' isomorphism defined by g) is the volume form and $K_t(p,q) \in End((S\otimes E)_q, (S\otimes E)_p)$. Now it follows from a fundamental idea of Hadamard [H], further developed by Minakshisundaram and Pleijel [MiPl], Seeley [Se], and Greiner [Gr] that we have an asymptotic expansion

$$K_t(p,p) \sim_{t\to 0+} \sum_{j\geq 0} t^{j-m/2} U_j(p)\ ,\ p \in M\ . \tag{1.7}$$

In local frames and coordinates, the endomorphisms U_j are recursively defined as polynomials in the derivatives of g and the data on E such that U_j contains $2j$ derivatives of the metric. Using this in (1.5) one finds

$$\int_M \mathrm{tr}_{S\otimes E}\ [\varrho\ U_j(p)]\ \mathrm{vol}_M(p) = \begin{cases} 0\ , & j < m/2\ , \\ \int_M \hat{A}(M) \wedge \mathrm{ch}\ E, & j = m/2\ . \end{cases} \tag{1.8}$$

The local index theorem now asserts that the identities (1.8) hold even pointwise.

Theorem 1.2 *For all $p \in M$,*

$$\mathrm{tr}_{S\otimes E}[\varrho\, U_j(p)] = \begin{cases} 0\,, & j < m/2\,, \\ \\ \hat{A}(M) \wedge \mathrm{ch}\, E(e_1, \ldots, e_m)(p), & j = m/2\,. \end{cases} \tag{1.9}$$

Thus, 'massive cancellations' occur upon taking pointwise supertraces to the effect that the final answer contains only two derivatives of all coefficients involved. Theorem 1.2 has been proved by Kotake [Ko] for the case of Riemann surfaces and later by Patodi for the Gauß-Bonnet operator [P1] and the Hirzebruch signature operator [P2]. Building on methods of invariant theory introduced by Gilkey [Gi], Atiyah, Bott, and Patodi [ABP] proved Theorem 1.2 for twisted Dirac operators, thus yielding also another proof of the full Atiyah-Singer index theorem. Their method did not allow a direct identification of $\mathrm{tr}_{S\otimes E}[\varrho\, U_{m/2}(p)]$ but only up to certain universal constants which had to be computed from examples. Following ideas of the physicists Alvarez-Gaumé [Al] and Friedan and Windy [FW], Getzler [Ge, BeGeVe] designed a direct proof based on a scaling argument and Mehler's formula. His proof has been modified by many authors in order to clarify the subtleties of the analysis involved; we mention only the approach of B. Simon [Si] which is somewhat close to what we are going to present.

Our starting point in this paper is an apparent asymmetry in the statement of the local index theorem: we require m derivatives to formulate it (i.e. to construct $U_{m/2}$) but the answer will contain only two. Thus it is natural to look for a (possibly weaker) statement which requires less regularity. Clearly, if we allow C^1-structures everywhere then D^E is still perfectly well defined, with the same properties as listed before and the same index. But we also can present the index as an integral, starting from (1.5), as follows. Choose a family $\{\chi_{t,p} \mid t \in (0,1], p \in M\} \subset C(M)$ with the following properties:

$$\chi_{t,p} \geq 0 \text{ and supp } \chi_{t,p} \subset B_{\sqrt{t}/3}(p), \tag{1.10a}$$

$$\text{the map } p \mapsto \chi_{t,p} \text{ is continuous}, \tag{1.10b}$$

$$\int_M \chi_{t,p}(q)\, \mathrm{vol}_M(p) = 1 \quad \text{for all } q \in M,\ t \in (0,1]\,, \tag{1.10c}$$

$$\lim_{t\to 0+} \int_M \chi_{t,p}(q)\, \mathrm{vol}_M(q) = 1 \text{ for all } p \in M\,. \tag{1.10d}$$

Such a family is easy to construct, cf. (2.6) below. Now the map

$$M \ni p \mapsto\ \varrho\chi_{t,p}\, e^{-t(D^E)^2} \in C_1(L^2(S\otimes E))\,,$$

(where C_p denotes the von Neumann-Schatten class of order $p > 0$), is continuous hence from (1.5)

$$\begin{aligned} \text{ind } D_+^E &= \int_M \operatorname{tr}_{L^2(S\otimes E)} \left[\varrho \chi_{t,p}\, e^{-t(D^E)^2}\right] \operatorname{vol}_M(p) \\ &=: \int_M F_t^E(p) \operatorname{vol}_M(p)\ . \end{aligned} \tag{1.11}$$

In the smooth case, we have from Theorem 1.2 and the conditions (1.10), uniformly in $p \in M$,

$$\lim_{t\to 0+} F_t^E(p) = \hat{A}(M) \wedge \text{ch } E(e_1, \ldots, e_m)(p)\ .$$

This leads us to the following definition: We say that the *weak local index theorem holds for* D_+^E if there is $F^E \in L^1(M)$ such that

$$\lim_{t\to 0+} F_t^E \ =: \ F^E \ \text{ in } L^1(M), \tag{1.12a}$$

such that

$$\text{ind } D_+^E = \int_M F^E(p) \operatorname{vol}_M(p). \tag{1.12b}$$

It is fairly obvious that no local index theorem can hold if we require just C^1-structures on the (compact smooth) manifold M. In particular, to identify F^E with the Atiyah-Singer integrand we have to make sure that the latter exists and is integrable. Our main result thus reads as follows.

Theorem 1.3 *Let M be a compact smooth spin manifold and E a smooth complex vector bundle over M. If we equip M with a $C^{1,1}$ metric, and E with a $C^{1,1}$ hermitian structure and a unitary Lipschitz connection then the weak local index theorem holds for D_+^E. Moreover, for almost all p in M,*

$$F^E(p) = \hat{A}(M) \wedge \text{ch } E(e_1, \ldots, e_m)(p)\ . \tag{1.13}$$

The original motivation to investigate this problem was to design a proof of the local index theorem à la Getzler which is general enough to carry over to stratified situations like wedges; we will return to this question in a future publication.

The assumptions of Theorem 1.3 are perhaps not optimal. It would be interesting to find the precise minimal regularity condition under which the theorem remains true.

The paper is organized as follows. In Section 2 we give the outline of the proof. In Section 3 we compute the transformation of $(D^E)^2$ under the scaling map. Section 4 contains the main analytic facts needed in the proof of Theorem 1.3 which we present in a more general version than actually needed here since we could not find easy references in the literature. All assertions used in Sec. 2 are proved in

Sec. 5 which deals with C_q–estimates for heat kernels of certain elliptic systems of second order. We avoid higher regularity essentially by employing throughout the v.Neumann-Schatten scale of operator ideals, the crucial norm estimate being given in Theorem 5.2

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2. Outline of the proof

From now on, we are dealing with a compact smooth spin manifold, M, equipped with a $C^{1,1}$ metric, g. That is, the metric is a C^1 section of the bundle of symmetric two-tensors and we can find a finite C^2 atlas, $(U_i, \varphi_i)_{i=1}^{N}$, for M with $D(\varphi_i \cdot \varphi_j^{-1})$ Lipschitz for all i, j and $(\varphi_i^{-1})^* g$ C^1, with all partial derivatives of first order Lipschitz. Similarly, we consider a smooth complex coefficient bundle, E, over M equipped with a $C^{1,1}$ hermitian structure and a Lipschitz unitary connection, ∇^E. Under these conditions it is well known that the exponential map $\exp_p : \{s \in T_pM \mid |s| < i(M)\} \to B_{i(M)}(p)$ is a Lipschitz homeomorphism. We may assume that M has injectivity radius larger than one, $i(M) > 1$.

Our convergence proof for F_t in (1.11) will make use of the "Getzler scaling", defined in (2.10) below. This requires two technical adjustments: first, we replace t by $\tilde{t} := \varepsilon^2 t$ with $\varepsilon, t \in (0, 1]$, and second, we replace the coefficient bundle E by

$$E(p) := (M \times S_p^*) \otimes E \ . \tag{2.1}$$

The product bundle $S_p^* := M \times S_p^*$ is equipped with the obvious hermitian structure and the flat connection, $E(p)$ carries the tensor product structures. Then, clearly,

$$\begin{aligned} F_t^E(p) &= 2^{-k} \operatorname{tr}_{L^2(S \otimes E(p))} \left[\varrho \chi_{t,p} \, e^{-t(D^{E(p)})^2} \right] \\ &=: 2^{-k} \tilde{F}_t^E(p) \ . \end{aligned} \tag{2.2}$$

Next we give an explicit description of a family $(\chi_{t,p})$ with the properties (1.10). Denote by d_M the Riemannian distance and define

$$\exp_p^*(\mathrm{vol}_M)(s) =: \Theta_p(s) ds, \qquad |s| < 1. \tag{2.3}$$

Then choose $\eta \in C_0^\infty(B_{1/3}(0))$ with $\eta \geq 0$, $\eta(s) = \eta(|s|)$, and

$$\int_{\mathbb{R}^m} \eta(s) ds = 1 \ . \tag{2.4}$$

Then, with

$$\eta_t(s) := t^{-m/2}\,\eta(s/\sqrt{t})\;, \tag{2.5}$$

we define

$$\chi_{t,p}(q) := \eta_t\,(d_M(p,q))\,(\Theta_q(\exp_q^{-1}p))^{-1}\;. \tag{2.6}$$

The first step in the proof of Theorem 1.3 consists in a suitable transformation of $(D^{E(p)})^2$, following Getzler. Fix $p \in M$ and $\varepsilon \in (0,1]$, then we obtain the diagram:

$$\begin{array}{ccc} L^2(B_1(0), \Lambda_p^* \otimes E_p) & \xrightarrow{\Psi^1_{\varepsilon,p}} & L^2(B_\varepsilon(0), \mathbb{C}\ell_p \otimes E_p) \\ & & \downarrow \Psi^2_{\varepsilon,p} \\ & & L^2(B_\varepsilon(0), S_p \otimes E(p)_p) \;\xrightarrow{\Psi^3_{\varepsilon,p}}\; L^2(S \otimes E(p)|B_\varepsilon(p)) \end{array} \tag{2.7}$$

To describe the various maps in (2.7) we fix a local orthonormal frame, $(e_i)^m$, for TM in $B_1(p)$ and write for any strictly ordered multiindex, $I = \{i_1, \ldots, i_\ell\} \subset \{1, \ldots, m\}$,

$$\mu(I) \;:=\; \ell\;, \tag{2.8a}$$
$$e_I(q) \;:=\; e_{i_1} \cdot \ldots \cdot e_{i_\ell}(q) \in \mathbb{C}\ell_q M\;, \tag{2.8b}$$
$$e_I^\flat(q) \;:=\; e_{i_1}^\flat \wedge \ldots \wedge e_{i_\ell}^\flat(q) \in \Lambda_q^* M, \qquad q \in B_1(p)\;. \tag{2.8c}$$

Here, again, $\flat : TM \to T^*M$ is the 'musical' isomorphism definded by the metric. Then

$$\Psi^1_{\varepsilon,p}(e_I^\flat(p) \otimes \omega)(s) := \varepsilon^{\mu(I)} e_I(p) \otimes \omega(s/\varepsilon),\; \omega \in L^2(B_1(0), E_p)\;,\; |s| < \varepsilon\;. \tag{2.9a}$$

For the definition of $\Psi^2_{\varepsilon,p}$ we recall the existence of a $\mathbb{C}\ell_p$-equivariant isomorphism

$$\alpha_p : \mathbb{C}\ell_p M \to S_p \otimes S_p^*\;.$$

Then

$$\Psi^2_{\varepsilon,p} g(s) := (\alpha_p \otimes I_{E_p})(g(s)), \quad g \in L^2(B_\varepsilon(0), \mathbb{C}\ell_p \otimes E_p),\; |s| < \varepsilon\;. \tag{2.9b}$$

Finally, we need the parallel transport along radial geodesics from p : if F is any complex vector bundle over M with $C^{1,1}$ hermitian structure and Lipschitz unitary connection, ∇^F, then we get parallel transport

$$\mathrm{P}_p^F(s) : F_p \to F_{\exp_p(s)}, \quad |s| < 1\;,$$

with Lipschitz dependence on s. We define

$$\Psi^3_{\varepsilon,p}\, g(\exp_p s) := \mathrm{P}_p^{S\otimes E(p)}(s)(g(s)),\; g \in L^2(B_\varepsilon(0), S_p \otimes E(p)_p),\; |s| < \varepsilon\;. \tag{2.9c}$$

Finally, let

$$\Phi_{\varepsilon,p} := \Psi^3_{\varepsilon,p}\ \Psi^2_{\varepsilon,p}\ \Psi^1_{\varepsilon,p}\ . \tag{2.10a}$$

We note the explicit formula

$$\begin{aligned}&\Phi_{\varepsilon,p}(e^\flat_I(p)\otimes\omega)(\exp_p s)\\ &= \varepsilon^{\mu(I)} P_p^{S\otimes S^*_p}(s)(\alpha_p(e_I(p))\otimes P^E_p(s)(\omega(s/\varepsilon))\ .\end{aligned} \tag{2.10b}$$

Now, with (2.2) we arrive at

$$\tilde F^E_{\varepsilon^2 t}(p) = \mathrm{tr}_{\Lambda^*_p\otimes L^2(B_1(0),E_p)}\left[\Phi^{-1}_{\varepsilon,p}\ \varrho\ \Phi_{\varepsilon,p}\ \Phi^{-1}_{\varepsilon,p}\ \chi_{\varepsilon^2 t,p}\ e^{-t\varepsilon^2(D^{E(p)})^2}\Phi_{\varepsilon,p}\right]\ . \tag{2.11}$$

We examine next the transformed heat kernel. Formally, $P_{\varepsilon,p}(t) := \Phi^{-1}_{\varepsilon,p}\ e^{-t\varepsilon^2(D^{E(p)})^2}\Phi_{\varepsilon,p}$ solves the equation

$$0 = \left[\frac{\partial}{\partial t} + \Phi^{-1}_{\varepsilon,p}\ \varepsilon^2(D^{E(p)})^2\ \Phi_{\varepsilon,p}\right] P_{\varepsilon,p}(t) =: \left[\frac{\partial}{\partial t} + \tau_{\varepsilon,p}\right] P_{\varepsilon,p}(t)\ .$$

$\tau_{\varepsilon,p}$ is a second order elliptic operator on $C^\infty_0(B_1(0),\Lambda^*_p\otimes E_p)$ but not symmetric in general. In Section 3 we will prove

Lemma 2.1 *We can write*

$$\tau_{\varepsilon,p} = -\sum_{\beta,|\gamma|\le 2} M_\beta \partial^\gamma_s A^{\beta\gamma}_{\varepsilon,p}, \tag{2.12}$$

where M_β denotes multiplication by s^β. Moreover,

$$A^{0\gamma}_{\varepsilon,p} =: A^\gamma_{\varepsilon,p} = a^{\gamma_1\gamma_2}_{\varepsilon,p} I_{\Lambda^*_p\otimes E_p} \tag{2.13a'}$$

and, for some $\lambda_1,\lambda_2>0$ and uniformly in ε and p,

$$\lambda_1|\xi|^2 \le \sum_{\gamma_1,\gamma_2} a^{\gamma_1\gamma_2}_{\varepsilon,p}\xi_{\gamma_1}\xi_{\gamma_2} \le \lambda_2|\xi|^2, \tag{2.13a''}$$

and we have the regularity assumptions

$$A^{\beta\gamma}_{\varepsilon,p} \in C^{|\gamma|}(B_1(0),\mathrm{End}\,\Lambda^*_p\otimes E_p) \quad \text{(the } C^2 \text{ case)} \tag{2.13b'}$$

or, more generally,

$$A^{\beta\gamma}_{\varepsilon,p} \in \begin{cases} C^1 & \text{if } |\gamma| = 2,\\ \mathrm{Lip} & \text{if } |\gamma| = 1,\\ L^\infty & \text{if } |\gamma| = 0 \quad \text{(the } C^{1,1} \text{ case).}\end{cases} \tag{2.13b''}$$

If we decompose the coefficients with respect to the basis $(e_I^\flat)$ *(as in* (2.20) *below) then*

$$[A^{\beta\gamma}_{\varepsilon,p}]_{IJ} = 0 \quad \text{if } |\beta| - |\gamma| - |\mu(I) - \mu(J)| < -2. \tag{2.13c}$$

Finally, we have a limit in the sense that, for $f \in C_0^\infty(B_1(0), \Lambda_p^* \otimes E_p)$,

$$\lim_{\varepsilon \to 0} \tau_{\varepsilon,p}\, f = \tau_{0,p}\, f\ ,$$

where $\tau_{0,p}$ *is the "Getzler operator":*

$$\tau_{0,p} = -\sum_{i=1}^m \left[\frac{\partial}{\partial s_i} - \frac{1}{8} \sum_{j=1}^m s_j\ \Omega_{ij}(p) \otimes I_{E_p} \right]^2 + \Omega^E(p)\ . \tag{2.13}$$

If we denote by R^M, R^E *the respective curvature tensors and by* $'w'$ *the operation of wedge multiplication, then we have*

$$\Omega_{ij}(p) = \sum_{k,\ell=1}^m R^M_{k\ell ij} w(e_k^\flat \wedge e_\ell^\flat(p))\ ,$$

$$\Omega^E(p) = \frac{1}{2} \sum_{i,j=1}^m w(e_i^\flat \wedge e_j^\flat(p)) \otimes R^E_{e_i,e_j}(p)\ .$$

Next we have to introduce "boundary conditions". It seems most natural in this context to use the Friedrichs extension as defined by Kato [Ka, p.325]. The details will be given in Section 4; let us thus define $T_{\varepsilon,p}$ as the Friedrichs extension of $\tau_{\varepsilon,p}$ in $\mathcal{H} := L^2(B_{2/3}(0), \Lambda_p^* \otimes E_p)$ with domain $C_0^\infty(B_{2/3}(0), \Lambda_p^* \otimes E_p)$. Then $T_{\varepsilon,p}$ generates an analytic semigroup, $e^{-tT_{\varepsilon,p}}$, which is a good replacement for $P_{\varepsilon,p}(t)$:

Theorem 2.2 *Denote by* C_p *the von Neumann-Schatten class of order* $p > 0$ *(such that* C_1 *is the trace class). Then, for* $\chi_1, \chi_2 \in C_0^\infty(B_{1/2}(0))$ *and* $N \in \mathbb{N}$,

$$||\chi_1(e^{-tT_{\varepsilon,p}} - P_{\varepsilon,p}(t))\chi_2||_{C_1(\mathcal{H}))} \le C_N(\varepsilon, p, \chi_1, \chi_2) t^N\ .$$

This result will also be proved in Section 4. To analyze $e^{-tT_{\varepsilon,p}}$ it is important to have a good approximation. For this, somewhat surprisingly, the heat kernel of the Getzler operator does not seem convenient. Instead, we look at the operator which consists of the scalar principal part in (2.12),

$$\tau^0_{\varepsilon,p} := -\sum_{|\gamma_1|=|\gamma_2|=1} \partial_s^\gamma\ A^\gamma_{\varepsilon,p}. \tag{2.14}$$

Its Friedrichs extension, $T^0_{\varepsilon,p}$, is defined as before and it turns out that the operators $T_{\varepsilon,p}$ and $T^0_{\varepsilon,p}$ all have the same domain. This is the basis for the following representation by a Neumann series, also to be proved in Section 4.

Theorem 2.3 *Introduce for $t \in (0,1]$*

$$R^0_{\varepsilon,p}(t) := e^{-tT^0_{\varepsilon,p}} , \tag{2.15a}$$

$$R^{\nu+1}_{\varepsilon,p}(t) := \int_0^t R^\nu_{\varepsilon,p}(t-u) \sum_{\beta,|\gamma|\leq 1} M_\beta \partial^\gamma_s A^{\beta\gamma}_{\varepsilon,p} e^{-uT^0_{\varepsilon,p}} du \ , \ \nu \in \mathbb{Z}_+ \, . \tag{2.15b}$$

Then we have

$$e^{-tT_{\varepsilon,p}} = \sum_{\nu\geq 0} R^\nu_{\varepsilon,p}(t) \, , \tag{2.16}$$

and the sum converges in operator norm, uniformly in $t, \varepsilon \in (0,1]$ and $p \in M$.

Next, a little calculation using (2.10b) shows that

$$\chi_{\varepsilon^2 t,p} \, \Phi_{\varepsilon,p} = \Phi_{\varepsilon,p} \varepsilon^{-m} \ \eta_t \ \zeta_{\varepsilon,p} \quad , \quad \zeta_{\varepsilon,p} \in C^\infty(B_1(0)) \, , \tag{2.17a}$$

where, uniformly in s, ε, p,

$$\lim_{\varepsilon\to 0} \zeta_{\varepsilon,p}(s) = 1 \, . \tag{2.17b}$$

Thus we obtain from (2.11) and Theorem 2.2 (with $N = m/2+1$), setting $H := L^2(B_1(0), E_p)$,

$$\tilde F^E_{\varepsilon^2 t}(p) = \varepsilon^{-m} \mathrm{tr}_{\Lambda^*_p \otimes H} \left[\Phi^{-1}_{\varepsilon,p} \ \varrho \ \Phi_{\varepsilon,p} \ \eta_t \ \zeta_{\varepsilon,p} \ e^{-tT_{\varepsilon,p}} \right] + O_\varepsilon(t) \, . \tag{2.18}$$

It remains to deal with the involution $\varrho_{\varepsilon,p} := \Phi^{-1}_{\varepsilon,p} \ \varrho \ \Phi_{\varepsilon,p}$. In preparation of the relevant statement we introduce a matrix decomposition of $A \in \mathcal{L}(\Lambda^*_p \otimes K), K$ a Hilbert space, via

$$A(e^\flat_I(p) \otimes x) =: \sum_J e^\flat_J(p) \otimes [A]_{JI}(x), \ [A]_{JI} \in \mathcal{L}(K) \, . \tag{2.19a}$$

Hence, with $\psi_I : x \mapsto e^\flat_I(p) \otimes x$ we have $[A]_{JI} = \psi^*_J A \psi_I$. In particular, we write

$$[A]_{top} := [A]_{\sim\emptyset,\emptyset} \, . \tag{2.19b}$$

Then we have the following variant of the Berezin-Patodi-Lemma.

Theorem 2.4 *For $t, \varepsilon \in (0,1], p \in M$, we have*

$$F^E_{\varepsilon^2 t}(p) = (-2\sqrt{-1})^k \ \mathrm{tr}_H \left[\eta_t \ \zeta_{\varepsilon,p} \ e^{-tT_{\varepsilon,p}} \right]_{top} + O_\varepsilon(t) \, . \tag{2.20}$$

Proof With the notation (2.20) we have

$$\begin{aligned} \varepsilon^{-m}\mathrm{tr}_{\Lambda_p^*\otimes H}\left[\varrho_{\varepsilon,p}\ \eta_t\ \zeta_{\varepsilon,p}e^{-tT_{\varepsilon,p}}\right] &= \varepsilon^{-m}\sum_{I,J}\mathrm{tr}_H\left[\varrho_{\varepsilon,p}\right]_{IJ}\left[\eta_t\ \zeta_{\varepsilon,p}\ e^{-tT_{\varepsilon,p}}\right]_{JI} \\ &=: G_{\varepsilon,t}(p)\ . \end{aligned} \tag{2.21}$$

For any multiindex $I\subset\{1,\ldots,m\}$ we denote by $\sim I$ the (strictly ordered) complementary multiindex and by sgn $(I,\sim I)$ the sign of the permutation $\{I,\sim I\}$ of $\{1,\ldots,m\}$. Then we have

$$\omega_{\mathbb{C}}\cdot e_I(p) = (-1)^{\mu(I)}\ e_I\cdot\omega_{\mathbb{C}}(p)$$

$$= (-1)^{\mu(I)}\sqrt{-1}^k e_I\cdot e_I\cdot\mathrm{sgn}(I,\sim I)\ e_{\sim I}(p)$$

$$= \sqrt{-1}^{k+\mu(I)(\mu(I)-1)}\mathrm{sgn}(I,\sim I)\ e_{\sim I}(p)\ ,$$

hence (2.10b) and the equivariance of α imply (with $*$ the Hodge star operator)

$$\begin{aligned} \varrho_{\varepsilon,p}(e_I^\flat(p)\otimes\omega) &= \varepsilon^{2\mu(I)-m}\sqrt{-1}^{k+\mu(I)(\mu(I)-1)} * e_I^\flat(p)\otimes\omega \\ &=: \varepsilon^{2\mu(I)-m}(\tau_p\otimes I)(e_I^\flat(p)\otimes\omega)\ . \end{aligned} \tag{2.22}$$

Here τ is the involution defining the signature operator on $\Omega(M)$. (2.22) and (2.23) yield

$$\begin{aligned} G_{\varepsilon,t}(p) &= \varepsilon^{-m}\sum_I\mathrm{tr}_H[\varrho_{\varepsilon,p}]_{I,\sim I}[\eta_t\ \zeta_{\varepsilon,p}\ e^{-tT_{\varepsilon,p}}]_{\sim I,I} \\ &= \sum_I\varepsilon^{-2\mu(I)}(-1)^k\sqrt{-1}^{k+\mu(I)(\mu(I)+1)}\mathrm{sgn}(\sim I,I)\mathrm{tr}_H\left[\eta_t\ \zeta_{\varepsilon,p}\ e^{-tT_{\varepsilon,p}}\right]_{\sim I,I} \\ &=: \sum_I G^I_{\varepsilon,t}(p)\ . \end{aligned} \tag{2.23}$$

Now observe that the theorem is proved if we can show that $G^I_{\varepsilon,t}(p)=G^\emptyset_{\varepsilon,t}(p)$ for all I. To see this, we recall from the construction of $\tau_{\varepsilon,p}$ (cf. Section 3) that the operators $B^{\beta\gamma}_{\varepsilon,p}$ in (2.12) act on Λ_p^* only through operators of the form

$$\prod_{i\in I}[\varepsilon^{-1}w(e_i^\flat(p))-\varepsilon\ i(e_i(p))]\ ,\quad I\subset\{1,\ldots,m\}\ ,$$

where i denotes interior multiplication, i.e. they act only through scaled Clifford multiplication. In view of Theorem 2.3, the same is true for $e^{-tT_{\varepsilon,p}}$ and we can write

$$e^{-tT_{\varepsilon,p}} =: \sum_J\prod_{i\in J}[\varepsilon^{-1}w(e_i^\flat(p))-\varepsilon i(e_i(p))]\otimes A^J_{\varepsilon,p}(t)\ .$$

A contribution to (2.24) can arise only if $J = \{1, \ldots, m\}$, and we obtain (with obvious notation)

$$\begin{aligned}
&\left[\eta_t \zeta_{\varepsilon,p} e^{-tT_{\varepsilon,p}}\right]_{\sim I,I} \\
&= (-1)^{\mu(I)} \varepsilon^{2\mu(I)-m} \mathrm{sgn}(\sim I, I) \left[w(e^\flat_{\sim I}(p)) i(e_I(p))\right]_{\sim I,I} \eta_t \zeta_{\varepsilon,p} A^{\sim\emptyset}_{\varepsilon,p}(t) \\
&= (-1)^{\mu(I)+\mu(I)(\mu(I)-1)/2} \mathrm{sgn}(\sim I, I)\, \varepsilon^{2\mu(I)-m} \eta_t \zeta_{\varepsilon,p} A^{\sim\emptyset}_{\varepsilon,p}(t) \ .
\end{aligned}$$

The proof is complete. □

It is natural to expect convergence in (2.21) as $\varepsilon \to 0$. However, η_t involves a factor $t^{-m/2}$ and we are taking traces. The estimate necessary to handle this problem is the analytic core of the paper and will be given in Section 5 below. As a simple consequence, also to be proved in Section 5, we obtain the following result.

Theorem 2.5 *There is a function ϕ with $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ such that*

$$||\eta_t \left[\zeta_{\varepsilon,p}\ e^{-tT_{\varepsilon,p}} - e^{-tT_{0,p}}\right]_{top} ||_{C_1(H))} \leq C(\phi(\varepsilon) + t^{1/2}) \ , \tag{2.24}$$

uniformly in $t \in (0,1]$ and $p \in M$.

The final step in the argument uses again Getzler's calculations based on Mehler's formula; it will be given in Section 4.

Lemma 2.6

$$\begin{aligned}
|(-2\sqrt{-1})^k \ \mathrm{tr}_H \left[\eta_t\ e^{-tT_{0,p}}\right]_{top} \quad &- \quad \hat{A}(M) \wedge \mathrm{ch}\ E(e_1, \ldots, e_m)(p)| \\
&= \quad O(\sqrt{t}) \ .
\end{aligned} \tag{2.25}$$

The preceding results now easily yield the

Proof of Theorem 1.3 Combining (2.21), (2.25), and (2.26) we find

$$\begin{aligned}
&|F^E_{\varepsilon^2 t}(p) - \hat{A}(M) \wedge \mathrm{ch}\ E(e_1, \ldots, e_m)(p)| \\
&\leq C_\varepsilon t + C(\phi(\varepsilon) + \sqrt{t}) \ .
\end{aligned}$$

□

3. Scaling

We now want to prove Lemma 2.1. This calculation is essentially well known but we have to redo it in our special context. So we fix $p \in M$ and a local orthonormal frame, $(e_i)_{i=1}^m$, for $TB_1(p)$ as before. We choose a canonically associated local orthonormal frame (σ_α) for $S|B_1(p)$ and a local orthonormal frame (τ_β) for $E|B_1(p)$; (σ^*_α) denotes the dual frame for $S^*|B_1(p)$. The first important fact we need is the Bochner-Lichnerowicz-Weitzenboeck formula [LM, p.164]:

$$\begin{aligned}&\left(D^{E(p)}\right)^2 = \\ &-\sum_{i=1}^m \left[\nabla^{S\otimes E(p)}_{e_i}\,\nabla^{S\otimes E(p)}_{e_i} - \nabla^{S\otimes E(p)}_{\nabla_{e_i}e_i}\right] + \kappa/4 + 1/2\sum_{i,j=1}^m c(e_i)c(e_j)\otimes R^E_{e_i,e_j}\ ,\end{aligned} \tag{3.1}$$

where $\nabla := \nabla^{TM}$ and κ is the scalar curvature of M. It is apparent from (3.1) that the main computation concerns $\nabla^{S\otimes E(p)}_{e_i}$. Thus we choose $\omega \in C^\infty(B_1(0), E_p)$ and observe that (2.10) implies

$$\begin{aligned}&\Phi_{\varepsilon,p}\Big(e^\flat_I(p)\otimes\omega\Big)\\ &=: \varepsilon^{\mu(I)}\,\alpha_p\widetilde{(e_I(p))}\otimes\widetilde{\omega^\varepsilon}\ ,\end{aligned} \tag{3.2}$$

where a tilde denotes parallel translates i. e.

$$\begin{aligned}\alpha_p\widetilde{(e_I(p))}(\exp_p s) &= P_p^{S\otimes S_p^*}(s)\,(\alpha_p(e_I(p))),\\ \widetilde{\omega^\varepsilon}(\exp_p s) &= P_p^E(s)(\omega(s/\varepsilon))\ ,\end{aligned}$$

are parallel along radial geodesics from p. Hence we obtain

$$\begin{aligned}&\nabla^{S\otimes E(p)}_{e_i}\,\Phi_{\varepsilon,p}(e_I^\flat\otimes\omega)\\ &= \nabla^{S\otimes E(p)}_{e_i}\,(\varepsilon^{\mu(I)}\,\alpha_p\widetilde{(e_I(p))}\otimes\widetilde{\omega^\varepsilon})\\ &= \varepsilon^{\mu(I)}\Big[(\nabla^{S\otimes S_p^*}_{e_i}\,\alpha_p\widetilde{(e_I(p))}\otimes\widetilde{\omega^\varepsilon} + \alpha_p\widetilde{(e_I(p))}\otimes\nabla^E_{e_i}\widetilde{\omega^\varepsilon}\Big]\\ &=: \mathrm{I}\ +\ \mathrm{II}\ .\end{aligned} \tag{3.3}$$

We evaluate I, noting that the spin connection is given by the following formula for *any* section $\sigma \in C^\infty(S|B_1(p))$ which is parallel along radial geodesics from p [LM, p.110]:

$$\nabla^S_e\sigma = \frac{1}{4}\sum_{j,k=1}^m \left\langle\nabla^{TM}_e e_j, e_k\right\rangle c\,(e_j\cdot e_k)\sigma\ ,\ e\in C^\infty(TB_1(p)). \tag{3.4}$$

We will write

$$\Gamma^i_{jk} := \left\langle \nabla^{TM}_{e_i} e_j, e_k \right\rangle , \tag{3.5}$$

and find, with $\beta_p : \Lambda^*_p \to \mathbb{C}\ell_p$ the canonical isomorphism:

$$\begin{aligned}
\mathrm{I} &= \varepsilon^{\mu(I)/4} \sum_{j,k=1}^{m} \Gamma^i_{jk}\, c(e_j \cdot e_k)\, \alpha_p \widetilde{(e_I(p))} \otimes \widetilde{\omega^\varepsilon} \\
&= \varepsilon^{\mu(I)/4} \sum_{j,k=1}^{m} \alpha_p \Big(c(e_j \cdot \widetilde{e_k(p))} e_I(p) \Big) \otimes \Big((\Gamma^i_{jk} \circ \widetilde{\exp_p})^{1/\varepsilon} \omega \Big)^{\varepsilon} \\
&= \varepsilon^{\mu(I)/4} \sum_{j,k=1}^{m} \Big(\alpha_p \beta_p \, (w(e^\flat_j) - i(e_j))(w(e^\flat_k) - i(e_k)) e^\flat_I(p) \Big) \otimes \\
&\qquad \otimes \Big((\Gamma^i_{jk} \circ \widetilde{\exp_p})^{1/\varepsilon} \omega \Big)^{\varepsilon} \\
&= \Phi_{\varepsilon,p} \Big(\frac{1}{4} \sum_{j,k=1}^{m} \Big(\Gamma^i_{jk} \circ \exp_p \Big)^{1/\varepsilon} \Big(\varepsilon^{-1} w(e^\flat_j) - \varepsilon i(e_j) \Big) \Big(\varepsilon^{-1} w(e^\flat_k) - \varepsilon i(e_k) \Big) e^\flat_I(p) \otimes \omega \Big) .
\end{aligned} \tag{3.6}$$

We further evaluate this using the well known Taylor expansion

$$\begin{aligned}
\Gamma^i_{jk}(\exp_p \varepsilon s) &= -\varepsilon/2 \sum_{\ell=1}^{m} s_\ell \Big[\langle R^M(e_i, e_\ell) e_j, e_k \rangle + \hat{\Gamma}^{i\ell}_{jk}(p; \varepsilon s) \Big] \\
&=: -\varepsilon/2 \sum_{\ell=1}^{m} s_\ell \Big(R^M_{i\ell jk}(p) + \hat{\Gamma}^{i\ell}_{jk}(p; \varepsilon s) \Big) ,
\end{aligned} \tag{3.7}$$

with some continuous functions $\hat{\Gamma}^{i\ell}_{jk} \in C(M \times B_1(0))$ satisfying $\hat{\Gamma}^{i\ell}_{jk}(p, 0) = 0$. It follows that

$$\begin{aligned}
\varepsilon\ \mathrm{I} &= \Phi_{\varepsilon,p} \Big[-1/8 \sum_{\ell=1}^{m} s_\ell (R^M_{i\ell jk}(p) + \hat{\Gamma}^{i\ell}_{jk}(p; \varepsilon s)) \cdot \\
&\qquad \cdot (w(e^\flat_j) - \varepsilon^2 i(e_j))(w(e^\flat_k) - \varepsilon^2 i(e_k)) \otimes \omega \Big] .
\end{aligned} \tag{3.8}$$

To evaluate II we observe that

$$\begin{aligned}
\nabla^E_{e_i} \widetilde{\omega^\varepsilon}(q) &= \nabla^E_{e_i} (P^E_p(\omega^\varepsilon \circ \exp_p^{-1}))(q) \\
&= P^E_p(\exp_p^{-1} q)(P^E_p(\exp_p^{-1} q)^{-1} \nabla^E_{e_i} P^E_p(\omega^\varepsilon \circ \exp_p^{-1})(q)) .
\end{aligned}$$

Using the terminology introduced in [Br, Lemma 5.2] we obtain a unitary connection $(P^E)^* \nabla^E =: \tilde{\nabla}^E$ on $C^\infty(B_1(0), E_p)$ with the property that

$$\nabla^E_{e_i} \widetilde{\omega^\varepsilon}(\exp_p s) = P^E_p(s)\, \tilde{\nabla}^E_{(T_s \exp_p)^{-1} e_i}\, \omega^\varepsilon(s) . \tag{3.9}$$

Writing $\tilde{e}_i(s) := (T_s\exp_p)^{-1}(e_i) \in T_s\mathbb{R}^m$, it follows that

$$\begin{aligned} \mathrm{II} \quad &= \varepsilon^{\mu(I)}\widetilde{\alpha_p(e_I(p))} \otimes \nabla^E_{e_i}\,\widetilde{\omega^\varepsilon} \\ &= \Phi_{\varepsilon,p}\Big(e^\flat_I(p) \otimes (\tilde{\nabla}^E_{\tilde{e}_i}\,\omega^\varepsilon)^{1/\varepsilon}\Big)\ . \end{aligned} \tag{3.10}$$

Now we have in the normal coordinates defined by $(e_i)_{i=1}^m$

$$\tilde{e}_i(s) =: \sum_{j=1}^m a_{ij}(s)\frac{\partial}{\partial s_j}\ ,$$

with $a_{ij} \in C^\infty(B_1(0)), a_{ij}(0) = \delta_{ij}$. Thus we obtain, with certain $\hat{\Gamma}^i \in C^\infty(B_1(0), \mathrm{End}\, E_p)$,

$$\Big(\tilde{\nabla}^E_{\tilde{e}_i}\omega^\varepsilon\Big)^{1/\varepsilon}(s) = \varepsilon^{-1}\sum_{j=1}^m a_{ij}(\varepsilon s)\frac{\partial\omega}{\partial s_j}(s) + \hat{\Gamma}^i(\varepsilon s)\omega(s)$$

or

$$\varepsilon\ \mathrm{II}\ = \Phi_{\varepsilon,p}\Big(e^\flat_I(p) \otimes \Big(\sum_{j=1}^m a_{ij}^{1/\varepsilon}\frac{\partial}{\partial s_j} + (\hat{\Gamma}^i)^{1/\varepsilon}\Big)\Big)\ . \tag{3.11}$$

Combining (3.8) and (3.11) we now find

$$\begin{aligned} &\varepsilon\Phi^{-1}_{\varepsilon,p}\,\nabla^{S\otimes E(p)}_{e_i}\,\Phi_{\varepsilon,p} = \sum_{j=1}^m a_{ij}(\varepsilon s)\,\tfrac{\partial}{\partial s_j} + \varepsilon I_{\Lambda^*_p}\otimes\hat{\Gamma}^i(\varepsilon s) \\ &-\tfrac{1}{8}\sum_{\ell=1}^m s_\ell(R^M_{i\ell jk}(p) + \hat{\Gamma}^{i\ell}_{jk}(p;\varepsilon s))(w(e^\flat_j) - \varepsilon^2 i(e_j))(w(e^\flat_k) - \varepsilon^2 i(e_k)) \otimes I_{E_p}\ . \end{aligned} \tag{3.12}$$

The calculation of $\tau_{\varepsilon,p}$ is thus almost complete, in view of (3.1). Only the last term deserves some further attention:

$$\begin{aligned} &\tfrac{1}{2}\sum_{i,j=1}^m c(e_i\cdot e_j)\otimes R^E_{e_i,e_j}\,\Phi_{\varepsilon,p}\,\Big(e^\flat_I(p)\otimes\omega\Big)(\exp_p s) \\ &= \varepsilon^{\mu(I)}\,\tfrac{1}{2}\sum_{i,j=1}^m c(e_i\cdot e_j)\otimes R^E_{e_i,e_j}\,\widetilde{\alpha_p(e_I(p))}\otimes\omega^\varepsilon\widetilde{(\exp_p s)} \\ &= \varepsilon^{\mu(I)}\,\tfrac{1}{2}\sum_{i,j=1}^m P^{S\otimes E(p)}_p(s)\Big(\alpha_p\beta_p((w(e^\flat_i) - i(e_i))(w(e^\flat_j) - i(e_j))\, e^\flat_I(p)\otimes \\ &\quad\otimes P^E_p(s)^{-1}\,R^E_{e_i,e_j}\,P^E_p(s)(\omega(\varepsilon^{-1}s))\Big) \\ &= \Phi_{\varepsilon,p}\Big(\tfrac{1}{2}\sum_{i,j=1}^m(\varepsilon^{-1}w(e^\flat_i(p)) - \varepsilon i(e_i(p)))(\varepsilon^{-1}w(e^\flat_j(p)) - \varepsilon i(e_j(p)))\otimes \\ &\quad\otimes((P^E_p)^{-1}\,R^E_{e_i,e_j}\,P^E_p)^{1/\varepsilon}\omega\Big)(\exp_p s)\ . \end{aligned} \tag{3.13}$$

From (3.1), (3.12), and (3.13) one now easily reads off the proof of Lemma 2.1. ∎

4. Off-diagonal decay of heat traces

We begin this section with some abstract results which are often used in the spectral analysis of elliptic operators, and which will lead to the proof of Theorem 2.2 and Lemma 2.6 ; many special versions are, of course, well known. But the point here is to use as little regularity as possible which forced us to present the treatment below. We work with the resolvent and transfer the results to the heat semigroup via Cauchy integrals.

Let H be a Hilbert space, $\mathcal{D}$ a dense subspace, and $T : \mathcal{D} \to H$ a closed m-sectorial operator in H. By definition [Ka, p.280] this means that the numerical range (and hence spec T) is contained in a sector

$$\{z \in \mathbb{C} \mid |\arg(z-\gamma)| \leq \theta\} , \qquad \gamma \in \mathbb{R},\ 0 \leq \theta < \pi/2 , \tag{4.1}$$

and that we have the resolvent estimate [Ka, p.490]

$$||(T+\lambda+\gamma)^{-1}|| \leq C_{\theta'}|\lambda|^{-1}, \quad |\lambda| \geq 1, |\arg\lambda| \geq \theta' > \theta. \tag{4.2}$$

In what follows we assume for simplicity that

$$\gamma \geq 1 . \tag{4.3}$$

We introduce the resolvent

$$G(z) := (T+z^2)^{-1},\ z \in Z_\delta := \{z \in \mathbb{C} \mid |z| \geq \delta, |\arg z| < (\pi-\theta')/2\} ,$$

where $\theta' > \theta$, and we use 'z^2' in view of our applications to differential operators of second order. The class of closed m-sectorial operators is natural since it contains eg. the Friedrichs extension of second order elliptic differential operators with scalar principal symbol.

To imitate the usual cut-off procedure we introduce a subspace, $\mathcal{S}$, of $\mathcal{L}(H)$ with the following properties (the "cut-off axioms"):

(CO1) For all $\phi \in \mathcal{S},\ \phi(\mathcal{D}) \subset \mathcal{D}$.

(CO2) For all $\phi \in \mathcal{S}$, the operators

$$T^{-1/2}[T,\phi], [T,\phi]T^{-1/2} : \mathcal{D} \to H$$

extend to H by continuity.

(CO3) There is a transitive relation ' $<$' on $\mathcal{S}$ such that $\phi_1 < \phi_2$ implies

$$\phi_1\phi_2 = \phi_2\phi_1 = \phi_1 , \tag{4.4a}$$

and

$$\text{there is } \phi_3 \in \mathcal{S} \text{ with } \phi_1 < \phi_3 < \phi_2 \text{ and} \tag{4.4b}$$

$$\begin{aligned} T^{-1/2}[T,\phi_1](I-\phi_3) &= (I-\phi_3)[T,\phi_1]T^{-1/2} \\ &= T^{-1/2}[T,\phi_2]\phi_3 = \phi_3[T,\phi_2]T^{-1/2} = 0 \ . \end{aligned}$$

(CO4) If $I_H \notin \mathcal{S}$, then for all $\phi \in \mathcal{S}$ there is $\phi' \in \mathcal{S}$ with $\phi < \phi'$.

(CO5) $\mathcal{S}^* = \mathcal{S}$.

We remark that in view of (4.3) and [Ka, Ch V, §3.11] the operators $T^{-\alpha}, \alpha \in (0,1)$, are well defined bounded operators in H. We use again the von Neumann-Schatten classes of compact operators which we denote by $C_p(H), p > 0$, with norm $||\cdot||_p$.

Lemma 4.1 *Assume that a subspace $\mathcal{S}$ of $\mathcal{L}(H)$ satisfies the assumptions* (CO1) *through* (CO5)*, and that in addition*

$$\phi T^{-1} \in C_{p_0}(H), \quad \text{for } \phi \in \mathcal{S} \text{ and some } p_0 > 0 \ .$$

Then, for all $\phi_1, \phi_2 \in \mathcal{S}$ with $\phi_1 < \phi_2$ and all $p > 0$ we have

$$\phi_1 G(z)(I-\phi_2) \in C_p(H), \quad z \in Z_\delta \ .$$

Moreover, we have the norm estimate

$$||\phi_1 G(z)(I-\phi_2)||_p \leq C(\phi_1,\phi_2,p,N)|z|^{-N} \ ,$$

valid for $z \in Z_\delta$ and all $N \in \mathbb{N}$.

Proof Using (4.4b) we determine $\phi_j \in \mathcal{S}, i \leq j \leq 4$, such that $\phi_1 < \phi_3 < \phi_4 < \phi_2$. Since $\phi G(z)u \in \mathcal{D}$ for $u \in H, \phi \in \mathcal{S}, z \in Z_\delta$, we obtain from

$$(T+z^2)\phi_3 G(z)(I-\phi_2) = [T,\phi_3]G(z)(I-\phi_2)$$

and (4.46) the representation

$$\phi_1 G(z)(I-\phi_2) = (\phi_1 T^{-1/2})(T^{1/2}G(z)T^{1/2})(T^{-1/2}[T,\phi_3])\phi_4 G(z)(I-\phi_2) \ . \tag{4.5}$$

Now we observe the easy estimates

$$||G(z)|| \leq C|z|^{-2} \ , \tag{4.6a}$$

$$||T^{1/2}G(z)T^{1/2}|| = ||I - z^2 G(z)|| \leq C \ , \tag{4.6b}$$

$$||\phi G(z)||_p \leq ||\phi T^{-1}||_p + ||\phi T^{-1}\ z^2 G(z)||_p \leq C||\phi T^{-1}||_p \ , \tag{4.6c}$$

where we have used the resolvent equation in (4.6c). Next we combine the integral representation [Ka, p.281]

$$T^{-1/2} = \frac{1}{2\pi i} \int\limits_{\Gamma} \xi^{-1/2} G((-\xi)^{1/2}) d\xi \ , \tag{4.7}$$

where Γ is (e.g.) the wedge $|\arg z| = \theta'$, $\theta' > \theta$, traversed upward, with the interpolation inequality

$$||A||_{p'} \leq ||A||^{1-p/p'} ||A||_p^{p/p'} \ , \tag{4.8}$$

valid for $A \in C_p(H)$ and all $p' > p > 0$. Then we deduce that for $p_1 > 2p_0$

$$||\phi G((-\xi)^{1/2})||_{p_1} \leq C|\xi|^{p_0/p_1 - 1} ||\phi T^{-1}||_{p_0}^{p_0/p_1} \leq C|\xi|^{p_0/p_1-1} \ ,$$

which gives with (4.7)

$$\phi T^{-1/2} \in C_{p_1}(H) \ . \tag{4.9}$$

Using (4.9) (with $p_1 = 3p_0$), (4.6b), (CO2), and (4.6a) in (4.5) we derive

$$||\phi_1 G(z)(I - \phi_2)||_{3p_0} \leq C(\phi_1, \phi_2)|z|^{-2}, z \in Z_\delta \ . \tag{4.10a}$$

Next we use the Hölder inequality for C_p-norms in (4.5) to prove by induction on $L \in \mathbb{N}$

$$||\phi_1 G(z)(I - \phi_2)||_{3p_0/L} \leq C(\phi_1, \phi_2, L)|z|^{-2} \ ,$$

i.e. for $p > 0$

$$||\phi_1 G(z)(I - \phi_2)||_p \leq C(\phi_1, \phi_2, p)|z|^{-2}, \ z \in Z_\delta \ . \tag{4.10b}$$

Finally, we rewrite (4.5) as

$$\phi_1 G(z)(I - \phi_2) = \phi_1 G(z) T^{1/2} (T^{-1/2}[T, \phi_3]) \phi_4 G(z)(I - \phi_2)$$

and use the estimate

$$||G(z)T^{1/2}|| \leq C|z|^{-1} \ , \ z \in Z_\delta \ , \tag{4.10c}$$

which follows from (4.7) and the resolvent equation by a straightforward estimate. Then another induction will prove the assertion. □

By symmetry, the same reasoning also proves the following statement.

Corollary 4.2 *Assume that all the assumptions of Lemma* 4.1 *hold with the only change that now*

$$T^{-1}\phi \in C_{p_0}(H) \ ,$$

for $\phi \in \mathcal{S}$ *and some* $p_0 > 0$. *Then, for* $\phi_1, \phi_2 \in \mathcal{S}$ *with* $\phi_1 < \phi_2$ *and all* $p > 0$ *we have*

$$(I - \phi_2)G(z)\phi_1 \in C_p(H), \ z \in Z_\delta \ ,$$

with norm estimate

$$||(I - \phi_2)G(z)\phi_1||_p \leq C(\phi_1, \phi_2, p, N)|z|^{-N} \ ,$$

$z \in Z_\delta, N \in \mathbb{N}$.

We will actually need the consequences of Lemma 4.1 for the heat operator,

$$e^{-tT} = \frac{1}{2\pi i} \int_{\Gamma} e^{-t\xi}(T-\xi)^{-1} d\xi \ . \tag{4.11}$$

Here we can choose Γ as in (4.7) but, by holomorphy, we can replace it by Γ_t, the path obtained from Γ by traversing the circle $|\xi| = t^{-1}$ outside Γ, and then Γ for $|\xi| \geq t^{-1}$. This gives

Lemma 4.3 *Under the assumptions of Lemma* 4.1 *or Corollary* 4.2 *we have for* $t \in (0,1]$ *and* $p > 0$

$$||\phi_1 e^{-tT}(I-\phi_2)||_p \leq C(\phi_1, \phi_2, p, N)t^N \tag{4.12a}$$

and

$$||(I-\phi_2)e^{-tT}\phi_1||_p \leq C(\phi_1, \phi_2, p, N)t^N \ , \tag{4.12b}$$

respectively.

Now we turn to the class of operators which will be the main object of study in the remainder of this section, and to which we will apply the abstract results above. These are certain second order elliptic operators on Riemannian manifolds which are sectorial. Thus, consider a (smooth) connected Riemannian manifold M of dimension m, a (smooth) hermitian bundle F of rank k over M, and a differential operator, τ, on $C_0^\infty(E)$. We assume that, in a bundle chart φ with local coordinates $s = (s_1, \dots s_m) \in B_1(0)$, we have

$$\tau_\varphi := \varphi \circ \tau \circ \varphi^{-1} = -\sum_{|\gamma| \leq 2} \partial_s^\gamma B_\varphi^\gamma, \tag{4.13}$$

where the coefficients B_φ^γ satisfy the assumptions (2.13a) and (2.13b").

These conditions are independent of the choice of φ. The following facts are proved by standard arguments.

Lemma 4.4 *Let* τ *satisfy the conditions* (4.13).
1) τ *has a formal adjoint on* $C_0^\infty(E), \tau^t$, *which is also elliptic of second order on* $C_0^\infty(E)$ *and satisfies* (4.13).
2) *For* $\phi \in C_0^\infty(M)$ *and* $u \in C_0^\infty(E)$ *we have*

$$||\phi u||_{H^1_{loc}(E)} \leq C_3(\phi)(\mathrm{Re}\,(\tau u, u) + ||u||^2) \ , \tag{4.14}$$

where C_3 *also depends on a choice of norm in* $H^1(E|\,\mathrm{supp}\ \phi)$.
3) *For* $\phi \in C^\infty(M)$ *with* supp $d\phi$ *compact and* $u \in C_0^\infty(E)$ *we have*

$$|(\tau\phi u, \phi u)| \leq C(\phi)(\mathrm{Re}\,(\tau u, u) + ||u||^2) \ . \tag{4.15}$$

With τ we associate the following sesquilinear forms on $C_0^\infty(E)$:

$$\mathfrak{t}[u,v] := (\tau u, v),\ \mathfrak{t}[u] := \mathfrak{t}[u,u]\ , \tag{4.16a}$$

$$\mathfrak{h}[u,v] := \left(\frac{1}{2}(\tau+\tau^t)u, v\right) =: (\mathcal{R}\tau u, v),\ \mathfrak{h}[u] := \mathfrak{h}[u,u]\ , \tag{4.16b}$$

$$\mathfrak{k}[u,v] := \left(\frac{1}{2i}(\tau-\tau^t)u, v\right) =: (\mathcal{I}\tau u, v),\ \mathfrak{k}[u] := \mathfrak{k}[u,u]\ , \tag{4.16c}$$

such that

$$\mathfrak{t}[u,v] = \mathfrak{h}[u,v] + i\mathfrak{k}[u,v]\ .$$

Observe that $\mathcal{I}\tau$ is a differential operator of first order.

To ensure sectoriality of $\mathfrak{t}$ (in the sense of [Ka, p.310]) we require, in addition to the conditions (4.13), the estimates

$$\mathfrak{h}[u] \geq \gamma||u||^2\ , \tag{4.17a}$$

$$|\mathfrak{k}[u]| \leq (\ \mathrm{tg}\ \theta)(\mathfrak{h}[u] - \gamma||u||^2),\ \theta \in [0,\pi/4)\ , \tag{4.17b}$$

for all $u \in C_0^\infty(E)$. Without loss of generality we may assume that $\gamma \geq 1$. Then $\mathfrak{h}$ is also sectorial and symmetric. So we can form the Friedrichs extension of both $\mathfrak{t}$ and $\mathfrak{h}$, to be denoted by T and H, respectively (cf. [Ka, p.325]). The forms $\mathfrak{t}$ and $\mathfrak{h}$ are closable, with closures $\tilde{\mathfrak{t}}$ and $\tilde{\mathfrak{h}}$, and it follows from (4.17b) that

$$\mathcal{D}(\tilde{\mathfrak{t}}) = \mathcal{D}(\tilde{\mathfrak{h}})\ .$$

Note that the estimates (4.14) and (4.15) extend to $\mathcal{D}(\tilde{h})$ by continuity.

In the setting just described we now have to verify the assumptions (CO1) - (CO5) for T, the Friedrichs extension of τ in $L^2(F)$. We put

$$\mathcal{S} := C_0^\infty(M)\ , \tag{4.18}$$

acting on $L^2(F)$ by multiplication.

For (CO1), we observe that by [Ka, p.322] we have

$$\mathcal{D}(T) = \{u \in \mathcal{D}(\tilde{\mathfrak{t}})\ |\ |(u,\tau^t v)| \leq C_u||v|| \text{ for all } v \in C_o^\infty(F)\}\ . \tag{4.19}$$

For $u \in \mathcal{D}(T)$ we can thus find a sequence $(u_n) \subset C_0^\infty(F)$ such that $u_n \to u$ in $L^2(F)$ and $\lim_{n,m\to\infty} \mathfrak{t}[u_n - u_m] = 0$. For $\phi \in \mathcal{S}$ we obtain $\phi u_n \to \phi u$ in $L^2(F)$ and, from (4.15),

$$\begin{aligned}
|\mathfrak{t}[\phi(u_n-u_m)]| \ &=\ |(\tau\phi(u_n-u_m),\ \phi(u_n-u_m))| \\
&\leq C(\phi)(\mathfrak{h}[u_n-u_m]\ +\ ||u_n-u_m||^2_{L^2(F)}) \\
&\leq C(\phi)(|\mathfrak{t}[u_n-u_m]|\ +\ ||u_n-u_m||^2_{L^2(F)})\ \to 0, \quad n,m\to\infty\ .
\end{aligned}$$

Hence $\phi u \in \mathcal{D}(\tilde{\mathfrak{t}})$. Next, for $v \in C_0^\infty(F)$ we get

$$(\phi u, \tau^t v) = (u, [\phi, \tau^t]v) + (Tu, v) ;$$

since $[\phi, \tau^t]$ is a compactly supported differential operator of order at most one on $C^\infty(F)$, we obtain from (4.14) the estimate

$$|(\phi u, \tau^t v)| \leq C_u ||v|| ,$$

as desired.

(CO2) we prove first for $H^{-1/2}$ in place of $T^{-1/2}$: for $u \in \mathcal{D}(\tilde{\mathfrak{h}}) = \mathcal{D}(H^{-1/2})$ we have in view of (4.14)

$$||[T, \phi]u||^2 \leq \; C||u||^2_{H^1(F|\operatorname{supp}\phi)} \leq C(||H^{1/2}u||^2 + ||u||^2) .$$

Setting $u := H^{-1/2}v$ gives the boundedness of $[T, \phi]H^{-1/2}$; applying the same argument to T^* gives the boundedness of $H^{-1/2}[T, \phi]$. To relate $H^{-1/2}$ to $T^{-1/2}$ we choose $\psi \in C_0^\infty(M)$ with $\psi = 1$ in a neigborhood of supp ϕ and note that $\psi\mathcal{D}(T) \subset \mathcal{D}(H)$, in view of (4.14) again and the identity

$$(\psi u, Hv) = (T\psi u + i\mathcal{I}\tau\phi u, v) ,$$

valid for $u \in \mathcal{D}(T)$ and $v \in C_0^\infty(F)$. Then we derive the representation

$$\begin{aligned}\psi(T-\lambda)^{-1} \; &= (H-\lambda)^{-1}(\psi + ([\tau, \psi] - i\mathcal{I}\tau\psi)(T-\lambda)^{-1}) \\ &=: (H-\lambda)^{-1}(\psi + A(\lambda)) .\end{aligned} \tag{4.20}$$

The arguments given above show that

$$||H^{-1/2}A(\lambda)|| \;\leq\; C|\lambda|^{-1}, \quad |\lambda| \geq 1 \text{ and } Re\lambda \leq 0 . \tag{4.21}$$

Using the representation (4.7) we deduce from (4.20) and (4.21) that

$$\psi T^{-1/2} =: H^{-1/2}B ,$$

for some bounded operator B. This gives the boundedness of $[T, \phi]T^{-1/2} =$ $[T, \phi]\psi T^{-1/2}$; applying the argument to T^* gives the boundedness of $T^{-1/2}[T, \phi]$.

The remaining properties, (CO3) - (CO5), are easy to see if we make the obvious choice for the relation ‘ $<$’:

$$\phi_1 < \phi_2 \text{ iff } \phi_2 = 1 \text{ in a neighborhood of } \operatorname{supp}\phi_1 . \tag{4.22}$$

To apply Lemma 4.3 to T, it only remains to show that $\phi T^{-1} \in C_p(L^2(F))$ for $\phi \in C_0^\infty(M)$ and some $p > 0$. So choose a compact manifold with boundary,

$M_1 \subset M$, such that supp $\phi \subset \overset{\circ}{M}_1$. Denote by H_{M_1} the Friedrichs extension of $\mathcal{R}\tau$ in $L^2(F|\overset{\circ}{M}_1)$; then it is well known that

$$H_{M_1}^{-1} \in C_p(L^2(F|\overset{\circ}{M}_1)) \text{ for all } p > m/2 \ .$$

But ϕT^{-1} maps into $\mathcal{D}(H_{M_1})$ and

$$H_{M_1}\phi T^{-1} = [T,\phi]T^{-1} + \phi - i\mathcal{I}\tau\phi T^{-1}$$

is bounded in view of the arguments proving (CO2).

We have proved:

Lemma 4.5 *If T is the Friedrichs extension of a second order differential operator with the properties* (4.13) *and* (4.17) *then the estimates* (4.12) *hold, for all $\phi_1, \phi_2 \in C_0^\infty(M)$ with $\phi_2 = 1$ in a neighborhood of* supp ϕ_1.

With these preparations we are ready to prove Theorem 2.2 and Lemma 2.6.

Proof of Theorem 2.2 Choose $\chi_1, \chi_2 \in C_0^\infty(B_{1/2}(0))$ with $\chi_1 > \chi_2$. By elliptic regularity, we have for $u \in \mathcal{D}(T_{\varepsilon,p})$

$$\chi_1 P_{\varepsilon,p}(t)\chi_2 u \in \mathcal{D}(T_{\varepsilon,p}),$$

and an easy calculation gives

$$(\partial_t + T_{\varepsilon,p})\chi_1 P_{\varepsilon,p}(t)\chi_2 u = [T_{\varepsilon,p},\chi_1]P_{\varepsilon,p}(t)\chi_2 u =: v(t), \quad t > 0, \tag{4.23a}$$

$$\lim_{t\to 0+} \chi_1 P_{\varepsilon,p}\chi_2 u = \chi_2 u. \tag{4.23b}$$

Now, for fixed ε and p we may write, in view of (CO2) above,

$$v(t) = A[(D^{E(p)})^2, \tilde\chi_1]e^{-\varepsilon^2 t(D^{E(p)})^2}((D^{E(p)})^2+1)^{-1}\tilde u), \tag{4.24}$$

for some bounded operator $A : L^2(S\otimes E(p)|B_\varepsilon(p)) \to L^2(B_1(0), \Lambda_p^*\otimes E_p)$, some function $\tilde\chi_1 \in C_0^\infty(B_\varepsilon(p))$, and some $\tilde u \in L^2(S\otimes E(p))$. The spectral theorem then readily shows that v is Hölder continuous in $[0,1]$ with exponent $1/2$; thus we can apply [Ka, Thm. IX, 1.27] to conclude the representation

$$\begin{aligned}\chi_1 P_{\varepsilon,p}(t)\chi_2 &= e^{-tT_{\varepsilon,p}}\chi_2 + \int_0^t e^{-(t-u)T_{\varepsilon,p}}[T_{\varepsilon,p},\chi_1]P_{\varepsilon,p}(u)\chi_2 du \\ &= e^{-tT_{\varepsilon,p}}\chi_2 + \int_0^t e^{-(t-u)T_{\varepsilon,p}}{T_{\varepsilon,p}}^{1/2}({T_{\varepsilon,p}}^{-1/2}[T_{\varepsilon,p},\chi_1])(I-\chi_3)P_{\varepsilon,p}(u)\chi_2 du,\end{aligned}$$

for any χ_2 with $\chi_1 > \chi_3 > \chi_2$ (cf. (CO3)).

Now we deduce from the Cauchy-Dunford representation analogous to (4.11) the estimate

$$\|e^{-(t-u)T_{\varepsilon,p}}T_{\varepsilon,p}{}^{1/2}\| \leq c(t-u)^{-1/2},$$

whereas the second factor in the integral is bounded by (CO2). The third factor can be written as

$$(I-\chi_3)P_{\varepsilon,p}(u)\chi_2 = \Phi_{\varepsilon,p}{}^{-1}((I-\tilde{\chi}_{3,\varepsilon})e^{-\varepsilon^2 u(D^{E(p)})^2}\tilde{\chi}_{2,\varepsilon}\Phi_{\varepsilon,p} =: \Phi_{\varepsilon,p}{}^{-1}Q_{\varepsilon,p}\Phi_{\varepsilon,p},$$

with $\tilde{\chi}_{2,\varepsilon} < \tilde{\chi}_{3,\varepsilon}$ in $C^\infty(M)$. Since the self–adjoint operator $(D^{E(p)})^2$ equals its Friedrichs extension from $C^\infty(S \otimes E(p))$ and satisfies (4.13) and also (4.17), by self–adjointness, we get from Lemma 4.5

$$\|(I-\chi_3)P_{\varepsilon,p}(u)\chi_2\|_{C_1} \leq \|\Phi_{\varepsilon,p}{}^{-1}\|\|\Phi_{\varepsilon,p}\|C(\varepsilon,p,N)u^N.$$

The proof is complete. □

Proof of Lemma 2.6 This proof is achieved by exactly the same arguments as above for Theorem 2.2, with only a few modifications. In $L^2(\mathbb{R}^m, \Lambda_p^* \otimes E_p)$ we introduce the integral operator, $P(t)$, with kernel

$$P(t;s_1,s_2) := (4\pi t)^{-m/2}P_1(\Omega(p)t)\ P_2(\Omega^E(p)t)\ P_3(\Omega(p)t;s_1,s_2), \tag{4.25}$$

$$P_1(t) := \left(\det\frac{2\Omega(p)t}{\sinh 2\Omega(p)t}\right)^{1/2}, \tag{4.26}$$

$$P_2(t) := \exp\left(-\frac{1}{4t} < \frac{2\Omega(p)t}{\tanh 2\Omega(p)t}(s_1-s_2), s_1-s_2 > +4it < \Omega(p)s_1, s_2 >)\right), \tag{4.27}$$

$$P_3(t;s_1,s_2) := \exp(-\Omega^E(p)t), \tag{4.28}$$

where $\Omega = (\Omega_{ij})$ and Ω^E are defined in Lemma 2.1. Then, for $\chi_1 \in C_0^\infty(B_{1/2}(0))$, it is easy to see that $\chi_1 P(t)$ maps into $\mathcal{D}(T_{0,p})$. Moreover, it follows from [BeGeVe, Ho] that $P(t)$ solves the heat equation associated with $\tau_{0,p}$ so we derive the analogue of (4.23).

The explicit formula (4.25) ensures Hölder continuity and off–diagonal decay so we arrive at

$$\|\chi_1(e^{-tT_{0,p}} - P(t))\chi_2\|_{C_1} \leq C(N,\chi_1,\chi_2)t^N,$$

for all $N \in \mathbb{N}$.

The proof of the lemma is completed recalling (2.20) and observing that $[P(t;s,s)]_{\mathrm{top}}$ is actually independent of t, hence can be evaluated at $t = (2\pi\sqrt{-1})^{-1}$. □

5. Estimates for $e^{-tT_{\varepsilon,p}}$

It remains to prove Theorems 2.3 and 2.5 which is the goal of this final section. The main point is to find a suitable representation of $e^{-tT_{\varepsilon,p}}$ which allows to read off the uniform estimates (and continuity properties) required in the proof of Theorem 2.5. This is achieved by comparing $T_{\varepsilon,p}$ with the Friedrichs extension, $T^0_{\varepsilon,p}$, of the principal part, $\tau^0_{\varepsilon,p}$ (cf. (2.15)). The advantage of this operator over $T_{0,p}$ lies in the fact that it is diagonal with respect to Λ^*_p; the corresponding Neumann series has enough structure to carry through the estimates we need.

Thus we have to consider the operators $T_{\varepsilon,p}$ and $T^0_{\varepsilon,p}$ in the coordinate Hilbert space $\mathcal{H} = L^2(B_{2/3}(0), \Lambda^*_p \otimes E_p)$, the Friedrichs extension of the differential operators $\tau_{\varepsilon,p}$ and $\tau^0_{\varepsilon,p}$ defined in (2.12) and (2.15), respectively. It follows from standard elliptic theory and the definition of the Friedrichs extension (4.19) that

$$\mathcal{D}(T_{\varepsilon,p}) = \mathcal{D}(T^0_{\varepsilon,p}) = H^2(B_{2/3}(0), \Lambda^*_p \otimes E_p) \cap H^1_0(B_{2/3}(0), \Lambda^*_p \otimes E_p). \tag{5.1}$$

Thus we obtain for $u \in \mathcal{H}$

$$(\partial_t + T_{\varepsilon,p})e^{-tT^0_{\varepsilon,p}}u = -\sum_{\beta,\gamma,|\gamma|\le 1} M_\beta \partial_s^\gamma A^{\beta,\gamma}_{\varepsilon,p} e^{-tT^0_{\varepsilon,p}}u, \tag{5.2a}$$

$$\lim_{t\to 0} e^{-tT^0_{\varepsilon,p}}u = u. \tag{5.2b}$$

Since $T^0_{\varepsilon,p}$ generates a holomorphic semigroup, the standard a priori inequality for $u \in \mathcal{D}(T^0_{\varepsilon,p})$ (cf. Lemma 4.4) gives for $|\gamma| \le 1$

$$\|\partial_s^\gamma e^{-tT^0_{\varepsilon,p}}\|_{\mathcal{H}} \le Ct^{-|\gamma|/2}, \tag{5.3}$$

(with C independent of ε and p) such that we obtain, as in the proof of Theorem 2.2,

$$e^{-tT_{\varepsilon,p}} = e^{-tT^0_{\varepsilon,p}} + \sum_{\beta,|\gamma|\le 1} \int_0^t e^{-(t-u)T_{\varepsilon,p}} M_\beta \partial_s^\gamma A^{\beta\gamma}_{\varepsilon,p} e^{-uT^0_{\varepsilon,p}} du. \tag{5.4}$$

This leads us to define, as in Theorem 2.3,

$$\begin{aligned} R^0_{\varepsilon,p}(t) &= e^{-tT^0_{\varepsilon,p}}, && (5.5a) \\ R^{\nu+1}_{\varepsilon,p}(t) &= \sum_{\beta,\gamma,|\gamma|\le 1} \int_0^t R^\nu_{\varepsilon,p}(t-u) M_\beta \partial_s^\gamma A^{\beta\gamma}_{\varepsilon,p} e^{-tT^0_{\varepsilon,p}} du, && (5.5b) \end{aligned}$$

one expects that the resulting Neumann series converges, as expressed in Theorem 2.3.

Proof of Theorem 2.3 The theorem follows if we prove that for positive constants C_0, C_1, independent of ε and p, we have the estimate

$$\|R^\nu_{\varepsilon,p}(t)\| \le C_0 C_1^\nu (\nu!)^{-1/2} t^{\nu/2}, \quad \nu \ge 0, 0 < t \le 1. \tag{5.5}$$

This is obvious for $\nu=0$, from (5.5a) and (5.3).

If the estimate holds for some $\nu\geq 0$ then (5.5b) and (5.3) give

$$\begin{aligned}\|R^{\nu+1}_{\varepsilon,p}(t)\|_{\mathcal{H}} &\leq C_2\int_0^t \|R^{\nu}_{\varepsilon,p}(t-u)\|_{\mathcal{H}}u^{-1/2}du\\ &\leq C_2C_0C_1^{\nu}(\nu!)^{-1/2}\int_0^t (t-u)^{\nu/2}u^{-1/2}du\\ &\leq C_0C_2C_1^{\nu}(\nu!)^{-1/2}t^{(\nu+1)/2}\frac{\Gamma(\nu/2+1)\Gamma(1/2)}{\Gamma(\nu/2+3/2)}\\ &\leq C_0C_1^{\nu}C_2C_3((\nu+1)!)^{-1/2}t^{(\nu+1)/2},\end{aligned}$$

where we have used the asymptotics of the Γ–function. The assertion follows with $C_1:=C_2C_3$ and $\|e^{-tT^0_{\varepsilon,p}}\|_{\mathcal{H}}=:C_0$. □

We will need trace estimates, however, to achieve the proof of Theorem 2.5. In what follows we work with the von Neumann Schatten classes of compact operators in $\mathcal{H}$; we denote them, as before, by $C_q(\mathcal{H})$ and write $\|\cdot\|_q:=\|\cdot\|_{C_q(\mathcal{H})}$, for $q>0$. In particular, we use $\|\cdot\|_\infty:=\|\cdot\|_{\mathcal{H}}$.

Lemma 5.1 *There are positive constants C_4, C_5, independent of ε and p, such that*

$$\|R^{\nu}_{\varepsilon,p}(t)\|_1\leq C_4C_5^{\nu}((\nu+1)!)^{-1/2}t^{(\nu-m)/2}. \tag{5.6}$$

Proof For $\nu=0$, we recall that $T^0_{\varepsilon,p}$ is actually scalar and hence satisfies the kernel estimate, for $s_1,s_2\in B_{2/3}(0)$,

$$|e^{-tT^0_{\varepsilon,p}}(s_1,s_2)|\leq Ct^{-m/2}e^{-C'|s_1-s_2|^2/t}, \tag{5.7}$$

again with C, C' independent of ε and p (cf. eg. [Da, p.89]). This implies the estimates

$$\|e^{-tT^0_{\varepsilon,p}}\|_2 \leq Ct^{-m/2}, \tag{5.8a}$$
$$\|e^{-tT^0_{\varepsilon,p}}\|_1 \leq \|e^{-t/2T^0_{\varepsilon,p}}\|_2^2\leq Ct^{-m/2}, \tag{5.8b}$$

and with (5.3), for $|\gamma|\leq 1$,

$$\|\partial_s^{\gamma}e^{-tT^0_{\varepsilon,p}}\|_1\leq\|\partial_s^{\gamma}e^{-t/2T^0_{\varepsilon,p}}\|_\infty\|e^{-t/2T^0_{\varepsilon,p}}\|_1\leq Ct^{-(m+|\gamma|)/2}. \tag{5.8c}$$

Hence the case $\nu=0$ is settled by (5.8b).

If the assertion is true for some $\nu\geq 0$ we use (5.5b) again, now splitting the integral in order to avoid nonintegrable singularities:

$$\|R^{\nu+1}_{\varepsilon,p}(t)\| \leq C_2\Big\{\int_0^{t/2}\|R^{\nu}_{\varepsilon,p}(t-u)\|_1u^{-|\gamma|/2}du$$

$$
\begin{aligned}
& + \int_{t/2}^{t} C_0 C_1^{\nu} (\nu!)^{-1/2} (t-u)^{\nu/2} \| \partial_s^{\gamma} e^{-uT^0_{\varepsilon,p}} \|_1 du \Big\} \\
\leq\; & C_2 (\nu!)^{-1/2} \Big\{ C_4 C_5^{\nu} t^{(\nu-m+2-|\gamma|)/2} 2^k B(\nu/2+1, 1/2) \\
& + C_0 C_1^{\nu} t^{(\nu-m+2-|\gamma|)/2} 2^k B(\nu/2+1,1/2) \Big\} \\
\leq\; & t^{(\nu+1-m)/2} ((\nu+1)!)^{-1/2} C_2 2^k [C_3 C_4 C_5^{\nu} + C_0 C_3 C_1^{\nu}] \\
\leq\; & C_4 C_5^{\nu+1} ((\nu+1)!)^{-1/2} t^{(\nu+1-m)/2},
\end{aligned}
$$

if C_4, C_5 is chosen appropriately. $\square$

From Lemma 5.1, we derive the interesting consequence that

$$
\begin{aligned}
\Big\| \eta_t \sum_{\nu=N}^{\infty} R^{\nu}_{\varepsilon,p}(t) \Big\|_1 \;&\leq\; C_4 t^{-m} \sum_{\nu \geq N} \frac{(C_5^2 t)^{\nu/2}}{(\nu!)^{1/2}} \|\eta\|_{L^\infty} && (5.9)\\
&\leq\; C t^{1/2}, && (5.10)
\end{aligned}
$$

if $N > 2m$. Consequently, the proof of Theorem 2.5 depends on properties of only finitely many R^{ν}. To examine those, we will now heavily invoke the weight structure of $\tau_{\varepsilon,p}$ as expressed in (2.13c). To use it, we will have to rewrite the basic recursion (5.5b) in a more complicated fashion. It has to incorporate the multiplication operators M_α, a cut–off in the t–variable, and also 'retarded arguments'. More precisely, we propose to estimate inductively the q–norms of the operator family

$$
\tilde{\eta}_t R^{\nu}_{\varepsilon,p}(u) M_\alpha, \quad 0 < u \leq t \leq 1, \varepsilon \in (0,1], p \in M, \alpha \in \mathbb{Z}_+^m, \nu \in \mathbb{Z}_+, \tag{5.11}
$$

where we have written $\eta_t =: t^{-m/2} \tilde{\eta}_t$, i.e. $\tilde{\eta}_t(x) = \eta(x/\sqrt{t})$.

We will prove:

Theorem 5.2 *There are positive constants C_ν, independent of ε and p, such that*

$$
\| [\tilde{\eta}_t R^{\nu}_{\varepsilon,p}(u) M_\alpha \partial_s^{\alpha'}]_{IJ} \|_q \leq C_{\nu\alpha q} t^{(|\mu(I)-\mu(J)|+|\alpha|)/2} (t/u)^{m/2q} u^{-|\alpha'|/2}, \tag{5.12}
$$

for $0 < u \leq t \leq 1, \varepsilon \in (0,1], p \in M, \alpha, \alpha' \in Z_+^m, |\alpha'| \leq 1, I, J \subset \{1, \ldots, m\}$, and $0 < q \leq \infty$.

Let us remark that (5.11) does not follow from (5.5) in the case $u = t, \alpha = 0, q = \infty$. Moreover, with Lemma 5.1 we have the following interesting estimate for the heat kernel.

Corollary 5.3 *For $0 < u \leq t \leq 1, \alpha, \alpha' \in \mathbb{Z}_+^m, |\alpha'| \leq 1$, we have*

$$
\| [\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}} M_\alpha \partial_s^{\alpha'}]_{IJ} \|_1 \leq C_\alpha t^{(|\mu(I)-\mu(J)+|\alpha|)/2} (t/u)^{m/2} u^{-|\alpha'|/2}, \tag{5.13}
$$

and, in particular,

$$
\| [\eta_t e^{-tT_{\varepsilon,p}}]_{\mathrm{top}} \|_1 \leq C. \tag{5.14}
$$

For the proof of Theorem 5.2 we need several lemmas to which we turn now.

The presence of the factor M_α on the right of (5.10) creates the operators $\partial_s^\gamma e^{-tT^0_{\varepsilon,p}} M_\alpha$ in the recursion, and we are forced to move M_α to the left. Thus we want to write (suppressing the dependence on ε, p in the right hand side for simplicity)

$$e^{-tT^0_{\varepsilon,p}} M_\alpha =: \sum_{\delta \leq \alpha} M_\delta S^\alpha_\delta(t), \tag{5.15}$$

such that $S^0_0(t) = e^{-tT_{\varepsilon,p}} = S^\alpha_\alpha(t)$. Clearly, $S^\alpha_\delta(t)$ is an integral operator with kernel

$$(s_1 - s_2)^{\alpha - \delta} e^{-tT^0_{\varepsilon,p}}(s_1, s_2) \tag{5.16}$$

which allows to estimate the C_q–norms for $2 \leq q \leq \infty$, in view of (5.7). To incorporate derivatives, we argue as often before to derive the representation

$$\begin{aligned} M_\alpha e^{-tT^0_{\varepsilon,p}} &= e^{-tT^0_{\varepsilon,p}} M_\alpha + \int_0^t e^{-(t-u)T^0_{\varepsilon,p}} [\tau^0_{\varepsilon,p}, M_\alpha] e^{-uT^0_{\varepsilon,p}} du \\ &= e^{-tT^0_{\varepsilon,p}} M_\alpha - \sum_{\beta, |\gamma| = |\gamma' + \gamma''| \leq 2, |\gamma''| \geq 1} \int_0^t e^{-(t-u)T^0_{\varepsilon,p}} M_{\alpha - \gamma''} C_{\gamma\gamma'} \partial_s^{\gamma'} A^{\beta\gamma}_{\varepsilon,p} e^{-uT^0_{\varepsilon,p}}\, du, \end{aligned}$$

with certain constants $C_{\gamma\gamma'}$, or for $\delta < \alpha$

$$S^\alpha_\delta(t) = \sum_{\beta, \gamma = \gamma' + \gamma'', |\gamma''| \geq 1} \int_0^t S^{\alpha + \gamma''}_\delta (t - u) C_{\gamma\gamma''} \partial_s^{\gamma''} A^{\beta\gamma}_{\varepsilon,p} e^{-uT^0_{\varepsilon,p}} du. \tag{5.17}$$

With this notation established, we now obtain from (5.5b) the following recursion:

$$\begin{aligned} & \left[\tilde\eta_t R^{\nu+1}_{\varepsilon,p}(u) M_\alpha \partial_s^{\alpha'}\right]_{IJ} = \\ &= \sum_{\beta, |\gamma| \leq 1, \delta, K} \int_0^u [\tilde\eta_t R^\nu_{\varepsilon,p}(u - v) M_\beta \partial_s^\gamma M_\delta]_{IK} [A^{\beta\gamma}_{\varepsilon,p} S^\alpha_\delta(v) \partial_s^{\alpha'}]_{KJ} dv \\ &= \sum_{\beta, |\gamma| \leq 1, \delta, K} \int_0^u [\tilde\eta_t R^\nu_{\varepsilon,p}(u - v)(M_{\beta+\delta} \partial_s^\gamma + \qquad (5.18) \\ & \qquad < \gamma, \delta > M_{\beta + \delta - \gamma})]_{IK} [A^{\beta\gamma}_{\varepsilon,p} S^\alpha_\delta(v) \partial_s^{\alpha'}]_{KJ} dv. \qquad (5.19) \end{aligned}$$

Thus we have to establish estimates on $S^\alpha_\delta(v) \partial_s^{\alpha'}$ first.

Lemma 5.4 *We have the estimates, uniformly in ε and p,*

$$\begin{aligned} \|S^\alpha_\delta(v) \partial_s^{\alpha'}\|_\infty &\leq C_\alpha v^{(|\alpha - \delta| - |\alpha'|)/2}, & (5.20) \\ \|\tilde\eta_t S^\alpha_\delta(v) \partial_s^{\alpha'}\|_q &\leq C_{\alpha q} t^{|\alpha - \delta|/2} (t/v)^{m/2q} v^{-|\alpha|/2}, & (5.21) \\ \|\tilde\eta_t S^\alpha_\delta(v) \partial_s^{\alpha'} (1 - \tilde\eta^1_t)\|_q &\leq C_{\alpha q} t^{|\alpha - \delta|/2} v^{-|\alpha|/2}. & (5.22) \end{aligned}$$

Here $0 < q \leq \infty$, $\delta, \alpha, \alpha' \in \mathbb{Z}^m_+$ with $\delta \leq \alpha$ and $|\alpha'| \leq 1, 0 < v \leq t \leq 1$, and $\eta^1 > \eta$ in the sense of (4.22) .

Proof (5.18) for $\alpha = 0$ follows from (5.3) by taking adjoints, and in general it follows easily, by induction on $|\alpha|$, from (5.16).

(5.19) and (5.20) are proved together by induction on α.

For (5.19) with $\alpha = 0$ we prepare the estimates

$$\begin{aligned} \|\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}}\|_2 &\leq C(t/u)^{m/4}, & (5.23a)\\ \|\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}}\tilde{\eta}^1_t\|_1 &\leq \|\tilde{\eta}_t e^{-u/2T^0_{\varepsilon,p}}\|_2 \|e^{-u/2T^0_{\varepsilon,p}}\tilde{\eta}^1_t\|_2 \\ &\leq C(t/u)^{m/2}, & (5.23b)\\ \|\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}}(1-\tilde{\eta}^1_t)\|_2 &\leq C_N(u/t)^N, \quad N \in \mathbb{N}, & (5.23c) \end{aligned}$$

which are easy consequences of (5.7). To proceed we write the commutator $[\tilde{\eta}_t, e^{-uT^0_{\varepsilon,p}}]$ in the now familiar way to obtain the representation

$$\begin{aligned} &\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}}(1-\tilde{\eta}^1_t) \\ &= -\sum_{\beta,|\gamma|\leq 2} \int_0^u \tilde{\eta}_t e^{-(u-v)T^0_{\varepsilon,p}} M_\beta [\partial_s^\gamma, \tilde{\eta}^2_t] A^{\beta\gamma}_{\varepsilon,p} e^{-vT^0_{\varepsilon,p}}(1-\tilde{\eta}^1_t)dv \qquad (5.24)\\ &=: \sum_{\substack{\beta,|\gamma|\leq 2\\ \delta\leq\beta,|\gamma'|\leq 1}} \int_0^u \tilde{\eta}_t e^{-(u-v)T^0_{\varepsilon,p}} M_\delta S^\beta_\delta\Big(\frac{u-v}{2}\Big)\partial_s^{\gamma'} \tilde{\eta}^3_t A^{\beta\gamma}_{\gamma'}(t) e^{-vT^0_{\varepsilon,p}}(1-\tilde{\eta}^1_t)dv \end{aligned}$$

where $\eta < \eta^2 < \eta^3 < \eta^1$ and

$$\|A^{\beta\gamma}_{\gamma'}(t)\|_{L^\infty(B_{2/3},\mathrm{End}\Lambda^*_p\otimes E_p)} \leq Ct^{(|\gamma|-2)/2}. \qquad (5.25)$$

Next we get from (5.21a) by interpolation

$$\|\tilde{\eta}_t e^{-(u-v)/2T^0_{\varepsilon,p}}\|_{2m} \leq C(t/(u-v))^{1/4}. \qquad (5.26)$$

Using (5.24), (5.21c), (5.18), and (5.23) together with the Hölder inequality for Schatten norms in (5.22) we arrive at

$$\|\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}}(1-\tilde{\eta}^1_t)\|_{2m/(m+1)} \leq C_N(u/t)^N. \qquad (5.27)$$

Upon iteration we see that the same estimate holds for the norm of order $2m/(m+L)$, $L \in \mathbb{N}$, hence, for all $q > 0$,

$$\|\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}}(1-\tilde{\eta}^1_t)\|_q \leq C_{N,q}(u/t)^N. \qquad (5.28)$$

Thus, with (5.21b),

$$\|\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}}\|_1 \leq C(t/u)^{m/2}. \qquad (5.29)$$

Now we can iterate as in the derivation of (5.21b), using the Hölder inequality for Schatten norms, to derive (5.19) and (5.20) with $\alpha = \alpha' = 0$.

Next, from (5.19) (with $\alpha = \alpha' = 0$) and (5.3) we get

$$\begin{aligned} \|\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}} \partial_s^{\alpha'}\|_q &\leq \|\tilde{\eta}_t e^{-u/2T^0_{\varepsilon,p}}\|_q \|e^{-u/2T^0_{\varepsilon,p}} \partial_s^{\alpha'}\|_\infty \\ &\leq C(t/u)^{m/2q} u^{-|\alpha'|/2}, \end{aligned}$$

which is (5.19) for $\alpha = 0$.

For (5.20) with $\alpha = 0$, we use the representation following from (5.22) again, starting this time with the estimate

$$\|\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}} \partial_s^{\alpha'}\|_{2m} \leq C(t/u)^{1/4} u^{-|\alpha'|/2}.$$

This leads to

$$\|\tilde{\eta}_t e^{-uT^0_{\varepsilon,p}} \partial_s^{\alpha'} (1 - \tilde{\eta}^1_t)\|_m \leq C u^{-|\alpha'|/2},$$

which gives (5.20) by iteration.

We turn to the inductive step and note that (5.19) with $q = \infty$ follows easily from (5.16) and the induction hypothesis. For general q, we use again (5.16) splitting it by writing $A^{\beta\gamma}_{\varepsilon,p} = ((1 - \tilde{\eta}^2_t) + \tilde{\eta}^2_t) A^{\beta\gamma}_{\varepsilon,p}$. The q–norm of the first integral is estimated using (5.19) and the induction hypothesis for (5.20). The second integral is split at $u/2$, using (5.20) on the first and (5.18) on the second factor in the integral from 0 to $u/2$, and the other way around in the integral from $u/2$ to u. This, clearly, completes the induction for (5.19).

An entirely analogous estimate gives (5.20). □

With these preparations we can give the

Proof of Theorem 5.2 The proof is by induction on ν, using the recursion (5.17), the induction hypothesis is formed by the estimate (5.11) and the parallel estimate

$$\| \left[\tilde{\eta}_t R^\nu_{\varepsilon,p} M_\alpha \partial_s^{\alpha'} (1 - \tilde{\eta}^1_t) \right]_{IJ} \|_q \leq C_{\nu\alpha q} t^{(|\mu(I) - \mu(J)| + |\alpha|)/2} u^{-|\alpha'|/2}. \tag{5.11'}$$

For $\nu = 0$ we write

$$\tilde{\eta}_t R^0_{\varepsilon,p}(u) M_\alpha \partial_s^{\alpha'} = \sum_{\delta \leq \alpha} M_\delta \tilde{\eta}_t S^\alpha_\delta(u) \partial_s^{\alpha'}.$$

Since $T^0_{\varepsilon,p}$ is scalar, (5.11) and (5.11') follow in this case from (5.19) and (5.20).

To establish the assertion (5.11) for $\nu + 1$, we insert $1 = 1 - \tilde{\eta}^1_t + \tilde{\eta}^1_t$ in (5.17) in front of $A^{\beta\gamma}_{\varepsilon,p}, \eta^1 > \eta$, and estimate the two resulting terms separately. For the q–norm of the first term we obtain the bound

$$\begin{aligned} &\sum_{\beta,\gamma,\delta \leq \alpha, K} \int_0^u C_{\nu\alpha} t^{(|\mu(I) - \mu(K)| + |\beta + \delta|)/2} (u - v)^{-|\gamma|/2} v^{(|\alpha - \delta| - |\alpha'|)/2} dv \\ &\leq \sum_{\beta,\gamma,\delta \leq \alpha, K} C_{\nu\alpha} t^{(|\mu(I) - \mu(K)| + |\beta + \delta|)/2} u^{(|\alpha - \delta| - |\alpha'| - |\gamma| + 2)/2} \\ &\leq \sum_{\beta,\gamma,\delta \leq \alpha, K} C_{\nu\alpha} t^{(|\mu(I) - \mu(K)| + |\alpha| + |\beta| - |\gamma| + 2)/2} u^{-|\alpha'|/2} \\ &\leq C_{\nu\alpha} t^{(|\mu(I) - \mu(J)| + |\alpha|)/2} u^{-|\alpha'|/2}, \end{aligned}$$

since in a nonzero term in (5.17) we have, by (2.13c),

$$|\beta| - |\gamma| + 2 \geq |\mu(K) - \mu(J)|,$$

hence

$$|\mu(I) - \mu(K)| + |\beta| - |\gamma| + 2 \geq |\mu(I) - \mu(J)|.$$

In the second term we split again the integral at $u/2$, using the q–norm on the factor whose norm remains integrable. For the integral from 0 to $u/2$ the q–norm is bounded analogously by

$$\begin{aligned}
&C_{\nu\alpha} \sum_{\beta,\gamma,\delta,K} \int_0^{u/2} t^{(|\mu(I)-\mu(K)|+|\beta+\delta|)/2} (t/(u-v))^{m/2q} (u-v)^{-|\gamma|/2} t^{|\alpha-\delta|/2} v^{-|\alpha'|/2} dv \\
&\leq C_{\nu\alpha} \sum_{\beta,\gamma,\delta,K} t^{(|\mu(I)-\mu(K)|+|\alpha|+|\beta|)/2} (t/u)^{m/2q} u^{(2-|\gamma|-|\alpha'|)/2} \\
&\leq C_{\nu\alpha} t^{(|\mu(I)-\mu(J)|+|\alpha|)/2} (t/u)^{m/2q} u^{-|\alpha'|/2}.
\end{aligned}$$

The second integral is estimated analogously by the same quantity; this establishes (5.11) for $\nu + 1$.

The splitting with $\tilde{\eta}_t^1$ leads, by the same arguments, to (5.11'). This completes the proof. ∎

It remains to discuss the continuity properties of $R^\nu_{\varepsilon,p}$ as a function of ε. This we do first in the C^2 case; then an approximation argument will handle the $C^{1,1}$ case. We now use the fact that we have obtained estimates for all q–norms with $q > 0$. Thus we obtain from (4.8) and Theorem 5.2

$$\begin{aligned}
&\left|\mathrm{tr}[\eta_t(R^\nu_{\varepsilon,p} - R^\nu_{0,p})(t)]_{\mathrm{top}}\right| \\
&\quad \leq t^{-m/2} \|\tilde{\eta}_t(R^\nu_{\varepsilon,p} - R^\nu_{0,p})(t)\|_\infty^{1/2} \|\tilde{\eta}_t(R^\nu_{\varepsilon,p} - R^\nu_{0,p})(t)\|_{1/2}^{1/2} \\
&\quad \leq C_\nu \|\tilde{\eta}_t(R^\nu_{\varepsilon,p} - R^\nu_{0,p})(t)\|_\infty^{1/2} t^{-m/4}. \qquad (5.30)
\end{aligned}$$

It is, therefore, enough to prove continuity in the operator norm of $R^\nu_{\varepsilon,p}$ at $\varepsilon = 0$; more precisely, we need

Lemma 5.5 *Under the regularity condition* (2.13b')*, there are positive functions* $C_{\nu\alpha}(\varepsilon), \nu \in \mathbb{Z}^m_+, \varepsilon \in (0,1]$*, with* $\lim_{\varepsilon\to 0} C_{\nu\alpha}(\varepsilon) = 0$ *and*

$$\|\left[\tilde{\eta}_t(R^\nu_{\varepsilon,p} - R^\nu_{0,p})(u) M_\alpha \partial_s^{\alpha'}\right]_{IJ}\|_\infty \leq C_{\nu\alpha}(\varepsilon) t^{(|\mu(I)-\mu(J)|+|\alpha|)/2} u^{-|\alpha'|/2}, \qquad (5.31)$$

for $0 < u \leq t \leq 1, p \in M, I, J \subset \{1, \ldots, m\}$*, and* $\alpha' \in \mathbb{Z}^m_+$ *with* $|\alpha'| \leq 1$.

Proof A straightforward estimate of the right hand side in (5.17), using the smoothness assumption (2.13b), shows that the lemma follows inductively from the following facts:

(1) the estimate (5.19) with $\nu = 0$,

(2) the estimate

$$\|(S^{\alpha}_{\delta}(\varepsilon; v) - S^{\alpha}_{\delta}(0; v))\partial_s^{\alpha'}\|_{\infty} \le C_{\alpha}(\varepsilon) v^{(|\alpha-\delta|-|\alpha'|)/2}, \tag{5.32}$$

where the function C_{α} satisfies $\lim_{\varepsilon\to 0} C_{\alpha}(\varepsilon) = 0$.

Using induction on $|\alpha|$ as before, we see that (5.29) follows readily from (5.16) once it is proved for $\alpha = 0$. To achieve this we invoke once more the solution of the inhomogeneous heat equation to obtain the identity

$$\begin{aligned}(e^{-uT^0_{\varepsilon,p}} &- e^{-uT^0_{0,p}})\partial_s^{\alpha'} \\ &= \sum_{|\gamma|=2} \int_0^u e^{-(u-v)T^0_{0,p}} \partial_s^{\gamma}(A^{\gamma}_{\varepsilon,p} - A^{\gamma}_{0,p}) e^{-vT^0_{\varepsilon,p}} \partial_s^{\alpha'} dv. \end{aligned} \tag{5.33}$$

Since $A^{\gamma}_{\varepsilon,p}$ is C^1 in ε and since the operators

$$(T^0_{0,p})^{-1} \sum_{|\gamma|=2} \partial_s^{\gamma}(A^{\gamma}_{\varepsilon,p} - A^{\gamma}_{0,p}) \quad \text{and} \quad \sum_{|\gamma|=2} \partial_s^{\gamma}(A^{\gamma}_{\varepsilon,p} - A^{\gamma}_{0,p})(T^0_{\varepsilon,p})^{-1}$$

are bounded, we obtain the desired estimate by splitting again the integral at $u/2$.

It remains only to prove (5.29) for $\nu = 0$. We have

$$\tilde{\eta}_t(e^{-uT^0_{\varepsilon,p}} - e^{-uT^0_{0,p}}) M_{\alpha} \partial_s^{\gamma'} = \sum_{\delta \le \alpha} M_{\delta} \tilde{\eta}_t (S^{\alpha}_{\delta}(\varepsilon; u) - S^{\alpha}_{\delta}(0; u)) \partial_s^{\gamma'},$$

so that the estimate follows from (5.30). The lemma is proved. □

We conclude this section with filling the last gaps in the proof.

Proof of Theorem 2.5 in the C^2 case
From Corollary 5.3 we get

$$\|[\eta_t(\zeta_{\varepsilon,p} - 1)e^{-tT_{\varepsilon,p}}]_{\mathrm{top}}\|_1 \le C \sup_{p,s} |\zeta_{\varepsilon,p}(s) - 1| =: \phi_1(\varepsilon),$$

with $\lim_{\varepsilon\to 0} \phi_1(\varepsilon) = 0$. Furthermore, using (5.9) and (5.29) we find

$$\begin{aligned} &\|[\eta_t(e^{-tT_{\varepsilon,p}} - e^{-tT_{0,p}})]_{\mathrm{top}}\|_1 \\ &\quad \le t^{-m/2} \sum_{\nu=0}^{2m} \|[\tilde{\eta}_t(R^{\nu}_{\varepsilon,p} - R^{\nu}_{0,p})(t)]_{\mathrm{top}}\|_1 + Ct^{1/2} \\ \le &\quad \sup_{\nu \le 2m} C_{\nu_0}(\varepsilon) + Ct^{1/2} \\ &\quad =: \phi_2(\varepsilon) + Ct^{1/2}. \end{aligned}$$

This completes the proof. □

We remark that in this case the convergence is even in $L^\infty(M)$.

Finally, we deal with the $C^{1,1}$ case using an approximation argument. For this, it is useful to rephrase the results of this section a little bit. So we look at all operators, τ, satisfying the conditions (2.13) in $L^2(B_{2/3}(0), F)$, for some finite dimensional complex vector space F. We denote this set by $\mathcal{T}$ and introduce the norm

$$\|\tau\|_{1,1} := \sum_{\beta,\gamma,\delta\leq\gamma} \|\frac{\partial^{|\delta|}}{\partial s^\delta} A^{\beta\gamma}(s)\|_{L^\infty(B_{2/3}(0))} + \lambda_1^{-1} + \lambda_2. \tag{5.34}$$

For $\tau \in \mathcal{T}$, we let T again be the Friedrichs extension.

Then we have, actually, proved the following facts.

Corollary 5.6 (1) *For $\tau \in \mathcal{T}$ we have an estimate*

$$\|[\eta_t e^{-tT}]_{\rm top}\|_1 \leq C = C(\|\tau\|_{1,1}), \quad \text{uniformly in } \ t \leq 1.$$

(2) *For any constants $\varepsilon > 0, C_1 > 0$ we can find $\delta > 0$ such that*

$$\|\tau\|_{1,1} + \|\tau'\|_{1,1} \leq C_1$$

and

$$\|\tau - \tau'\|_{1,1} \leq \delta$$

imply, uniformly in $t \leq 1$,

$$\|[\eta_t(e^{-tT} - e^{-tT'})]_{\rm top}\|_1 \leq \varepsilon.$$

Proof We simply have to go through the proofs of Theorem 5.2 and Lemma 5.5 to see that they work literally in this situation, too. □

This allows us to give the

Proof of Theorem 2.5 in the $C^{1,1}$ case

We choose a C^2 approximation of all data which locally approximates in the norm (5.32). This leads to functions $F_t^E, F_t^{n,E}$ according to (1.11) and also $F^E, F^{n,E}$ where

$$F^E(p) = F^E(p;t) = {\rm tr}[\eta_t e^{-tT^0_{0,p}}]_{\rm top},$$

and $F^{n,E}(p)$ is defined similarly. Then we derive from Corollary 5.6 that, for given $\varepsilon > 0$,

$$|F_t^E(p) - F_t^{n,E}(p)| \leq \varepsilon, \tag{5.35a}$$

if $n \geq n(\varepsilon)$ and $t \leq t(\varepsilon)$, uniformly in p. Next, it follows from the first case of Theorem 2.5 that

$$\lim_{t\to 0} |F_t^{n,E}(p) - F^{n,E}(p)| = 0, \tag{5.35b}$$

uniformly in p but not necessarily in n. Finally, another application of Corollary 5.6 shows that

$$|F^E(p) - F^{n,E}(p)| \leq C$$

for some constant C, uniformly in n and a.e. in p, and

$$\lim_{n\to\infty} |F^E(p) - F^{n,E}(p)| = 0$$

for almost all p. Thus we derive from Lebesgue's theorem

$$\lim_{n\to\infty} \|F^E - F^{n,E}\|_{L^1(M)} = 0. \tag{5.35c}$$

The estimates (5.34) together yield

$$\lim_{t\to 0} F_t^E = F^E \quad \text{in} \quad L^1(M)$$

as desired. □

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J. Brüning
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6

10099 Berlin
Germany
Email: bruening@mathematik.hu-berlin.de