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On the spectral geometry of algebraic curves

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1. Introduction

Spectral geometry investigates the relationship between the geometry of Riemannian manifolds and the spectral data of certain self-adjoint operators, naturally associated to the geometry. The main analytic tool is the de Rham complex,

$$0 \to \Omega_0^0(M) \xrightarrow{d_0} \Omega_0^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{m-1}} \Omega_0^m(M) \to 0\,, \tag{1.1}$$

where $\Omega_0^j(M) = C_0^\infty(\Lambda^j T^*M)$ denotes the smooth j-forms with compact support on the smooth manifold M, of dimension m, and d is the exterior derivative. If M carries a Riemannian metric then we have an associated scalar product on $\Omega_0(M)$, so (1.1) may be viewed as a differential complex of densely defined operators in the Hilbert space $L^2(\Lambda^*M)$. The metric also determines on $\Omega_0^j(M)$ the transposed operators, d_j^t, and the Laplacians, $\Delta_j = d_j^t d_j + d_{j-1} d_{j-1}^t$. The latter are nonnegative symmetric operators in $L^2(\Lambda^j T^*M)$ and hence admit self-adjoint extensions. If M happens to be complete then Δ_j is essentially self-adjoint on $\Omega_0^j(M)$; thus, the natural spectral data are provided by the closures, $\bar{\Delta}_j$. Their analysis, however, can be complicated by the presence of essential spectrum. Another complication arises if M is incomplete because then Δ_j may have infinitely many self-adjoint extensions. A natural choice of such extensions is again linked with (1.1): if we find a Hilbert complex extending the de Rham complex (i.e. a choice of closed extensions, D_j, for the d_j preserving the complex property) then its Laplacians are automatically self-adjoint. This is always possible since we can choose $D_j = d_{j,\min} := \bar{d}_j$ or $D_j = d_{j,\max} := (d_j^t)^*$. This seems most natural if $d_{j,\min} = d_{j,\max}$ for all j, which we refer to as the *case of uniqueness* (it should be noted that this uniqueness does not imply essential self-adjointness for the Laplacians). Certainly, if M is compact we have uniqueness and all Laplacians are discrete.

In this paper, we study a family, $\mathcal{M}$, of possibly incomplete Riemannian manifolds which, nevertheless, displays uniqueness and discreteness in the sense just described. This family consists of (the regular part of) all algebraic curves, equipped with metrics which are induced from some hermitian metric on complex projective space – e.g. the Fubini-Study metric – via projective embeddings. By the uniformization theorem, any compact orientable Riemannian 2-manifold can be obtained in this way. Thus, this setting generalizes the case of compact orientable surfaces, the spectral geometry of which is highly developed. Of course, the difficulties encountered are caused by the singularities of the manifold. Our main result says that from the spectral data we can decide whether or not the given algebraic curve has singularities (other than multiple points). Thus, even though the singularities of an algebraic curve are "small" in the sense that we are in the case of uniqueness – as opposed e.g. to the case of manifolds with boundary – they can be detected from the spectrum of the Laplacians; in fact, we will show that they have a rather drastic effect on the shape of the heat asymptotics.

To explain our results in greater detail, we recall the following fact which was proved in [BL2], Lemma 3.1 and Thm. 3.7, and is the starting point of our analysis (note that algebraic curves are conformally conic in the sense of [BL2], Sec. 2).

Theorem 1.1. *Let $M \in \mathcal{M}$. Then we are in the case of uniqueness, i.e.*

$$d_{j,\min} = d_{j,\max}, \quad 0 \leqq j \leqq 2 .$$

If T_j, $0 \leqq j \leqq 2$, denotes the corresponding self-adjoint Laplacian then T_0 equals the Friedrichs extension of Δ_0.

Finally, all T_j are discrete (i.e. have empty essential spectrum).

The only assertion which is not obvious from loc. cit. is the discreteness of T_j. This can be seen as follows: by [BL2], Sec. 2, a conformally conic manifold is quasi-isometric to a compact manifold with an isolated conic singularity. By [BL1], Lemma 2.17 and Sec. 3, discreteness of the T_j is invariant under quasi-isometries. Now, for a compact manifold with isolated conic singularities the discreteness of the T_j is well known (see e.g. [C3], Sec. 3 and 4).

By Poincaré duality (cf. [BL1], Lemmas 2.16, 3.7 and 4.2) we see that T_2 is unitarily equivalent to T_0. Moreover, the Hodge decomposition [BL1], Cor. 2.5 implies a unitary equivalence between $T_1 | (\ker T_1)^\perp$ and two copies of $T_0 | (\ker T_0)^\perp$. Thus, a complete set of spectral data is provided by $\operatorname{spec} T_0$ (with multiplicities) and β_1, where $\beta_j := \dim \ker T_j$, $0 \leqq j \leqq 2$. In place of β_1 we can use the L^2-Euler characteristic defined as

$$\chi_{(2)}(M) := 2\beta_0 - \beta_1 ;$$

it has been computed explicitly for the case under discussion in [BL2], Cor. 5.14, using the result in [BPS], Thm. 4.1; cf. also [N2].

These spectral data are translated into geometric information by certain averaging processes, usually obtained from the asymptotic expansion of the trace of convenient parameter-dependent functions of T_0. Well known examples are the heat kernel, the ζ-function, and the resolvent, the latter providing slightly stronger information than the other two (cf. Lemmas 2.1 and 2.2 for a precise statement). We will work with the resolvent throughout but formulate the result in terms of the heat kernel, for convenience. It involves some basic information on the structure of the singularities of M. In fact, let $M \in \mathcal{M}$ and denote by Σ the (finite) set of singularities relative to some projective embedding (which also

induces the metric on M, from some hermitian metric on complex projective space). For $p \in \Sigma$, let $L(p)$ be the number of irreducible components, locally near p, and let

$$N_i(p), \quad 1 \leqq i \leqq L(p) ,$$

be the multiplicity of the ith component. Then we have the following spectral information.

Theorem 1.2. (1) *For $t > 0$, e^{-tT_0} is trace class and we have the asymptotic expansion*

$$\text{(1.2)} \quad \operatorname{tr}(e^{-tT_0}) \sim_{t\to 0+} \sum_{j\geqq 0} a_j t^{j-1} + \sum_{j \geqq 1} b_j t^{j-1} \log t + \sum_{\substack{p\in\Sigma \\ 1 \leqq i \leqq L(p)}} \sum_{j \geqq 0} c_j(i,p)\, t^{j/2N_i(p)} .$$

(2) *In* (1.2), *we have*

$$\text{(1.3)} \quad a_0 = \frac{\operatorname{vol} M}{4\pi},$$

and

$$\text{(1.4)} \quad b_1 = 0 .$$

(3)

$$\text{(1.5)} \quad \lim_{t\to 0+} (\operatorname{tr} e^{-tT_0} - a_0 t^{-1}) - \chi_{(2)}(M)/6 = \frac{1}{12} \sum_{\substack{p\in\Sigma \\ 1 \leqq i \leqq L(p)}} (N_i(p) + N_i(p)^{-1} - 2) .$$

(4) *There are $M \in \mathcal{M}$ with $b_2 \neq 0$. More precisely, among the generalized parabolas $C^{k,l}$ (defined after Lemma 4.6), b_2 distinguishes the parabolas of type $C^{1,l}$, $l \in \mathbb{N}$.*

(5) *There are $M \in \mathcal{M}$ with $c_2(i,p) \neq 0$, for some $p \in \Sigma$ and $1 \leqq i \leqq L(p)$.*

The proof of this theorem is the main goal of this work; it will be given at the end of Sec. 4. We remark that in the nonsingular case the coefficients b_j and $c_j(i,p)$ all vanish. Their appearance is dictated by our method of proof but, as usual, the resulting formulae are difficult to evaluate. Nevertheless, (1.5) reveals the presence of the singularities: the right hand side vanishes if and only if all $N_i(p)$ are equal to one, and in this case we recover the familiar computation of a_1 (cf. e.g. [BGM], p. 222). The logarithmic terms turn out to be quite subtle (cf. Section 4 for a more detailed discussion) but the class $\mathcal{M}$ seems to provide the most simple and natural setting for their appearance. It seems that very little information on the spectral distribution of T_0 was obtained before. During the preparation of this paper we obtained the preprint [Y] where the leading asymptotics of $\operatorname{tr} e^{-tT_0}$ are established, by quite different methods; this result was also proved in [L1], Sec. 2.4, (4.5). In [LT], an upper bound for the heat kernel of T_0 is given, even for algebraic varieties of arbitrary dimension.

We turn to our method of proof which is governed by the nature of the singularities. In fact, for algebraic curves we can describe the singularities very explicitly as follows (cf. [BPS], Sec. 2). We assume $M \hookrightarrow \mathbb{CP}^n$, for some n, with singular locus Σ, a finite set. For $p \in \Sigma$ we choose homogeneous coordinates $[z_0, \dots, z_n]$ in $\mathbb{CP}^n$ such that $p = [1, 0, \dots, 0]$.

Moreover, we choose a neighborhood U_p of p in $\mathbb{CP}^n$ such that $U_p \cap \Sigma = \{p\}$ and $U_p \setminus \{p\} \cap M = \bigcup_{1 \le i \le L(p)} U_{ip}$ where the U_{ip}, the *components of M near p*, are connected and mutually disjoint, and such that, for each component $V := U_{ip}$, we can find a biholomorphic map

$$\psi : D_\varepsilon := \{z \in \mathbb{C} \mid 0 < |z| < \varepsilon\} \to V, \tag{1.6}$$

$$\psi(z) = [1, P_1(z), \ldots, P_n(z)], \quad P_i \text{ holomorphic in } D_\varepsilon,$$

where we have with some $k \in \mathbb{N}$, $1 \le k \le n$,

$$\begin{aligned} P_i(z) &\equiv 0, \quad 1 \le i \le k-1, \\ P_k(z) &= z^{N_k}, \\ P_j(z) &= \sum_{l \ge N_j} a_{jl} z^l, \quad k+1 \le j \le n. \end{aligned} \tag{1.7}$$

Here $N_j \in \mathbb{N}$ with $N_k < N_{k+1} < \cdots < N_n$; $N_k =: N_i(p)$ is called the *multiplicity* of the ith component.

Since T_0 is the Friedrichs extension, we can restrict the analysis to a single component, U, with multiplicity N, cf. Lemma 2.4. Any hermitian metric on $\mathbb{CP}^n$ will induce a metric on U which, in polar coordinates $(x, \varphi) \in (0, \varepsilon) \times [0, 2\pi)$, takes the form

$$g(x, \varphi) = \alpha^2(x^{1/N}, \varphi)\, dx \otimes dx + \beta^2(x^{1/N}, \varphi)\, N^2 x^2\, d\varphi \otimes d\varphi\,. \tag{1.8}$$

Here $\alpha, \beta \in C^\infty([0, \varepsilon) \times S^1)$ with $\alpha(0, \cdot) = \beta(0, \cdot) = 1$. This leads us to a regular singular Sturm-Liouville operator with operator coefficients as our model operator. In fact, in the coordinates above we have, with $\tilde\alpha(x, \varphi) := \alpha(x^{1/N}, \varphi)$, $\tilde\beta(x, \varphi) := \beta(x^{1/N}, \varphi)$,

$$\Delta_0 = -(x\tilde\alpha\tilde\beta)^{-1} \partial_x (x\tilde\beta/\tilde\alpha) \partial_x - N^{-2} x^{-2} (\tilde\alpha\tilde\beta)^{-1} \partial_\varphi (\tilde\alpha/\tilde\beta) \partial_\varphi\,,$$

acting on $C_0^\infty((0, \varepsilon) \times S^1)$ in the Hilbert space $L^2(\mathbb{R}_+ \times S^1, Nx\tilde\alpha\tilde\beta\, dx\, d\varphi)$. Here

$$\partial_x := \frac{\partial}{\partial x}, \quad \partial_\varphi := \frac{\partial}{\partial \varphi}.$$

With the unitary transformation

$$\Phi : L^2(\mathbb{R}_+ \times S^1) \to L^2(\mathbb{R}_+ \times S^1, Nx\tilde\alpha\tilde\beta\, dx\, d\varphi), \quad f \mapsto (Nx\tilde\alpha\tilde\beta)^{-1/2} f,$$

we obtain the operator

$$\begin{aligned} \tau_\varepsilon &:= \Phi^{-1} \Delta_0 \Phi \\ &= -(x\tilde\alpha\tilde\beta)^{-1/2} \partial_x (x\tilde\beta/\tilde\alpha) \partial_x (x\tilde\alpha\tilde\beta)^{-1/2} - N^{-2} x^{-2} (\tilde\alpha\tilde\beta)^{-1/2} \partial_\varphi (\tilde\alpha/\tilde\beta) \partial_\varphi (\tilde\alpha\tilde\beta)^{-1/2} \\ &=: -\partial_x^2 + X^{-2} A_0 + R_\varepsilon\,, \end{aligned} \tag{1.9}$$

acting on $C_0^\infty((0, \varepsilon), C^\infty(S^1))$ in the Hilbert space $\mathscr{H} := L^2(\mathbb{R}_+, L^2(S^1))$. Here X is multiplication by the coordinate function x, and

$$A_0 := -\frac{1}{N^2} \frac{\partial^2}{\partial \varphi^2} - \frac{1}{4}.$$

The "perturbation" R_ε is built from the operators

$$U_0 := \mathrm{id}_{L^2(S^1)}, \quad U_1 = U_1(\gamma) := \Omega^\gamma X^{-1} (A_0 + I)^{1/2}, \quad U_2(\gamma) := \Omega^\gamma \partial_x\,, \tag{1.10}$$

where Ω is multiplication by $\omega(x) := x/(x+1)$ and $\gamma = \gamma(N)$ is some positive number. Then, with certain operator functions $C_{ij}^\varepsilon \in C([0, \varepsilon), \mathscr{L}(L^2(S^1)))$, we have

$$R_\varepsilon := \sum_{i,j=0}^{2} U_i^* C_{ij}^\varepsilon U_j\,. \tag{1.11}$$

The dependence on ε is singled out for the following reason: if we measure the strength of the perturbation R_ε by

$$\delta(\tau_\varepsilon) := \sum_{i,j=0}^{2} \| C_{ij}^\varepsilon \|_{\mathscr{L}(\mathscr{H})}\,, \tag{1.12}$$

then $\lim_{\varepsilon \to 0} \delta(\tau_\varepsilon) = 0$.

By a suitable cut off, we may hence assume that $\delta(\tau_\varepsilon)$ is as small as we please. Thus (1.9) is well defined and semibounded on $C_0^\infty((0, \infty), C^\infty(S^1))$ with values in $\mathscr{H}$. Again, we can reduce the asymptotic analysis to the operator T_ε, the Friedrichs extension of τ_ε in $\mathscr{H}$. This is almost a regular singular operator in the sense of [BS2], only the perturbation R_ε is considerably stronger than allowed in these papers. Our attempt to generalize the analysis to the present case lead us to a new approach, bypassing the explicit construction of the Neumann series. The necessary a priori estimates are reduced instead to estimates for the operators (1.10), rendering a much simpler proof also in the cases treated before. In particular, the Bessel functions make no appearance in our analysis any more.

Once the a priori estimates are established, the existence part of Theorem 1.2 follows from the Singular Asymptotics Lemma [BS1] essentially as before.

The plan of the paper is as follows. In Section 2 we prove the existence of the asymptotic expansion using the estimates mentioned above; they are proved in Section 3. The explicit calculations are carried out in Section 4.

Most of the results in this paper have been announced in [BL3].

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2. The existence of the asymptotic expansion

Asymptotic expansions for self-adjoint elliptic operators occur in various disguises, notably resolvent or heat traces and zeta functions. These are more or less equivalent and related in a well known way. Nevertheless, since we have to switch back and forth between them we recapitulate the basic facts and fix some notation, for the convenience of the reader.

Consider first a self-adjoint operator, T, in some Hilbert space H which is bounded below; without loss of generality we assume

$$(2.1)\qquad (Tu, u) \geqq \|u\|^2, \quad u \in \mathscr{D}(T).$$

We assume, moreover, that for some $q > 0$

$$(2.2)\qquad T^{-1} \in C_q(H),$$

where C_q denotes the von Neumann-Schatten class of order q. It then follows from the resolvent equation and the Hölder inequality for C_q-norms that

$$(2.3)\qquad (T+z^2)^{-n} \in C_1(H), \quad q < n \in \mathbb{N},$$

with uniform trace norm estimate in

$$(2.4)\qquad Z_\delta := \{z \in \mathbb{C} \mid |\arg z| < \delta\}, \quad 0 < \delta < \pi/2.$$

Here we have chosen to work with z^2 instead of z since all our applications will address second order operators. In this general framework we now want to assume an asymptotic expansion of the type

$$(2.5)\qquad \operatorname{tr}_H(T+z^2)^{-n} \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} A^r_{jk}(n)\, z^{\alpha_j - 2n} \log^k z,$$

as $z \to \infty$ in Z_δ, $n > q$. Here $k(j) \in \mathbb{Z}_+$ for all j, $(\alpha_j)_{j \in \mathbb{Z}_+}$ is a sequence of complex numbers with $\operatorname{Re}\alpha_j \to -\infty$, and $A^r_{jk}(n)$ are the coefficients (where "r" refers to "resolvent").

(2.5) has implications for the zeta function, ζ_T, of T defined by

$$(2.6)\qquad \zeta_T(s) := \sum_{\lambda \in \operatorname{spec} T \setminus \{0\}} \lambda^{-s}, \quad \operatorname{Re} s > q,$$

which are easily derived from the representation

$$(2.7)\qquad \binom{n-1-s}{n-1} \zeta_T(s) = \frac{\sin \pi s}{\pi} \int_1^{\infty} z^{n-1-s} \operatorname{tr}(T+z)^{-n} dz + \tilde{\zeta}_T(s),$$

valid for $\operatorname{Re} s > n$, with $\tilde{\zeta}_T$ entire.

Lemma 2.1. *Under the assumptions* (2.1), (2.2), *and* (2.5), ζ_T *extends meromorphically to the whole plane, with poles at most at the points* $\alpha_j/2$, $j \in \mathbb{Z}_+$.

The principal part of ζ_T *at* $\alpha_j/2$ *is given by*

$$(2.8)\qquad \frac{(n-1)!\sin \pi s}{\pi} \left(\prod_{l=1}^{n-1} (l-s)\right)^{-1} \sum_{k=0}^{k(j)} 2^{-k} k!\, A^r_{jk}(n)(s-\alpha_j/2)^{-k-1}.$$

Thus, the poles of ζ_T have order at most $k(j)$ if $\alpha_j/2 \in \mathbb{Z} \setminus \{1, \ldots, n-1\}$, and $k(j)+1$ otherwise.

(2.5) also implies an asymptotic expansion for the heat trace, $\operatorname{tr}(e^{-tT})$, as $t \to 0+$, following directly from the Cauchy integral

$$(2.9)\qquad e^{-tT} = t^{1-n} \frac{(n-1)!}{2\pi i} \int_\Gamma e^{-t\mu}(T-\mu)^{-n} d\mu;$$

here Γ is composed from the two rays $c_\pm(t) := te^{\pm i\pi/4}$, $t \geqq 1$, and the unit circle, traversed upward.

Lemma 2.2. *Under the assumptions* (2.1), (2.2), *and* (2.5), *we have the asymptotic expansion*

$$(2.10\text{a})\qquad \operatorname{tr}(e^{-tT}) \sim_{t \to 0+} \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} A^h_{jk}\, t^{-\alpha_j/2} \log^k t,$$

where

$$(2.10\text{b})\qquad A^h_{jk} = (-1)^k (n-1)! \sum_{l=k}^{k(j)} A^r_{jl}(n) 2^{-l} \binom{l}{l-k} \left(\frac{d}{d\alpha}\right)^{l-k} (\Gamma(-\alpha))^{-1}\big|_{\alpha = \alpha_j/2 - n}.$$

We note in particular that

$$(2.10\text{c})\qquad A^h_{j,k(j)} = (-2)^{-k(j)} \frac{\Gamma(n)}{\Gamma(n-\alpha_j/2)} A^r_{j,k(j)}(n).$$

The expansion (2.5) seems, in this generality, slightly stronger than the properties expressed in Lemmas 2.1 and 2.2. It is known to hold, typically, for symmetric elliptic operators e.g. for elliptic differential operators on compact manifolds [S1], elliptic boundary value problems for these [S2], or, recently, for classical elliptic pseudodifferential operators on compact manifolds and certain boundary value problems as well [GS]. In view of our applications we now consider a Riemannian manifold, M, of dimension m, a hermitian vector bundle, E, over M, and a symmetric elliptic differential operator of second order, τ, acting on $C_0^\infty(E)$. We then assume that

$$(2.11)\qquad (\tau u, u)_{L^2(E)} \geqq \|u\|^2_{L^2(E)}, \quad u \in C_0^\infty(E),$$

such that the Friedrichs extension, T, of τ satisfies (2.1). Since now M may be noncompact it may happen that (2.2) fails but that

$$(2.12)\qquad \varphi_1 T^{-1} \varphi_2 \in C_q(L^2(E)),$$

for certain $\varphi_1, \varphi_2 \in \mathscr{L}$,

$$\mathscr{L} := \{\psi \in C^\infty(M) \mid \operatorname{supp} d\psi \text{ compact in } M\}. \tag{2.13}$$

If, in addition, the analogue of (2.5) holds for $\varphi(T+z^2)^{-n}$ we will write

$$\operatorname{tr}_{L^2(E)} \varphi(T+z^2)^{-n} \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} A^r_{jk}(n;\varphi) z^{\alpha_j - 2n} \log^k z, \tag{2.14}$$

with similar notation for the heat trace. Here, the coefficients $A^r_{jk}(n;\varphi)$ are considered as distributions on $\mathscr{L}$.

If we restrict attention to $C_0^\infty(M) \subset \mathscr{L}$ then we can say considerably more about the coefficients in (2.14) due to the fact that the Schwartz kernel of the resolvent admits a pointwise expansion on the diagonal.

Lemma 2.3 ([S1], [Gr], [G], Sec. 1.7). *For $\varphi \in C_0^\infty(M)$, we have the expansions*

$$\operatorname{tr}_{L^2(E)} \varphi(T+z^2)^{-n} \sim \sum_{j=0}^{\infty} B^r_j(n;\varphi) z^{m-2(j+n)} \tag{2.15a}$$

and

$$\operatorname{tr}_{L^2(E)} \varphi e^{-tT} \sim \sum_{j=0}^{\infty} B^h_j(\varphi) t^{-m/2+j}. \tag{2.16a}$$

The distributions $B^r_j(n;\varphi)$ and $B^h_j(\varphi)$ are actually smooth functions i.e., with vol_M the volume form

$$B^r_j(n;\varphi) =: \int_M B^r_j(n;p)\varphi(p)\,\mathrm{vol}_M(p), \tag{2.15b}$$

$$B^h_j(\varphi) =: \int_M B^h_j(p)\varphi(p)\,\mathrm{vol}_M(p), \tag{2.16b}$$

where

$$B^h_j(p) = \frac{\Gamma(n)}{\Gamma(n+j-m/2)} B^r_j(n;p). \tag{2.17}$$

In view of Lemma 2.3 it is reasonable that we try to decompose M as

$$M := M_1 \cup U, \tag{2.18}$$

where M_1 is a compact manifold with boundary $N := \partial M_1 = \partial U$ and U, the "singular set", is open. The following result shows together with Lemma 2.3 that we can indeed reduce the expansion problem to U. To be more precise we denote by $R_U : L^2(E) \to L^2(E \mid U)$ the restriction, by $E_U : L^2(E \mid U) \to L^2(E)$ the extension by zero, and by T_U the Friedrichs extension of $\tau \mid C_0^\infty(E \mid U)$ in $L^2(E \mid U)$. Then we have the following result [B], Lemma 4.1 (cf. also [L2], Thm. 2.10).

Lemma 2.4. *Put*

$$\mathscr{L}_U := \{\psi \in C^\infty(M) \mid \operatorname{supp} d\psi \text{ compact in } U, \psi = 0 \text{ near } \partial U\}, \tag{2.19}$$

and assume that, for $\psi_1, \psi_2 \in \mathscr{L}_U$, either

$$\psi_1 T_U^{-1} \psi_2 \in C_q(L^2(E \mid U)) \tag{2.20a}$$

or

$$\psi_1 T^{-1} \psi_2 \in C_q(L^2(E)). \tag{2.20b}$$

Then for all $\psi \in \mathscr{L}_U$, $p > 0$, $z \in Z_\delta$ with $|z| \geqq 1$, and $N \in \mathbb{N}$ there is a constant $c = c(p,N)$ such that

$$\|\psi[(T_U+z^2)^{-1} - R_U(T+z^2)^{-1}E_U]\|_p \leqq c(p,N)|z|^{-N}. \tag{2.21}$$

Observing that $E_U R_U$ is the orthogonal projection in $L^2(E)$ onto $L^2(E \mid U)$ we see that Lemma 2.2 and Lemma 2.4 do in fact reduce the expansion problem in Theorem 1.2 to the expansion of $\operatorname{tr}_{L^2(E|U)}[\psi(T_U+z^2)^{-n}]$ where $U =: U_\varepsilon$ is $(0,\varepsilon) \times S^1$ equipped with the metric (1.8). Indeed, assume that for $\psi \in \mathscr{L}_U$ we have (2.20a) and an asymptotic expansion

$$\operatorname{tr}_{L^2(E|U)}(\psi(T_U+z^2)^{-n}) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} A^r_{jk}(n;\psi) z^{\alpha_j-2n} \log^k z.$$

Then we can choose in particular $\psi = 1 - \varphi$ with $\varphi \in C_0^\infty(M)$ and obtain from Lemma 2.3 and Lemma 2.4 the expansion

$$\begin{aligned} \operatorname{tr}_{L^2(E)}((T+z^2)^{-n}) &\sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} A^r_{jk}(n;\psi) z^{\alpha_j-2n} \log^k z \\ &\quad + \sum_{j=0}^{\infty} B^r_j(n; 1-\psi) z^{m-2(j+n)} \\ &=: \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} A^r_{jk}(n) z^{\beta_j-2n} \log^k z. \end{aligned} \tag{2.22}$$

Next we specialize our considerations using (1.9). Thus, from now on we assume that *T_U is unitarily equivalent to the Friedrichs extension, $\tilde{T}_\varepsilon$, of*

$$\tau_\varepsilon := -\partial_x^2 + X^{-2}A_0 + R_\varepsilon$$

in $L^2((0,\varepsilon),H)$ with domain $\bigcap_{k\geqq 1} C_0^\infty((0,\varepsilon), \mathscr{D}(A_0^k))$, where ε can be made arbitrarily small, with R_ε in (1.11). We fix a convenient ε, the size of which will be determined in Sec. 3. We consider the Friedrichs extension, $\tilde{T}$, of τ_ε in $\mathscr{H} := L^2(\mathbb{R}_+, H)$ with domain

$$\mathscr{H}^\infty := \bigcap_{k\geqq 1} C_0^\infty((0,\infty), \mathscr{D}(A_0^k)).$$

($\tilde{T}$ depends also on ε but we suppress this to simplify the notation.) Note that $(I+A_0)^{-1}\in C_p(H)$ for every $p>1/2$; we fix one such p in what follows. The general setting above applies also to T_ε, $\tilde{T}_\varepsilon$ and we can invoke [B], loc. cit., once more, using this time $\mathscr{L}:=C_0^\infty(\mathbb{R}_+)$. Thus the expansion problem is finally reduced to the study of

$$t(\varphi;z):=\operatorname{tr}_{\mathscr{H}}\varphi(\tilde{T}+z^2)^{-n},\quad \varphi\in C_0^\infty(\mathbb{R}),\quad z\in Z_\delta. \tag{2.23}$$

We abbreviate $G(z):=(\tilde{T}+z^2)^{-1}$; we are going to show that the Singular Asymptotics Lemma (SAL) of [BS1] applies to establish the existence of an asymptotic expansion of t of type (2.14). It is convenient at this point to introduce the scaled family $\tilde{T}_s$, $s\in[0,1]$, as the Friedrichs extension of

$$\tau_{\varepsilon,s}:=s^2\,U_s\tau U_s^*\ \text{in}\ L^2(\mathbb{R}_+,H),\quad \tau:=\tau_\varepsilon, \tag{2.24}$$

where

$$U_s v(x):=s^{1/2}v(sx),\quad s\in[0,1]. \tag{2.25}$$

We put $G_s^n(z):=(\tilde{T}_s+z^2)^{-n}$, $G_1:=G$. Then we see from Theorem 3.9 below that for $\alpha>2$ and $n>2$

$$V(s,z):=(1+X)^{-2(\alpha+1)}XG_s^m(z)X\in C_1(\mathscr{H}),$$

with uniform trace norm estimate in s and z; the same estimate shows that, in addition,

$$[\partial_x,V(s,z)]\in C_1(\mathscr{H}).$$

Thus the Trace Lemma in [BS2], Appendix, applies, and we deduce that $G_s^n(z)$ has a continuous kernel

$$G_s^n(z;x,y)\in C_1(H),\quad x,y>0. \tag{2.26}$$

It follows, moreover, that

$$t(\varphi;z)=\int_0^\infty \varphi(x)\operatorname{tr}_H G^n(z;x,x)\,dx.$$

Since U_s is unitary, we find the scaling relation

$$G^n(z;x,y)=s^{2n-1}G_s^n(sz;x/s,y/s), \tag{2.27}$$

hence

$$t(\varphi;z)=\int_0^\infty \varphi(x)x^{2n-1}\operatorname{tr}_H G_x^n(xz;1,1)\,dx. \tag{2.28}$$

Now we encounter a problem since G_x is not differentiable with respect to x in $[0,1]$. In fact, writing

$$\tau_{\varepsilon,s}=-\partial_x^2+X^{-2}A_0+R_{\varepsilon,s},$$

we have from (1.11) the representation

$$R_{\varepsilon,s}=\sum_{0\le i,j\le 2} U_i^*C_{i,j}^{\varepsilon,s}U_j. \tag{2.29}$$

Here the operators U_i are defined in (1.10), $C_{i,j}^{\varepsilon,s}$ is a smooth function of $s^{1/N}$ in $[0,1]$, and we have $\delta(\tau_{\varepsilon,s})=\delta(\tau_\varepsilon)$. Moreover, it is easily checked that

$$\sum_{0\le i,j\le 2}\left\|(1+X)^{-k/N}\left(\frac{\partial}{\partial s}\right)^k C_{i,j}^{\varepsilon,s^N}\right\|\le c(k). \tag{2.30}$$

This suggests that $G_{s^N}(z)$ is smooth in s and z, so we substitute $x=y^N$ in (2.27) to get

$$t(\varphi;z)=\int_0^\infty N\varphi(y^N)y^{2nN-1}\operatorname{tr}_H G_{y^N}^n((yw)^N;1,1)\,dy, \tag{2.31}$$

with $w:=z^{1/N}$. Hence we shall apply the SAL to the function

$$\sigma(y,\zeta):=Ny^{2nN-1}\varphi(y^N)\operatorname{tr}_H G_{y^N}^n(\zeta^N;1,1),\quad y\in\mathbb{R}_+,\quad \zeta\in Z_{\delta/N}. \tag{2.32}$$

We will verify the assumptions of the SAL in the following sequence of lemmas.

Lemma 2.5 (Smoothness). *Regarded as a map from $\mathbb{R}_+$ to the holomorphic functions in $Z_{\delta/N}$, σ is smooth.*

Proof. We show first that, for $M>2+j/N$, $j\in\mathbb{Z}_+$, and $\beta\in\mathbb{R}_+$ the map

$$\mathbb{R}_+\ni y\mapsto F(y;z):=(1+X)^{-M-\beta}G_{y^N}(z)(1+X)^\beta\in\mathscr{L}(\mathscr{H})$$

is j times differentiable as a map into the holomorphic functions in Z_δ with values in $C_{p+1/2}(\mathscr{H})$. In fact, the kth derivative is a sum of terms of the type

$$(1+X)^{-M-\beta}G_{y^N}(z)\left[\prod_{l=1}^{k'}S_l'^*C^l(y)S_l''G_{y^N}(z)\right](1+X)^\beta. \tag{2.33}$$

Here $k'\le k$, the operators S_l', S_l'' are in $\mathscr{S}$ (cf. Definition 3.1 below), and we have

$$C^l\in C^\infty([0,1],\mathscr{L}(\mathscr{H}))$$

with estimates

$$\|(1+X)^{-\alpha_l}C^l(y)\|\le c(l)<\infty. \tag{2.34}$$

Finally,

$$\sum_{l=1}^{k'}\alpha_l=\frac{k}{N}. \tag{2.35}$$

Thus, we can apply Theorem 3.9 to (2.33): distributing suitable powers of $(1+X)$ inside the product, we see that the first and the last factor is in $C_{2p+1}(\mathscr{H})$ while all others are bounded, with uniform norm estimates.

To establish (2.33) for $j=1$ we observe first that $\mathscr{D}(T_{y^N})$ is independent of $y \in [0,1]$ for ε sufficiently small; this follows from Theorem 3.2 below. Thus we obtain the identity

$$G_{y_1^N}(z) - G_{y_2^N}(z) = G_{y_1^N}(z)(R_{\varepsilon, y_2^N} - R_{\varepsilon, y_1^N})\, G_{y_2^N}(z)\,.$$

From this we obtain an integral representation

$$F(y_1; z) - F(y_2; z) =: \int_{y_2}^{y_1} F_1(t; z)\, dt$$

with $F_1 \in C_{p+1/2}(\mathscr{H})$, by Theorem 3.9; the Bochner integral on the right converges in $\mathscr{L}(\mathscr{H})$ and in $C_{p+1/2}(\mathscr{H})$. In general, (2.33) is easily proved by induction on $j \geqq 1$, using the representation (2.29).

Using the "Hölder inequality" for Schatten norms we obtain the analogous result for the map

$$\mathbb{R}_+ \ni y \mapsto (1+X)^{-M-\beta} G^n_{y^N}(z)(1+X)^{\beta} \in C_1(\mathscr{H})\,, \tag{2.36}$$

for $z \in Z_\delta$, $\beta \in \mathbb{R}_+$, and $M > 2n + j/N$. Repeating the above arguments, using again the Trace Lemma and noting that $\Omega^\gamma \partial_x \in \mathscr{S}$ for $\gamma > 0$, we find that all derivatives up to order j of the operator function $X(1+X)^{-M-\beta} G^n_{y^N}(z)(1+X)^{\beta} X$ have a continuous kernel in $\mathbb{R}_+ \times \mathbb{R}_+$ with values in $C_1(H)$. Thus the same is true of the operator function

$$\varphi_1 G^n_{y^N}(z)\varphi_2 \quad \text{if} \quad \varphi_i \in C_0^\infty(0,\infty), \quad i = 1, 2\,.$$

The estimate (4) in [BS2], p. 425, then implies that $G^n_{y^N}(z; 1, 1) \in C_1(H)$ has derivatives of any order in $y \in \mathbb{R}_+$ which are holomorphic functions in $Z_{\delta/N}$ with values in $C_1(H)$. This completes the proof of the lemma. □

In the following statements we leave aside the general notation introduced at the beginning of this section, in order to simplify the writing.

Lemma 2.6 (Interior asymptotic expansion). *$\sigma(y,\zeta)$ has an asymptotic expansion*

$$\sigma(y,\zeta) \sim_{\zeta \to \infty} \sum_{j=0}^{\infty} \sigma_j(y)\, \zeta^{N(2-2j-2n)}, \quad \zeta \in Z_{\delta/N}\,, \tag{2.37a}$$

in the sense of [BS1], (1.2a) (*note that* $\dim M = 2$). *In fact,* $\operatorname{tr}_H G^n_{y^N}(\zeta^N; 1, 1)$ *has an asymptotic expansion*

$$\operatorname{tr}_H G^n_{y^N}(\zeta^N; 1, 1) \sim_{\zeta \to \infty} \sum_{j=0}^{\infty} \tilde{a}_j(y)\, \zeta^{N(2-2j-2n)}, \quad \zeta \in Z_{\delta/N}\,, \tag{2.38}$$

which can be differentiated with respect to y. The asymptotics for all derivatives are uniform in $y \in [0,1]$, $\tilde{a}_j \in C^\infty([0,1])$, *and*

$$\sigma_j(y) = N y^{2nN-1} \varphi(y^N)\, \tilde{a}_j(y)\,. \tag{2.37b}$$

Proof. This can be proved as Theorem 4.3 in [BS2]. □

Lemma 2.7 (Integrability condition). *For* $j \in \mathbb{Z}_+$, $\zeta \in Z_{\delta/N}$ *with* $|\zeta| = c_0$ *there is a constant* $c(c_0, j)$ *such that for* $\Theta \in [0,1]$

$$\int_0^1 \int_0^1 s^j \left| \frac{\partial^j \sigma}{\partial y^j}(\Theta s t, s\zeta) \right| ds\, dt \leqq c(c_0, j)\,.$$

Proof. We have, by (2.32) and (2.27),

$$\begin{aligned} \sigma(st, s\zeta) &= N(st)^{2nN-1} \varphi((st)^N) \operatorname{tr}_H G_{(st)^N}((s\zeta)^N; 1, 1) \\ &= N s^{N-1} t^{2nN-1} \varphi((st)^N) \operatorname{tr}_H G_{t^N}(\zeta^N; s^N, s^N)\,, \end{aligned}$$

hence

$$s^j \left(\frac{\partial^j \sigma}{\partial y^j}\right)(st, s\zeta) = N s^{N-1} \partial_t^j [t^{2nN-1} \varphi((st)^N) \operatorname{tr}_H G_{t^N}(\zeta^N; s^N, s^N)]\,.$$

We conclude, again with the Trace Lemma, that for $\psi \in C_0^\infty(\mathbb{R})$ with $\psi | [0,1] = 1$ and $t \in [0,1]$

$$\begin{aligned} \int_0^1 s^j \left| \frac{\partial^j \sigma}{\partial y^j}(st, s\zeta) \right| ds &\leqq c \sum_{k=0}^{j} \int_0^1 s^{N-1} \partial_t^k [t^{2nN-1} \operatorname{tr}_H G_{t^N}(\zeta^N; s^N, s^N)]\, ds \\ &= c \sum_{k=0}^{j} \int_0^1 \operatorname{tr}_H \partial_t^k [t^{2nN-1} \psi G_{t^N}(\zeta^N; s, s)]\, ds \\ &\leqq c \sup_{0 \leqq k \leqq j} \| \partial_t^k [t^{2nN-1} \psi G_{t^N}(\zeta^N)] \|_{C_1(\mathscr{H})} \\ &\leqq c \sup_{0 \leqq k \leqq j,\, t \in [0,1]} \| \partial_t^k (\psi G_{t^N}(\zeta^N)) \|_{C_1(\mathscr{H})}\,. \end{aligned}$$

The last expression is finite in view of (2.36) (with $\beta = 0$) which completes the proof of the lemma. □

We remark that the full strength of the estimate in Theorem 3.9 is needed only in the proof of Lemma 2.7. The SAL together with Theorem 2.4 now gives the following result.

Theorem 2.8. *For* $n > 1$ *and* $\psi \in C_0^\infty(\mathbb{R})$, *with* $\psi = 1$ *near* 0, *we have the following asymptotic expansion as* $z \to \infty$ *in* Z_δ:

$$\begin{aligned} &\operatorname{tr}_{L^2(E)}(\psi(T+z^2)^{-n}) \sim \operatorname{tr}_{\mathscr{H}}(\psi G^n(z)) \\ &\qquad \sim \sum_{j=0}^{\infty} a_j^r(\psi)\, z^{2-2j-2n} + \sum_{j=1}^{\infty} b_j^r z^{2-2j-2n} \log z + \sum_{j=0}^{\infty} c_j^r z^{-j/N-2n}\,. \end{aligned}$$

Here,

(2.39a) $$a_j^r(\psi) = \fint_0^\infty \sigma_j(y)\, y^{N(2-2j-2n)}\, dy = \fint_0^\infty x^{1-2j}\psi(x)\,\tilde{a}_j(x^{1/N})\, dx\,,$$

(2.39b) $$b_j^r = \frac{\sigma_j^{(N(2n+2j-2)-1)}(0)}{(N(2n+2j-2)-1)!\,N} = \frac{1}{(2(j-1)N)!}\,\tilde{a}_j^{(2(j-1)N)}(0)\,,$$

(2.39c) $$c_j^r = \fint_0^\infty \frac{\zeta^{j+2nN-1}}{(j+2nN-1)!}\,\sigma^{(j+2nN-1)}(0,\zeta)\, d\zeta\,.$$

Note that the integrals in (2.39a), (2.39c) do not exist in the usual sense, but need regularization (cf. [BS1], p. 135, for the definition) for which we use here the notation $\fint$. However, (2.37b) and (2.39a) imply that no regularization is needed for $j=0$, hence $\int = \fint$ in this case.

3. Resolvent estimates for regular singular operators

Our basic estimate, Theorem 3.9 below, applies to the class of operators introduced above in (1.9), (1.10), and (1.11). We give an abstract treatment of their main properties in this section. Thus let H be an arbitrary Hilbert space and A_0 a self-adjoint operator in H satisfying

(3.1) $$A_0 \geqq -\frac{1}{4}$$

and, with $C_p(H)$ the von Neumann-Schatten class in H,

(3.2) $$(A_0+I)^{-1} \in C_p(H) \quad \text{for some } p>0\,.$$

The differential operator

(3.3) $$\tau_0 := -\partial_x^2 + X^{-2}A_0\,,$$

where $\partial_x u(x) = \frac{\partial u}{\partial x}(x)$, $X^{-2}u(x) = x^{-2}u(x)$, is well defined and symmetric in

$$\mathscr{H} := L^2(\mathbb{R}_+, H)$$

with domain

(3.4) $$\mathscr{H}^\infty := \bigcap_{k\in\mathbb{N}} C_0^\infty((0,\infty), \mathscr{D}(A_0^k))\,.$$

It is easily seen that $\mathscr{H}^\infty$ is dense in $\mathscr{H}$, and that τ_0 maps $\mathscr{H}^\infty$ to itself. Moreover, Hardy's inequality implies that $\tau_0 \geqq 0$ (this can also be seen from (3.13)).

In our applications, we meet perturbations of τ_0 of the type (1.11). Precisely, we build the perturbation on $\mathscr{H}^\infty$ from the operators (1.10),

$$U_0 = I,\quad U_1(\gamma) = \Omega^\gamma X^{-1}(A_0+I)^{1/2},\quad U_2(\gamma) = \Omega^\gamma \partial_x\,,$$

where Ω denotes multiplication by $\omega(x) := x/(x+1)$ and γ is positive; each U_i maps $\mathscr{H}^\infty$ into itself. Abstractly, we now introduce linear operators, τ, as follows:

(3.5a) τ maps $\mathscr{H}^\infty$ to $\mathscr{H}$,

(3.5b) $(\tau u, u) \geqq 0$ for $u \in \mathscr{H}^\infty$,

(3.5c) $$(\tau u, v) = (\tau_0 u, v) + \sum_{i,j=0}^{2} (C_{ij} U_i u, U_j v) \text{ for } u, v \in \mathscr{H}^\infty,$$

where the coefficients C_{ij} are bounded on $\mathscr{H}$ and commute with multiplication by functions on $\mathbb{R}_+$. Moreover we assume that the C_{ij} have support in a compact interval $[0,\varepsilon]$. We measure the "strength" of the perturbation again by

(3.6) $$\delta(\tau) = \sum_{i,j=0}^{2} \|C_{ij}\|_{\mathscr{L}(\mathscr{H})}\,.$$

By (3.5b), we can form the Friedrichs extension, T, of τ which is our model operator. In view of the remark after (1.12), for the application described in the previous sections we may assume that $\delta(\tau)$ is as small as we want.

Our main goal are certain weighted estimates for the resolvent of T; as before we write

$$G(z) := (T+z^2)^{-1}\,.$$

For the case $\gamma = 1$ these estimates have been derived in [BS2] by an explicit construction of $G(z)$ via the Neumann series. This construction is complicated by the fact that τ is a singular perturbation of τ_0, moreover, the corresponding estimates for $G_0(z) := (T_0 + t^2)^{-1}$, T_0 the Friedrichs extension of τ_0, make extensive use of Bessel functions. Here we propose a much simpler approach based on the following useful concept.

Definition 3.1. A linear operator, S, in $\mathscr{H}$ will be called *controlled by* τ if the following is true:

(3.7a) $$\mathscr{D}(S) = \mathscr{H}^\infty;$$

S is *transposable on* $\mathscr{H}^\infty$ in the sense that there is a linear operator, S^t, with

$$\mathscr{D}(S^t) \supset \mathscr{H}^\infty$$

and

(3.7b) $$(Su, v) = (u, S^t v) \quad \text{for } u, v \in \mathscr{H}^\infty;$$

for some constant, $c(S)$, we have the estimate

(3.7c) $$\|Su\|^2 \leqq c(S)[(\tau u, u) + \|u\|^2] =: c(S)\|u\|_\tau^2\,.$$

The set of all such operators is denoted by $\mathscr{S}$.

Remarks. (1) $\mathcal{S}$ is a linear space, and every $S \in \mathcal{S}$ is closable since, by (3.7b), S^* is densely defined.

(2) The closure of $\mathcal{H}^\infty$ under the norm $\|\cdot\|_\tau$ is $\mathcal{D}(T^{1/2})$. In view of (3.7c), each $S \in \mathcal{S}$ extends to $\mathcal{D}(T^{1/2})$ by continuity. This extension is closed, and (3.7c) persists to hold on $\mathcal{D}(T^{1/2})$. *We will always identify $S \in \mathcal{S}$ with this canonical extension.* Moreover, for $S \in \mathcal{S}$ the operators $S(T+I)^{-1/2}$ and $(T+I)^{-1/2}S^*$ are bounded on $\mathcal{H}$.

(3) Clearly, $\mathcal{L}(\mathcal{H}) \subset \mathcal{S}$. These elements are regarded as trivial and will be assigned the weight $\sigma(S) := 0$. Every unbounded element, S, of $\mathcal{S}$ will be given the weight $\sigma(S) := 1$. Such elements exist: a crucial example is $T^{1/2}$.

(4) If $B \in \mathcal{L}(\mathcal{H})$ and $B^*(\mathcal{H}^\infty) \subset \mathcal{H}^\infty$ then, for $S \in \mathcal{S}$, we have $BS \in \mathcal{S}$.

Our first task is to identify the operators U_i in (1.10) as elements of $\mathcal{S}$.

Theorem 3.2. *Let $\gamma > 0$. If $\delta(\tau)$ is sufficiently small, then $U_i(\gamma) \in \mathcal{S}$, $0 \leqq i \leqq 2$.*

Proof. $i = 0$ is trivial in view of Remark (3) above.

Next we want to show that, for $\gamma > 0$ fixed,

$$\|U_i(\gamma)u\|^2 \leqq c_i \|u\|_{\tau_0}^2, \quad u \in \mathcal{H}^\infty,\ i = 1, 2. \tag{3.8}$$

Once (3.8) is proved we deduce from (3.5c) (with $c_0 := 1$)

$$|\|u\|_\tau^2 - \|u\|_{\tau_0}^2| \leqq \delta(\tau)(\max_{0 \leqq i \leqq 2} c_i)\|u\|_{\tau_0}^2 .$$

If we require

$$\delta(\tau) \max_{0 \leqq i \leqq 2} c_i \leqq \frac{1}{2},$$

then we obtain

$$\frac{1}{2}\|u\|_{\tau_0}^2 \leqq \|u\|_\tau^2 \leqq \frac{3}{2}\|u\|_{\tau_0}^2 . \tag{3.9}$$

(3.8) and (3.9) together imply (3.7c); moreover, we have

$$U_1(\gamma)^t = U_1(\gamma), \quad U_2(\gamma)^t = -U_2(\gamma) + \gamma(X+1)^{-1}(A_0 + I)^{-1/2} U_1(\gamma).$$

Hence the theorem follows from (3.8).

The proof of (3.8) is done in two steps. First we prove it for

$$u \in \mathcal{H}^\infty \cap C_0^\infty((1/2, \infty), H),$$

then we prove it for $u \in \mathcal{H}^\infty \cap C_0^\infty((0,1), H)$. Hence it remains to note only that for $\varphi \in C_0^\infty((-1,1))$ with $\varphi = 1$ near $[-1/2, 1/2]$ we have – as a simple consequence of (3.14) below – the inequality

$$\|\varphi u\|_{\tau_0}^2 + \|(1-\varphi)u\|_{\tau_0}^2 \leqq c(\varphi)\|u\|_{\tau_0}, \quad u \in \mathcal{H}^\infty. \tag{3.10}$$

Thus (3.8) will be proved.

Step 1. Consider $u \in \mathcal{H}^\infty$ with support in $(1/2, \infty)$. From the identity

$$(\tau_0 u, u) = \|u'\|^2 + \|X^{-1}(A_0 + \lambda)^{1/2}u\|^2 - \lambda\|X^{-1}u\|^2$$

(valid for all $u \in \mathcal{H}^\infty$ and all $\lambda \geqq 1/4$) we deduce, with $\lambda = 1/4$,

$$\|U_2(\gamma)u\|^2 \leqq \|u'\|^2 \leqq \|u\|_{\tau_0}^2,$$

and, with $\lambda = 1$,

$$\|U_1(\gamma)u\|^2 \leqq \|X^{-1}(A_0 + I)^{1/2}u\|^2 \leqq 4\|u\|_{\tau_0}^2 .$$

Step 2. Now we introduce the self-adjoint and discrete operator

$$B_0 := -\frac{1}{2} + \left(A_0 + \frac{1}{2}\right)^{1/2}, \quad \mathcal{D}(B_0) = \mathcal{D}((A_0 + I)^{1/2}). \tag{3.11}$$

Then it is easily checked that the linear operator D on $\mathcal{H}^\infty$, defined by

$$Du := \partial_x u + X^{-1}B_0, \tag{3.12}$$

satisfies

$$D^*Du = \tau_0 u, \quad u \in \mathcal{H}^\infty. \tag{3.13}$$

Consequently, for $u \in \mathcal{H}^\infty$,

$$\|u\|_{\tau_0}^2 = \|Du\|^2 + \|u\|^2 . \tag{3.14}$$

Choose now an orthonormal basis, $(e_i)_{i \in \mathbb{N}}$, of H such that $B_0 e_i = b_i e_i$, $b_i \geqq -1/2$. For $u_i \in C_0^\infty((0,1))$ we put

$$u_i'(x) + \frac{b_i}{x}u_i(x) =: v_i(x) =: D_i u_i(x),$$

and obtain, since $u_i(0) = u_i(1) = 0$,

$$u_i(x) = \int_0^x (y/x)^{b_i} v_i(y)\,dy =: P_{b_i,r} v_i(x)$$

$$= -\int_x^1 (y/x)^{b_i} v_i(y)\,dy =: P_{b_i,a} v_i(x).$$

On the space $\mathcal{H}^0 := \{\sum_{i \geqq 1} u_i e_i \mid u_i \in C_0^\infty(0,1), u_i \neq 0$ only for finitely many $i\}$ we then obtain the identity $u = PDu$, where

$$P = P_{-1/2,a} \oplus \bigoplus_{b_i > -1/2} P_{b_i,r} .$$

Schur's test shows that P extends to $\mathscr{H}$ by continuity and that, moreover,

$$\sup_{b_i > -1/2} \|(b_i^2 + b_i + 1)^{1/2} \Omega^\gamma X^{-1} P_{b_i,r}\| < \infty . \tag{3.15a}$$

To handle the case $b_i = -1/2$ we estimate

$$|P_{-1/2,a} v(x)| = \left| \int_x^1 (y/x)^{-1/2} v(y)\, dy \right| \leqq x^{1/2} |\log x| \|v\|_{L^2(0,1)} ;$$

this implies that

$$\|\Omega^\gamma X^{-1} P_{-1/2,a}\| < \infty . \tag{3.15b}$$

(3.15) now says that $U_1(\gamma) P$ extends from $\mathscr{H}^0$ to $L^2([0,1], H)$ by continuity, thus

$$\|U_1(\gamma) u\|^2 = \|U_1(\gamma) P D u\|^2 \leqq C_1 \|Du\|^2 \leqq C_1 \|u\|_{\tau_0}^2, \quad u \in \mathscr{H}^0 . \tag{3.16a}$$

Observing that

$$U_2(\gamma) P_{b_i,r/a} v_i(x) = (\Omega^\gamma D_i - B_0 (A_0 + I)^{-1/2} U_1(\gamma)) P_{b_i,r/a} v_i(x) = (\Omega^\gamma - B_0 (A_0 + I)^{-1/2} U_1(\gamma) P_{b,r/a}) v_i(x) ,$$

we also obtain that $U_2(\gamma) P$ extends boundedly, hence

$$\|U_2(\gamma) u\|^2 \leqq C_2 \|u\|_{\tau_0}^2, \quad u \in \mathscr{H}^0 . \tag{3.16b}$$

Finally, we note that $\mathscr{H}^0$ is dense in $\mathscr{H}^\infty \cap C_0^\infty((0,1), H)$ with respect to $\|\cdot\|_{\tau_0}^2$ so that (3.16) holds also on this space. □

The first step towards the estimate in Theorem 3.9 is the following boundedness result.

Lemma 3.3. *For $S_i \in \mathscr{S}$, $i = 1, 2$, and $z \in Z_\delta$, the operator $S_1 G(z) S_2^* : \mathscr{D}(S_2^*) \to \mathscr{H}$ extends to $\mathscr{H}$ by continuity, with uniform norm bound in Z_δ:*

$$\|S_1 G(z) S_2^*\|_{\mathscr{L}(\mathscr{H})} \leqq c(S_1, S_2, \delta) . \tag{3.17}$$

Proof. From Remark (2) above we know that for $S \in \mathscr{S}$

$$S(I+T)^{-1/2}, \quad (I+T)^{-1/2} S^* \in \mathscr{L}(\mathscr{H}) .$$

Consequently, for $u \in \mathscr{D}(S_2^*)$,

$$\|S_1 G(z) S_2^* u\| \leqq \|S_1 (I+T)^{-1/2}\| \|(I+T)^{1/2} G(z) (I+T)^{1/2}\| \|(I+T)^{-1/2} S_2^* u\| \leqq c(S_1, S_2, \delta) \|u\| ,$$

using the Spectral Theorem. □

Next we have to bring in the weight operators $(I+X)^\mu$, $\mu \in \mathbb{R}$. The following result is an obvious consequence of (3.14) and (3.9).

Lemma 3.4. *If $g \in C^\infty((0,\infty))$ satisfies*

$$\sup_{x>0} |g(x)| + \sup_{x>0} |g'(x)| < \infty ,$$

then multiplication by g on $\mathscr{H}^\infty$ extends to a continuous map of $\mathscr{D}(T^{1/2})$ to itself.

In particular, the lemma applies to $g(x) = (1+x)^\mu$, $\mu \leqq 0$.

Next we want to introduce the operators $(1+X)^\mu G(z) (1+X)^\nu$, which are a priori not well-defined. As a preparation we need the

Lemma 3.5. *For $\mu \in \mathbb{R}$ and $u \in \mathscr{H}^\infty$ we have*

$$(I+X)^\mu G(z) u \in \mathscr{D}(T^{1/2}) .$$

Proof. First we recall the following notation introduced above: for a real function, denoted by a lower case letter, we denote the corresponding multiplication operator by the corresponding capital letter. By the remark following the preceding lemma it suffices to prove the assertion for $\mu \in \mathbb{Z}_+$. We choose R so large that $\tau | [R, \infty) = \tau_0$, and we choose $\psi \in C^\infty(\mathbb{R})$ with $\psi | [R+1, \infty) = 1$, $\psi | [-\infty, R] = 0$. Then by Lemma 3.4 we have

$$(1 - \Psi)(I+X)^\mu G(z) u \in \mathscr{D}(T^{1/2})$$

for $\mu \in \mathbb{R}$.

Now we show

$$\Psi (I+X)^\mu G(z) u \in \mathscr{D}(T), \quad \mu \in \mathbb{Z}_+ ,$$

by induction on μ. For $\mu = 0$ this is obvious. Next we choose a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R})$ with

(i) $0 \leqq \varphi_n \leqq 1$, $\varphi_n | (-\infty, n] = 1$,

(ii) $\varphi_n(x) = 0$, $x \geqq 2n$,

(iii) $\|\varphi_n^{(j)}\|_\infty \leqq c/n^j$, $j = 1, 2$.

Then $\Phi_n''(I+X) \to 0$ in $\mathscr{L}(\mathscr{H})$ and $\Phi_n'(I+X) \to 0$ strongly since for $\xi \in \mathscr{H}$ we have

$$\|\Phi_n'(I+X)\xi\|^2 \leqq c \int_n^{2n} \|\xi(x)\|_H^2 dx \to 0, \quad n \to \infty .$$

Moreover, we have again by Lemma 3.4,

$$w_n := \Phi_n \Psi (I+X)^{\mu+1} G(z) u \in \mathscr{D}(T^{1/2}) . \tag{3.18}$$

Now we compute

$$(\tau+z^2)\Phi_n\Psi(I+X)^{\mu+1}G(z)u=\Phi_n\Psi(I+X)^{\mu+1}u+[\tau,\Phi_n\Psi(I+X)^{\mu+1}]G(z)u,$$

$$[\tau,\Phi_n\Psi(I+X)^{\mu+1}]=[\tau,\Phi_n(I+X)]\Psi(I+X)^{\mu}$$
$$+\Phi_n(I+X)[\tau,\Psi](I+X)^{\mu}+\Phi_n\Psi(I+X)[\tau,(I+X)^{\mu}],$$

$$[\tau,\Phi_n(I+X)]=-\Phi_n''(I+X)-2\Phi_n'-2\Phi_n'(I+X)\partial-2\Phi_n\partial.$$

By the induction hypothesis, $(I+X)^{\mu}G(z)u\in\mathscr{D}(T^{1/2})$, and since $\partial\Psi=\Psi'+\Psi\partial\in\mathscr{S}$ we find that $[\tau,\Phi_n(I+X)]\Psi(I+X)^{\mu}G(z)u\in\mathscr{H}$ and that it converges as $n\to\infty$.

Furthermore, since

$$(I+X)[\tau,(I+X)^{\mu}]=\mu(\mu+1)(I+X)^{\mu-1}-2\mu\partial(I+X)^{\mu}$$

and $[\tau,\Psi]\in\mathscr{S}$ with $\operatorname{supp}[\tau,\Psi]\subset[R,R+1]$ we analogously conclude that

(3.19) $$v_n:=(\tau+z^2)\Phi_n\Psi(I+X)^{\mu+1}G(z)u\in\mathscr{H},$$

and that $v:=\lim_{n\to\infty}v_n$ exists in $\mathscr{H}$.

Thus, (3.18) and (3.19) imply that $w_n\in\mathscr{D}(T)$. Moreover,

$$w_n=\Phi_n\Psi(I+X)^{\mu+1}G(z)u=G(z)v_n\to G(z)v,\quad n\to\infty,$$

so w_n converges in $\mathscr{D}(T)$ to $\Psi(I+X)^{\mu+1}G(z)u$. □

We introduce the operator $[G(z)]_{\mu\nu}:\mathscr{H}^{\infty}\to\mathscr{H}$,

(3.20) $$[G(z)]_{\mu\nu}u:=(I+X)^{\mu}G(z)(I+X)^{\nu}u\in\mathscr{D}(T^{1/2})\subset\mathscr{H},$$

$z\in Z_{\delta}$. By the preceding lemma, $[G(z)]_{\mu\nu}$ is well defined.

Theorem 3.6. *Let* $\mu+\nu\leqq 0$.

(1) $[G(z)]_{\mu\nu}$ *defines a bounded operator in* $\mathscr{H}$ *(to be denoted by the same symbol).*

(2) *The operator* $[G(z)]_{\mu\nu}$ *and its adjoint map into* $\mathscr{D}(T^{1/2})$.

(3) *For* $S_i\in\mathscr{S}$, $i=1,2$, $z\in Z_{\delta}$, *the operator* $S_1[G(z)]_{\mu\nu}S_2^*:\mathscr{D}(S_2^*)\to\mathscr{H}$ *extends to* $\mathscr{H}$ *with uniform norm bound,*

(3.21) $$\|S_1[G(z)]_{\mu\nu}S_2^*\|_{\mathscr{L}(\mathscr{H})}\leqq c(S_1,S_2,\delta,\mu,\nu).$$

Proof. Again, the proof is divided into several steps. First, we prove the result for $\nu\in\mathbb{Z}_+$, by induction on ν. Next, we prove it for $\nu\in\mathbb{R}_+$, finally for arbitrary real ν.

1. *Step.* For $\nu=0$, $\mu\leqq 0$, $[G(z)]_{\mu 0}$ is clearly a bounded operator which maps into $\mathscr{D}(T^{1/2})$, in view of Lemma 3.4. Using the continuity of multiplication with $(1+x)^{\mu}$ in $\mathscr{D}(T^{1/2})$, we can proceed as in the proof of Lemma 3.3: for $u\in\mathscr{D}(S_2^*)$,

$$\|S_1[G(z)]_{\mu 0}S_2^*u\|_{\mathscr{L}(\mathscr{H})}\leqq\|S_1(I+T)^{-1/2}\|\,\|(I+T)^{1/2}(I+X)^{\mu}(I+T)^{-1/2}\|$$
$$\cdot\|(I+T)^{1/2}G(z)(I+T)^{1/2}\|\,\|(I+T)^{-1/2}S_2^*u\|$$
$$\leqq c(S_1,S_2,\delta,\mu,0)\|u\|.$$

Since $[G(z)]_{\mu 0}^*=G(\bar z)(I+X)^{\mu}$ we conclude again from Lemma 3.4 that $[G(z)]_{\mu 0}^*$ also maps into $\mathscr{D}(T^{1/2})$.

Now assume that the theorem has been proved for $\nu\in\mathbb{Z}_+$, $\nu\leqq\nu_0$. Then we compute for $u,v\in\mathscr{H}^{\infty}$ and $\gamma>0$

(3.22) $$([\tau+z^2,(I+X)^{\nu_0+1}]u,v)$$
$$=(D(I+X)^{\nu_0+1}u,Dv)-(Du,D(I+X)^{\nu_0+1}v)$$
$$+\sum_{i,j=0}^{2}\{(C_{ij}U_i(\gamma)(I+X)^{\nu_0+1}u,U_j(\gamma)v)$$
$$-(C_{ij}U_i(\gamma)u,U_j(\gamma)(I+X)^{\nu_0+1}v)\}$$
$$=:\sum_{k,l=1}^{M}(V_ku,W_l(I+X)^{\nu_0}v),$$

for certain operators $V_k,W_l\in\mathscr{S}$, $1\leqq k,l\leqq M$, and some $M\in\mathbb{N}$. In fact, this follows from $[C_{ij},(I+X)^{\nu}]=0$, $0\leqq i,j\leqq 2$, $\nu\in\mathbb{R}$, and the identities

$$[\tilde U,(I+X)^{\nu_0+1}]=(I+X)^{\nu_0}\tilde U_r=\tilde U_l(I+X)^{\nu_0},\quad \tilde U_{r/l}\in\mathscr{S},$$

which are valid for $\tilde U=D,\,U_1(\gamma),\,U_2(\gamma)$, and $\nu_0\in\mathbb{Z}$.

The computation (3.22) is in fact valid for $v\in\mathscr{H}$ with $(I+X)^{\mu}v\in\mathscr{D}(T^{1/2})$ for all μ, thus it holds by the induction hypothesis with $G(\bar z)(I+X)^{\mu}v$ in place of v. We find

$$((\tau+z^2)u,[G(z)]_{\mu,\nu_0+1}^*v)$$
$$=((\tau+z^2)u,(G(\bar z)(I+X)^{\mu+\nu_0+1}-\sum_{k,l=1}^{M}G(\bar z)V_k^*W_l[G(z)]_{\mu\nu_0}^*)v).$$

Observe now that for $\xi\in\mathscr{D}(T^{1/2})$ the identity $((\tau+z^2)u,\xi)=0$ for all $u\in\mathscr{H}^{\infty}$ implies $\xi=0$ since $\mathscr{H}^{\infty}$ is dense in $\mathscr{D}(T^{1/2})$. The induction hypothesis thus implies

(3.23) $$[G(z)]_{\mu,\nu_0+1}^*v=(G(\bar z)(I+X)^{\mu+\nu_0+1}-\sum_{k,l=1}^{M}G(\bar z)V_k^*W_l[G(z)]_{\mu\nu_0}^*)v,$$

and the right hand side defines a bounded operator with adjoint

$$(3.24)\qquad [G(z)]_{\mu,\nu_0+1} = (I+X)^{\mu+\nu_0+1}G(z) - \sum_{k,l=1}^{M} [G(z)]_{\mu\nu_0} W_l^* V_k G(z).$$

(3.24) and the induction hypothesis imply (3.21) for $[G(z)]_{\mu,\nu_0+1}$.

The desired mapping properties follow from (3.24) which implies that for

$$S \in \mathscr{S}, \quad u \in \mathscr{D}(T^{1/2}), \quad v \in \mathscr{D}(S^*)$$

we have the estimate

$$|(T^{1/2}u, [G(z)]_{\mu,\nu_0+1} S^* v)| \leqq C\|u\|\|v\|,$$

since $T^{1/2} \in \mathscr{S}$.

2. *Step.* Since $[G(z)]_{\mu,\nu} = (I+X)^{\mu+\nu}[G(z)]_{-\nu,\nu}$, it is enough to consider the case $\mu = -\nu, \nu \geqq 0$. We apply complex interpolation to the function

$$([G(z)]_{-\nu,\nu}u, v), \quad u, v \in \mathscr{H}^\infty, \quad \operatorname{Re}\nu \in [\nu_0, \nu_0+1],$$

to see that $[G(z)]_{-\nu,\nu}$ is a bounded operator. Then we can consider

$$(S_1[G(z)]_{-\nu,\nu} S_2^* u, v), \quad S_i \in \mathscr{S}, \quad u \in \mathscr{D}(S_2^*), \quad v \in \mathscr{D}(S_1^*),$$

and the same method proves (3.21).

The mapping property of $[G(z)]_{-\nu,\nu}$ follows as above, for the adjoint we use the representation (3.23).

3. *Step.* If $\mu+\nu \leqq 0, \nu < 0$, we use the identity $[G(z)]_{\mu\nu} = [G(\bar z)]^*_{\nu\mu}$. □

To complete our estimates we have to bring in von Neumann-Schatten norms. As in [BS2], we construct a suitable comparison operator as follows.

Lemma 3.7. *Let $\varphi \in C^\infty \mathbb{R})$ satisfy*

$$\varphi(x) > 0 \quad \text{for} \quad x > 0,$$

and

$$(3.25)\qquad \varphi(x) = \begin{cases} x, & \text{for } x \text{ near } 0, \\ x^{2\alpha}, & \alpha > 1, \text{ for } x \text{ near } \infty. \end{cases}$$

Then the differential operator

$$(3.26)\qquad \varrho := -\partial_x \varphi \partial_x + A_0 + I$$

is bounded below by 3/4 on $\mathscr{H}^\infty$ so that its Friedrichs extension, R, exists. Then $R \geqq 3/4$ and with p from (3.2)

$$(3.27)\qquad R^{-1} \in C_{p+1/2}(\mathscr{H}).$$

Proof. We only have to prove (3.27). We introduce the unitary transformation

$$\Phi_1 : L^2(\mathbb{R}_+, H, \varphi^{-1/2}dx) \ni u \mapsto \varphi^{-1/4}u \in \mathscr{H}$$

and compute

$$\begin{aligned}\Phi_1^* \varrho &= -\varphi^{1/4}\partial_x\varphi\partial_x\varphi^{-1/4} + A_0 + I \\ &= -\varphi^{1/2}\partial_x\varphi^{1/2}\partial_x + A_0 + I + B,\end{aligned}$$

where B is multiplication by

$$b(x) := (4\varphi)^{-1}\left(\varphi\varphi'' - \frac{1}{2}\varphi'^2\right)(x).$$

For x near ∞ we thus find

$$\begin{aligned}b(x) &= \frac{1}{2}\alpha(2\alpha-1)x^{2\alpha-2} - \frac{1}{2}\alpha^2 x^{2\alpha-2} \\ &= \frac{3\alpha^2-2\alpha}{4}x^{2(\alpha-1)}.\end{aligned}$$

We put

$$s(x) := \int_0^x \varphi^{-1/2}(t)\,dt$$

and

$$c(\varphi) := s(\infty) = \int_0^\infty \varphi^{-1/2}(t)\,dt < \infty.$$

Then, under the unitary transformation

$$\Phi_2 : L^2([0, c(\varphi)], H) \ni u \mapsto u \circ s \in L^2(\mathbb{R}_+, H, \varphi^{-1/2}dx),$$

we obtain

$$\Phi_2^*\Phi_1^*\varrho = -\partial_x^2 + B \circ s^{-1} + A_0 + I =: \tilde\varrho.$$

Again, $\tilde\varrho$ is $\geqq 3/4$ on $\bigcap_{k\geqq 0} C_0^\infty((0, c(\varphi)), \mathscr{D}(A_0^k))$, and its Friedrichs extension, $\tilde R$, is unitarily equivalent to R. For $\tilde R$ we can separate variables; observing that $-\partial_x^2 + B \circ s^{-1}$ has eigenvalues $\lambda_n \geqq n^2$ we obtain as in [BS2], Lemma 3.5, that $\tilde R^{-1} \in C_{p+1/2}(\mathscr{H})$. □

Lemma 3.8. *With α in* (3.25) *we have*

$$R^{1/2}(I+X)^{-\alpha} \in \mathscr{S}.$$

Proof. For $u \in \mathscr{H}^\infty$ we have, by straightforward estimates,

$$\|R^{1/2}(I+X)^{-\alpha}u\|^2 = \|\varphi^{1/2}\partial_x(I+X)^{-\alpha}u\|^2 + \|(I+X)^{-\alpha}(A_0+I)^{1/2}u\|^2$$
$$\leqq 2\|\varphi^{1/2}(I+X)^{-\alpha}\partial_x u\|^2 + 2\|\alpha\varphi^{1/2}(I+X)^{-\alpha-1}u\|^2$$
$$+\|(I+X)^{-\alpha}(A_0+I)^{1/2}u\|^2$$
$$\leqq c(R)\sum_{i=1}^{2}\|U_i(1/2)u\|^2. \quad \square$$

We are now ready to prove our fundamental estimate. One further bit of notation: for $S_1, S_2 \in \mathcal{S}$ we write

$$\bar\sigma(S_1, S_2) := 2 - \sigma(S_1) - \sigma(S_2),$$

and we put $C_\infty(\mathcal{H}) := \mathcal{L}(\mathcal{H})$.

Theorem 3.9. *Fix $\alpha > 1$. For $z \in Z_\delta$, $S_1, S_2 \in \mathcal{S}$, and $\mu + \nu \leqq -\alpha\bar\sigma(S_1, S_2)$ we have the estimate*

$$\|S_1[G(z)]_{\mu\nu}S_2^*\|_{C_{(2p+1)/\bar\sigma(S_1,S_2)}(\mathcal{H})} \leqq c(\delta, S_1, S_2, \mu, \nu).$$

Proof. If $\bar\sigma = 0$ then the assertion follows from (3.21). Assume $\sigma(S_1) = 0$, $\sigma(S_2) = 1$, and write

$$S_1 = S_1 R^{-1/2} R^{1/2}(I+X)^{-\alpha}(I+X)^{\alpha}.$$

Then the assertion follows again from (3.21) since $\alpha + \mu + \nu \leqq 0$ and $R^{-1/2} \in C_{2p+1}(\mathcal{H})$. The remaining cases are treated similarly. $\square$

4. Computations

We begin this section with the proof of (1.8). In the parametrization ψ, described in (1.6), the given hermitian metric on $\mathbb{CP}^n$ is defined by a positive form in $\Omega^{1,1}(M)$,

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n} h_{ij}(z)\,dz_i \wedge d\bar z_j,$$

so the induced metric on a single branch U is given by

$$\psi^*\omega = \frac{\sqrt{-1}}{2}\sum_{i,j=k}^{n} h_{ij}\circ\psi(z)\,dP_i(z)\wedge d\overline{P_j(z)}$$
$$= \frac{\sqrt{-1}}{2}\sum_{i,j=k}^{n} h_{ij}\circ\psi(z)\,P_i'(z)\overline{P_j'(z)}\,dz\wedge d\bar z$$
$$=: \frac{\sqrt{-1}}{2}h(z,\bar z)\,dz\wedge d\bar z.$$

In view of (1.7) it is easy to see that

$$h(z,\bar z) = h_{kk}(0)\,N_k^2|z|^{2(N_k-1)} + O(|z|^{2N_k-1}),$$

so in polar coordinates (r,θ) for D_ε we obtain the induced Riemannian metric

$$g := h(r,\theta)(dr^2 + r^2 d\theta^2)$$

where

$$h(r,\theta) = h_{kk}(0)\,N_k^2 r^{2(N_k-1)} + O(r^{2N_k-1}).$$

Now we substitute $x := (h_{kk}(0))^{1/2} r^{N_k}$ so that

$$h(r,\theta) =: \left(\frac{dx}{dr}\right)^2 h_1(r,\theta), \quad h_1(0,\theta) \equiv 1,$$

and obtain

$$g = h_1(x^{1/N_k}(h_{kk}(0))^{-1/2N_k},\theta)(dx^2 + N_k^2 x^2 d\theta^2)$$
$$=: \alpha(x^{1/N_k},\theta)^2(dx^2 + N_k^2 x^2 d\theta^2),$$

as desired.

Having established the existence part (1.2) of Theorem 1.2 we now turn to the explicit computations. We begin with the coefficients of type (2.39a), $a_j^r(\varphi)$, which can be calculated in terms of the coefficients $\tilde a_j$ in the expansion (2.38). These, in turn, can be related to the coefficients B_j^r in (2.15b) which are, at least in principle, explicitly computable in terms of the geometry. So we look again at a single component, U, near a singularity, of multiplicity N, and we assume as before that U is isometric to $(0,\varepsilon)\times[0,2\pi)$ with the metric (1.8). We denote by x the coordinate in $(0,\varepsilon)$ and by $\pi: U \to (0,\varepsilon)$ the projection onto the first factor. Then we have

Lemma 4.1. *For $j \geqq 0$, $x \in (0,\varepsilon)$,*

$$x^{1-2j}\tilde a_j(x^{1/N}) = \int_{\pi^{-1}(x)} \alpha B_j^r(n;\cdot) =: \overline{B_j^r}(x). \tag{4.1}$$

Proof. From Lemma 2.3 we have the expansion

$$\operatorname{tr}_{L^2(M)}[\varphi(T_0+z^2)^{-n}] \sim \sum_{j\geqq 0} z^{2(1-j-n)}\int_M \varphi(p)B_j^r(n;p)\operatorname{vol}_M(p), \tag{4.2}$$

valid for $\varphi \in C_0^\infty(M)$; let $\varphi = \psi\circ\pi$, $\psi \in C_0^\infty(0,\varepsilon)$. Then we deduce from (2.31), (2.37a), and (2.37b)

$$\operatorname{tr}_{L^2(M)}[\varphi(T_0+z^2)^{-n}] = \int_0^\infty Ny^{2nN-1}\psi(y^N)\operatorname{tr}_{L^2(S^1)}G^n_{y^N}((yz^{1/N})^N;1,1)\,dy \tag{4.3}$$
$$\sim \sum_{j\geqq 0} z^{2(1-j-n)}\int_0^\infty Ny^{N(2-2j)-1}\psi(y^N)\tilde a_j(y)\,dy$$
$$= \sum_{j\geqq 0} z^{2(1-j-n)}\int_0^\infty x^{1-2j}\psi(x)\tilde a_j(x^{1/N})\,dx.$$

But

$$\int_U \varphi(p) B_j^r(n;p)\,\mathrm{vol}_M(p) = \int_0^\varepsilon \int_0^{2\pi} \psi(x) B_j^r(n;x,\theta)\, xN\alpha\beta(x^{1/N},\theta)\,d\theta\,dx$$

$$= \int_0^\infty \psi(x)\left[\int_{\pi^{-1}(x)} \alpha B_j^r(n;\cdot)\right]dx\,. \quad \square$$

Corollary 4.2.

$$A_{00}^h = a_0 = \frac{\mathrm{vol}\,M}{4\pi}\,.$$

Proof. We only have to observe that for $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ near 0, $\varphi := \psi \circ \pi$, from (2.39a) and (4.1)

$$a_0^r(\varphi) = \int_0^\varepsilon x\varphi(x)\,\tilde{a}_0(x^{1/N})\,dx$$

$$= \int_0^\varepsilon \varphi(x) \int_{\pi^{-1}(x)} \alpha B_0^r(n;\cdot)\,dx$$

$$= \int_U \varphi B_0^r(n;p)\,\mathrm{vol}_M(p) = A_{00}^r(\varphi)\,,$$

since the integrals converge. So, by Lemma 2.3,

$$A_{00}^r = \int_M B_0^r(n;p)\,\mathrm{vol}_M(p)\,.$$

In view of (2.10c) and (2.17) we also have

$$A_{00}^h = \int_M B_0^h(p)\,\mathrm{vol}_M(p)\,,$$

implying the assertion since $B_0^h = (4\pi)^{-1}$. $\square$

We have to study the limiting behavior of the geometry of $\pi^{-1}(x)$ in order to proceed. We introduce the curve

$$c_x : [0, 2\pi] \ni t \mapsto (x,t) \in U\,.$$

Lemma 4.3. *In the natural orientation of U we have*

$$\lim_{x\to 0} \int_{\pi^{-1}(x)} \kappa_g(c_x) = 2\pi N\,,$$

where κ_g denotes the geodesic curvature.

Proof. We use the coordinates from (1.8) such that the basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial\theta}\right\}$ is oriented. With $e_1(t) := \dot{c}_x(t)/|\dot{c}_x(t)|$ and $e_2(t) := *_M e_1(t)$ we have

$$\kappa_g(c_x) = \langle \nabla_{e_1(t)} e_1(t), e_2(t)\rangle = \langle e_1(t), [e_2, e_1](t)\rangle\,.$$

Since

$$e_1(t) = (Nx\beta)^{-1}\frac{\partial}{\partial\theta}\bigg|_{c_x(t)}, \quad e_2(t) = -\alpha^{-1}\frac{\partial}{\partial x}\bigg|_{c_x(t)},$$

we compute

$$\langle e_1(t), [e_2, e_1](t)\rangle = \left\langle -\alpha^{-1}\frac{\partial}{\partial x}(Nx\beta)^{-1}\frac{\partial}{\partial\theta}, (Nx\beta)^{-1}\frac{\partial}{\partial\theta}\right\rangle$$

$$= \frac{x\beta}{\alpha}\,\frac{\beta + (1/N)\,x^{1/N}\partial_x\beta}{(x\beta)^2}$$

$$= \left(\beta + (1/N)\,x^{1/N}\frac{\partial\beta}{\partial x}\right)(\alpha x\beta)^{-1}(x^{1/N},\theta)\,.$$

This implies

$$\int_{\pi^{-1}(x)} \kappa_g(c_x) = \int_0^{2\pi}\left(\beta + (1/N)\,x^{1/N}\frac{\partial\beta}{\partial x}\right)(\alpha x\beta)^{-1} Nx\beta(x^{1/N},\theta)\,d\theta$$

$$= N\int_0^{2\pi}\alpha^{-1}\left(\beta + (1/N)\,x^{1/N}\frac{\partial\beta}{\partial x}\right)(x^{1/N},\theta)\,d\theta$$

$$= 2\pi N + O(x^{1/N})\,. \quad \square$$

Note that the orientation of c_x is opposite to the boundary orientation required in the Gauß-Bonnet Theorem. We turn to the computation of a_1; note that this term is not well defined by the statement of Theorem 1.2.

Lemma 4.4. *With ψ as in Theorem* 2.8, $\varphi := \psi \circ \pi$, *we have*

$$a_1 := a_1^h(\psi) + A_{10}^h(1-\varphi)$$

$$= \frac{1}{6}\chi_{(2)}(M) + \frac{1}{6}\sum_{\substack{p\in\Sigma\\ 1\le i\le L(p)}} (N_i(p) - 1)\,.$$

Proof. For simplicity of notation, we assume that we have only one singularity and one singular branch of multiplicity N; the general case is an obvious extension. Then we have from (2.39a) and (2.10c)

$$a_1 = a_1^h(\psi) + A_{10}^h(1-\varphi) = a_1^r(\psi) + A_{10}^r(1-\varphi) \tag{4.4}$$

$$= \oint_0^\infty x^{-1}\psi(x)\,\tilde{a}_1(x^{1/N})\,dx + \int_M (1-\varphi(p))\,B_1^r(n;p)\,\mathrm{vol}_M(p)\,.$$

Now it is well known (cf. [BGM], p. 222) that

$$B_1^r(n;p) = B_1^h(p) = (12\pi)^{-1}K_M(p), \quad p \in M\,, \tag{4.5}$$

where K_M denotes the Gauß curvature. From Lemma 4.1 we get

$$x^{-1}\tilde{a}_1(x^{1/N}) = \int_{\pi^{-1}(x)} \alpha(12\pi)^{-1} K_M\,,$$

hence, with $U_\delta := (0,\delta)\times S^1 \subset U$, we obtain as $\delta \to 0$

$$\begin{aligned}(4.6)\qquad \int_{U\setminus U_\delta} K_M(p)\operatorname{vol}_M(p) &= 12\pi \int_\delta^\varepsilon x^{-1}\tilde{a}_1(x^{1/N})\,dx\\ &= -\tilde{a}_1(0)12\pi\log\delta + O(1)\\ &= 2\pi\chi(U\setminus U_\delta) - \int_{\pi^{-1}(\varepsilon)} \kappa_g(c_\varepsilon) + \int_{\pi^{-1}(\delta)} \kappa_g(\delta)\\ &= O(1)\,;\end{aligned}$$

here we have used the Gauß-Bonnet Theorem for surfaces with boundary [dC], p. 274, and Lemma 4.3 in the last two identities, and χ denotes the Euler characteristic.

The identities (4.6) yield

$$(4.7)\qquad \tilde{a}_1(0) = 0\,,$$

so we can replace ψ with $\psi_\delta :=$ characteristic function of $[0,\delta]$ in (4.4), $\delta \in (0,\varepsilon)$. It follows that

$$(4.8)\qquad \lim_{\delta\to 0} \int_0^\infty x^{-1}\psi_\delta(x)\tilde{a}_1(x^{1/N})\,dx = 0\,.$$

Next we write $M_\delta := M\setminus U_\delta$. Using the Gauß-Bonnet Theorem and Lemma 4.3 again we find

$$\begin{aligned}(4.9)\qquad \frac{1}{2\pi}\int_{M_\delta} K_M(p)\operatorname{vol}_M(p) &= \chi(M_\delta) + \frac{1}{2\pi}\int_{\pi^{-1}(\delta)} \kappa_g(c_\delta)\\ &= \chi(M_\delta) + N + O(\delta^{1/N})\,.\end{aligned}$$

To evaluate $\chi(M_\delta)$ we use the Mayer-Vietoris sequence in L^2-cohomology ([C2], Lemma 4.3, [Z], Prop. 1.18) to obtain

$$(4.10)\qquad \chi_{(2)}(M) = \chi_{(2)}(M_\delta) + \chi_{(2)}(U_\delta) - \chi_{(2)}(S^1)\,.$$

Since S^1 and $\bar{M}_\delta$ are compact manifolds with boundary we have (cf. e.g. [BL1], Thm. 4.1)

$$\begin{aligned}\chi_{(2)}(S^1) &= \chi(S^1) = 0\,,\\ \chi_{(2)}(M_\delta) &= \chi(M_\delta)\,.\end{aligned}$$

U_δ with the metric (1.8) is quasi-isometric to the metric cone, CS^1, over S^1, hence its L^2-cohomology is known ([C2], Lemma 3.4, [Z], 2.41) to be

$$H^k_{(2)}(CS^1) = \begin{cases}\mathbb{R}, & k=0\,,\\ 0, & k>0\,.\end{cases}$$

so

$$\chi_{(2)}(U_\delta) = 1\,.$$

Using this in (4.10) we derive

$$(4.11)\qquad \chi(M_\delta) = \chi_{(2)}(M) - 1\,,$$

hence from (4.11) and (4.9)

$$(4.12)\qquad \lim_{\delta\to 0}\frac{1}{2\pi}\int_{M_\delta} K_M(p)\operatorname{vol}_M(p) = \chi_{(2)}(M) + N - 1\,.$$

Finally, we plug (4.5), (4.8), and (4.12) into (4.4) completing the proof. □

We remark in passing that $\chi_{(2)}(M) = \chi(\tilde{M})$, $\tilde{M}$ the normalization of M.

Next we turn to the logarithmic terms. From (2.39 b) and (4.7) we get

$$(4.13)\qquad b_1^r = \tilde{a}_1(0) = 0\,.$$

In principle, the coefficients b_j^r are computable by algebraic manipulations involving only the derivatives of the function α in (1.8) at $x=0$. In fact, from (2.39 b) and Lemma 4.1 the following assertion is obvious.

Lemma 4.5. *The functions $\overline{B_j^r}$ defined in* (4.1) *have asymptotic expansions in powers of $x^{1/N}$ as $x\to 0$. If we denote by* Res *the coefficient of x^{-1} in this expansion then*

$$(4.14)\qquad b_j^r = \operatorname{Res}\overline{B_j^r}(0)\,.$$

This relation explains the logarithmic coefficients more geometrically; it reflects the conic scaling (cf. also [C1]).

To do explicit computations, we observe that in the special case of (1.8) where

$$(4.15)\qquad \alpha(x^{1/N},\theta) = \beta(x^{1/N},\theta) = h(x^{1/N})\,,$$

for some $h \in C^\infty([0,\varepsilon))$ with $h(0)=1$, we can substitute

$$(4.16\,a)\qquad y(x) := \int_0^x h(t^{1/N})\,dt\,,\quad y\in[0,y(\varepsilon))\,,$$

to arrive at the metric

$$(4.16\,b)\qquad dy^2 + h_1(y)^2 d\theta^2\,,\quad h_1(y(x)) = Nxh(x^{1/N})\,.$$

These metrics are known as 'warped products'; observe that they are of the type (1.8), too, with $\alpha\equiv 1$ and β independent of θ.

The first candidate is b_2^r, and we have from (4.14), (2.17) and Lemma 4.5

$$b_2^r = \operatorname{Res} \overline{B_2^r}(0) := \operatorname{Res}\left[\int_{\pi^{-1}(x)} \alpha B_2^r(n;\cdot)\right](0)$$

$$= n \operatorname{Res}\left[\int_{\pi^{-1}(x)} \alpha B_2^h\right](0).$$

The coefficient B_2^h has been computed in [BGM], p. 225, [G], Theorem 4.8.18; it is given in terms of the Gauß curvature as

$$(4.17) \qquad B_2^h(p) = \frac{1}{60\pi}(-\Delta K_M(p) + K_M(p)^2).$$

Now, for a metric of type (4.16) we have (cf. e.g. [O'N], p. 214)

$$(4.18) \qquad K_M(y,\theta) = -\frac{h_1''(y)}{h_1(y)},$$

and consequently

$$(4.19) \qquad \overline{B_2^r}(y) = 2\pi n h_1 B_2^h(y) = \frac{n}{30}\left(-\left(h_1\left(\frac{h_1''}{h_1}\right)'\right)' + \frac{(h_1'')^2}{h_1}\right)(y).$$

Next we use (cf. Lemma 4.7 below) that for $f \in C^\infty(0,\varepsilon)$ with an asymptotic expansion

$$(4.20) \qquad f(y) \sim_{y\to 0+} \sum_{j \geqq -k} f_j y^{j/N},$$

we must have

$$(4.20\,\mathrm{b}) \qquad \operatorname{Res} f'(0) = 0.$$

This implies

Lemma 4.6. *If in* (1.8) *we have*

$$\alpha \equiv 1, \quad \beta(x,\theta) \equiv h(x) \quad \text{with} \quad h \in C^\infty([0,\varepsilon]),$$

then

$$b_2^r = \frac{n}{30} \operatorname{Res} \frac{(h_1'')^2}{h_1}(0).$$

To produce an example with $b_2^r \neq 0$ we introduce the family $C^{l,k}$ of 'generalized parabolas', where $l, k \in \mathbb{N}$, $(l,k) = 1$. They are defined by

$$C^{l,k} = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 \mid z_0^{k+l} = z_1^k z_2^l\}.$$

It is easy to see that at most the points $p_1 := [0,1,0]$ and $p_2 := [0,0,1]$ can be singular, with only one branch and multiplicities $N(p_1) = l$, $N(p_2) = k$.

Interchanging z_1 and z_2 if necessary and using the explicit parametrization

$$(4.21) \qquad \mathbb{C} \to C^{k,l}, \quad z \mapsto [z^l, 1, z^{k+l}]$$

it is enough to study the case $l = N(p_1) =: N$ near p_1. Then we see easily that the Fubini-Study metric on $\mathbb{CP}^2$ induces on a pointed neighborhood $U_\varepsilon = (0,\varepsilon) \times S^1$ a metric of type (4.15) where

$$(4.22) \quad h(x) = (1 + x^{2N} + x^{2(N+k)})^{-1}\left(1 + \left(\frac{k+N}{N}\right)^2 x^{2k} + \left(\frac{k}{N}\right)^2 x^{2(k+N)}\right)^{1/2}.$$

In order to do the explicit calculations it is convenient to digress a little and to treat some simple properties of the residue. First, we give a slightly more general definition of the residue: consider the linear space, $\mathscr{F}$, of all functions $f \in C^\infty(0,\infty)$ admitting a finite asymptotic expansion of the type

$$(4.23) \qquad \int_\varepsilon^1 f(t)\,dt = \sum_{\substack{0 \leqq j \leqq j_0 \\ 0 \leqq k \leqq k(j)}} F_{jk}\, \varepsilon^{\alpha_j} \log^k \varepsilon + O(1),$$

as $\varepsilon \to 0$; here $j_0 \in \mathbb{N}$ and $\alpha_j \in \mathbb{C}$. Denote by $\mathscr{F}_{\mathrm{as}}$ the subspace of all f having an expansion of type (4.23) themselves,

$$(4.24) \qquad f(\varepsilon) = \sum_{\substack{0 \leqq j \leqq j_0 \\ 0 \leqq k \leqq k(j)}} f_{jk}\, \varepsilon^{\alpha_j} \log^k \varepsilon + O(1);$$

$\mathscr{F}_{\mathrm{as},0}$ contains, by definition, those $f \in \mathscr{F}_{\mathrm{as}}$ which have $k(j) = 0$ for all j. Then we put for $f \in \mathscr{F}$

$$(4.25) \qquad \operatorname{Res} f(0) := F_{j_1,1}, \quad \text{if} \quad \alpha_{j_1} = 0.$$

If f is defined only near 0 then the residue is defined using a suitable cut-off. The following result is then obvious.

Lemma 4.7. *Let* $f \in \mathscr{F}_{\mathrm{as},0}$. *Then*:

(1) $\operatorname{Res} f'(0) = 0$.

(2) *If* $f \in \mathscr{F}_{\mathrm{as},0}$ *and* $f(x) = f_{00}x^{\alpha_0} + O(x^{\alpha_0+\varepsilon})$ *with* $\alpha_0, \varepsilon > 0$ *then we have for* $g \in \mathscr{F}$

$$\operatorname{Res}(g \circ f) f'(0) = \alpha_0 \operatorname{Res} g(0).$$

Next we use these to express b_2^r in terms of h.

Lemma 4.8. *We have, with*

$$k(x) := h^{-1}(x)\partial_x(h^{-1}(x)xh'(x)) = h^{-3}(x)(hh'(x) + xhh''(x) - xh'(x)^2),$$

$$(4.26) \qquad b_2^r = \frac{n}{30N^3(2N-2)!}\sum_{j=0}^{N-2}\binom{2N-2}{2j+1} k^{(2j+1)}(0)\, k^{(2(N-2-j)+1)}(0)$$

$$=: n\, d(N,h),$$

where $d(N,h)$ is a universal polynomial in the derivatives of h up to order $2N-2$, with coefficients depending on N. In particular, $b_2^r=0$ if $N=1$, and if $h(y)=\tilde{h}(y)(1+O(y^{2N-1}))$ then

$$d(N,h)=d(N,\tilde{h})\,.$$

Proof. Using $y(x)=\int\limits_0^x h(t^{1/N})\,dt \cong x+\sum\limits_{j\geq 1}\frac{h^{(j)}(0)}{j!}\frac{x^{1+j/N}}{1+j/N}$ in Lemma 4.7 (2) we find

$$\begin{aligned}\operatorname{Res}\frac{(h_1'')^2}{h_1}(0)&=\operatorname{Res}\left(\frac{(h_1'')^2}{h_1}\circ y\,y'\right)(0)\\&=\operatorname{Res}\left[\frac{h(x^{1/N})}{xNh(x^{1/N})}\left(\frac{1}{h(x^{1/N})}\frac{d}{dx}\frac{1}{h(x^{1/N})}\frac{d}{dx}Nxh(x^{1/N})\right)^2\right](0)\\&=N\operatorname{Res}\left[\frac{1}{xh(x^{1/N})^2}\left(\frac{d}{dx}\frac{x}{h(x^{1/N})}\frac{d}{dx}h(x^{1/N})\right)^2\right](0)\,.\end{aligned}$$

Continuing with $x=z^N$ we end up with

$$\begin{aligned}\operatorname{Res}\frac{(h_1'')^2}{h_1}(0)&=N^{-3}\operatorname{Res}\left[\frac{1}{z^{2N-1}h(z)^2}\left(\frac{d}{dz}\frac{zh'(z)}{h(z)}\right)^2\right](0)\\&=N^{-3}\operatorname{Res}[z^{1-2N}k(z)^2](0)\\&=\frac{1}{N^3(2N-2)!}\left(\frac{d}{dz}\right)^{2N-2}k(z)^2|_{z=0}\\&=\frac{1}{N^3(2N-2)!}\sum_{j=0}^{2N-2}\binom{2N-2}{j}k^{(j)}(0)\,k^{(2N-2-j)}(0)\,.\end{aligned}$$

It remains to note that h is an even function and hence k is odd at 0. From this, the assertions of the lemma are obvious. □

We apply these calculations to (4.22); from Lemma 4.8 we see that we may replace h by

$$\tilde{h}(x):=\left(1+\left(\frac{k+N}{N}\right)^2x^{2k}\right)^{1/2}=:(1+\gamma x^{2k})^{1/2}\,.$$

Then we obtain from the prescription of Lemma 4.8

$$\tilde{k}(x)=2\gamma k^2x^{2k-1}(1+\gamma x^{2k})^{-5/2}\,.$$

Thus, $\tilde{k}^{(2l+1)}(0)\neq 0$ at most if $l+1\equiv 0 \bmod k$. Hence, the sum in (4.26) contains a nonzero term at most if $N\equiv 0\bmod k$ which can only happen if $k=1$ since $(N,k)=(l,k)=1$ by assumption. If $k=1$ then

$$\tilde{k}^{(2j+1)}(0)=2\left(\frac{N+1}{N}\right)^2(2j+1)!\left(\frac{N+1}{N}\right)^{2j}\binom{-5/2}{j}$$

hence

$$\begin{aligned}b_2^r&=\frac{4n\left(\frac{N+1}{N}\right)^{2N}}{30N^3(2N-2)!}\sum_{j=0}^{N-2}\binom{2N-2}{2j+1}(2j+1)!(2(N-2-j)+1)!\binom{-5/2}{j}\binom{-5/2}{N-2-j}\\&=\frac{4n(N+1)^{2N}}{30N^{2N+3}}\sum_{j=0}^{N-2}\binom{-5/2}{j}\binom{-5/2}{N-2-j}=\frac{2n(N+1)^{2N}}{15N^{2N+3}}\binom{-5}{N-2}.\end{aligned}$$

We have proved

Lemma 4.9. *Consider $C^{k,l}\subset\mathbb{CP}^2$, $(k,l)=1$, equipped with the metric induced from the Fubini-Study metric on $\mathbb{CP}^2$. Then we have $b_2^r=0$ unless $k=1$ or $l=1$.*

If $k=1$, $l=N$ or $l=1$, $k=N$, $N\geqq 2$, we have

$$b_2^r=\frac{2n(N+1)^{2N}}{15N^{2N+3}}\binom{-5}{N-2}.$$

Using (2.5), with $\alpha_j=2-2j$, and (2.10c), and observing that $|b_2^r|$ in Lemma 4.9 is strictly increasing as a function of $N\geqq 1$ we obtain

Corollary 4.10. *If $C^{k,l}$ is as in Lemma 4.9 then*

$$b_2=A_{21}^h=\begin{cases}-\dfrac{(N+1)^{2N}}{15N^{2N+3}}\dbinom{-5}{N-2}, & k=1 \text{ or } l=1 \text{ and } N>1\,,\\ 0, & \text{otherwise}\,.\end{cases}$$

In particular, b_2 distinguishes the curves $C^{1,l}$ among the $C^{k,l}$.

Finally, we turn to the coefficients of type (2.39c).

Lemma 4.11. *At a singular branch, U, of multiplicity N we have*

$$c_0^r=\frac{1}{12}(N^{-1}-N)\,. \tag{4.27}$$

Proof. In view of (2.32) and (2.39c) we have

$$\begin{aligned}c_0^r&=⨍_0^\infty N\zeta^{2nN-1}\operatorname{tr}_H G_0^n(\zeta^N;1,1)\,d\zeta\\&=\operatorname{Res}_0|_{s=0}\int_0^\infty N\zeta^{s+2nN-1}\operatorname{tr}_H G_0^n(\zeta^N;1,1)\,d\zeta\\&=\operatorname{Res}_0|_{s=0}\int_0^\infty \zeta^{s/N+2n-1}\operatorname{tr}_H G_0^n(\zeta;1,1)\,d\zeta\\&=:\operatorname{Res}_0|_{s=0}F(s/N)\\&=\operatorname{Res}_0|_{s=0}F(s)\,.\end{aligned}$$

It has been shown in [BS 2], Sec. 7, that F is holomorphic in the strip

$$-2n < \operatorname{Re} s < -2p-1$$

for any $p > 1/2$ and $n > p+1/2$, and that it extends meromorphically to the whole plane. Moreover, c_0^r has been computed in [BS 2], (7.12) and (7.16). For the answer we need the ζ-function, ζ_T, of $T = (A_0 + 1/4)^{1/2} = \left(-\frac{1}{N^2}\partial_\theta^2\right)^{1/2}$ which is

$$\zeta_T(s) = 2\sum_{k\geq 1}\left(\frac{k}{N}\right)^{-s} = 2N^s\zeta_R(s), \quad \operatorname{Re} s > 1,$$

where ζ_R denotes the Riemann ζ-function. Plugging this into [BS 2], (7.16), gives

$$\begin{aligned} c_0^r &= -\frac{1}{2}\operatorname{Res}_0|_{s=-1} 2N^s\zeta_R(s) + \frac{1}{4}B_1 \operatorname{Res}_1|_{s=1} 2N^s\zeta_R(s) \\ &= \frac{N^{-1}}{12} - \frac{N}{12}, \end{aligned}$$

using $\operatorname{Res}_1\zeta_R(1) = 1$ and $B_1 = -\frac{1}{6}$. □

The coefficients c_j^r with $j \geq 1$ are of considerable interest since they multiply the "nonstandard" powers $z^{-j/N-2n}$; if they are nonzero then they reveal the occuring multiplicities of the singularities of C. We first express these coefficients as "regularized traces".

Lemma 4.12. *We have*

$$\begin{aligned} c_j^r &= \frac{1}{j!}\left(\frac{d}{dy}\right)^j\Big|_{y=0} \fint_0^\infty \operatorname{tr}_H G_{yN}^n(1; x, x)\, dx \\ &=: \frac{1}{j!}\left(\frac{d}{dy}\right)^j\Big|_{y=0} \widetilde{\operatorname{tr}}_{L^2(\mathbb{R}_+, H)} G_{yN}^n(1). \end{aligned}$$

Proof. We have from (2.32) and (2.39c)

$$\text{(4.28)} \qquad \begin{aligned} c_j^r &= \frac{N}{j!}\operatorname{Res}_0|_{s=0}\int_0^\infty \zeta^{s+j+2nN-1}\left(\frac{d}{dy}\right)^j\Big|_{y=0} \operatorname{tr}_H G_{yN}^n(\zeta^N; 1, 1)\, d\zeta \\ &=: \operatorname{Res}_0|_{s=0}\left(\frac{1}{j!}\left(\frac{d}{dy}\right)^j\Big|_{y=0} F_j(y; s)\right). \end{aligned}$$

Using the substitution $x = \zeta^N$ and (2.27) with $s = 1/x$ we find

$$\begin{aligned} F_j(y; s) &= N\int_0^\infty \zeta^{s+j+2nN-1}\operatorname{tr}_H G_{yN}^n(\zeta^N; 1, 1)\, d\zeta \\ &= \int_0^\infty x^{(s+j)/N+2n-1}\operatorname{tr}_H G_{yN}^n(x; 1, 1)\, dx \\ &= \int_0^\infty x^{(s+j)/N}\operatorname{tr}_H G_{yN/x}^n(1; x, x)\, dx. \end{aligned}$$

Changing variables by $\tilde{y} := x^{-1/N}y$ in (4.28) we arrive at the assertion. □

To proceed, we assume now that we can rewrite the operator τ_ε in (1.9) in the form

$$\text{(4.29)} \qquad \tau_\varepsilon = -\partial_x^2 + X^{-2}A(x^{1/N}),$$

where the family A satisfies the following conditions (cf. [BS 2], p. 373):

(4.30a) $A(0) = A_0 = -N^{-2}\partial_\theta^2 - \frac{1}{4}$, $A(x) \equiv A_0$ for $x \geq 2\varepsilon$;

(4.30b) $A(x) \geq -\frac{1}{4}$, $x \geq 0$;

(4.30c) $\mathscr{D}(A(x)) = \mathscr{D}(A_0)$, $x \geq 0$;

(4.30d) the function $\mathbb{R}_+ \ni x \mapsto A(x)(A_0 + I)^{-1} \in \mathscr{L}(H)$ is smooth.

Then we have the following more explicit formula.

Lemma 4.13. *Assume that* τ_ε *has the form* (4.29) *and that* A *satisfies the assumptions* (4.30). *Then if* $A^{(l)}(0) = 0$, $1 \leq l \leq j-1$, *we have*

$$\text{(4.31)} \qquad c_j^r = \frac{(-n)}{j!}\widetilde{\operatorname{tr}}_{L^2(\mathbb{R}_+, H)}[X^{j/N-2}A^{(j)}(0)G_0^{n+1}(1)].$$

Proof. By Lemma 4.12, we have to consider the derivatives of

$$\text{(4.32)} \qquad \begin{aligned} \tilde{F}_j(y; s) &:= \int_0^\infty x^{s/N}\operatorname{tr}_H G_{yN}^n(1; x, x)\, dx \\ &= \operatorname{tr}_{L^2(\mathbb{R}_+, H)}[X^{s/N}G_{yN}^n(1)]. \end{aligned}$$

Now it is enough to recall the identity (cf. Lemma 2.5)

$$\frac{d}{dy}G_{yN}(1) = -G_{yN}(1)X^{1/N-2}A'(yx^{1/N})G_{yN}(1),$$

which implies in view of our assumption

$$\widetilde{\mathrm{tr}}_{L^2(\mathbb{R}_+,H)}\left[\left(\frac{d}{dy}\right)^j|_{y=0}G_{y^N}(1)\right]=(-n)\,\widetilde{\mathrm{tr}}_{L^2(\mathbb{R}_+,H)}[X^{j/N-2}A^{(j)}(0)G_0^{n+1}(1)]\,.$$

Using this in (4.32) implies the lemma. □

In view of the techniques of [BS2], Sec. 7, we thus can hope to compute c_j^r from (4.31) if $A^{(j)}(0)$ is nonzero and commutes with $A(0)$. This case does, in fact, occur if the metric has the special form (4.16b).

We consider first a one-dimensional example. We put for $\nu \geqq 0$

$$l_\nu := -\partial_x^2 + X^{-2}\left(\nu^2-\frac{1}{4}\right), \tag{4.33}$$

and we denote by L_ν the Friedrichs extension of l_ν in $L^2(\mathbb{R}_+)$. Furthermore, we put

$$G^{n,\nu}(z) := (L_\nu + z^2)^{-n}\,.$$

Proposition 4.14. *Consider the operator*

$$\tau := -\partial_x^2 + X^{-2}a(X^{1/N})\,, \tag{4.34}$$

where $a \in C^\infty(\mathbb{R})$, $a(x) \geqq -1/4$, $a(x)=a(0)=\nu^2-1/4$ *for* $x \geqq x_0$. *Then if* $a^{(l)}(0)=0$, *for* $1 \leqq l \leqq j-1$ *and some* $j<N$, *we have*

$$c_j^r = -\frac{a^{(j)}(0)\,\Gamma(n+j/2N)\,\Gamma(1/2-j/2N)}{j!\,4\sqrt{\pi}(n-1)!}\,\frac{\Gamma(\nu+j/2N)}{\Gamma(\nu+1-j/2N)}. \tag{4.35}$$

In particular, if $a^{(j)}(0)\neq 0$ *then* $c_j^r \neq 0$. *More generally, we have*

$$\fint_0^\infty x^s G^{n+1,\nu}(1;x,x)\,dx = \frac{\Gamma(n+1+s/2)\,\Gamma(-1/2-s/2)\,\Gamma(\nu+1+s/2)}{4\sqrt{\pi}\,n!\,\Gamma(\nu-s/2)}, \tag{4.36}$$

for $\max(-2-2n,-2-2\nu)<\operatorname{Re}s<-1$.

Proof. We first prove (4.36). Denote by I_ν, K_ν the modified Bessel functions. Following [BS2], p. 418, we find for $\max(-2-2n,-2-2\nu)<\operatorname{Re}s<-1$

$$\begin{aligned}\fint_0^\infty x^s G^{n+1,\nu}(1;x,x)\,dx &= \frac{1}{n!}\fint_0^\infty x^s\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^n xI_\nu(x\zeta)\,K_\nu(x\zeta)|_{\zeta=1}\,dx\\ &= \frac{1}{n!}\fint_0^\infty x^{s+2n+1}\left(-\frac{1}{2x}\frac{\partial}{\partial x}\right)^n I_\nu(x)\,K_\nu(x)\,dx\\ &= \frac{\Gamma((w+1)/2)\,\Gamma(n-w/2)}{4\sqrt{\pi}\,n!}\,\frac{\Gamma(\nu+z(w)+1)}{\Gamma(\nu-z(w))},\end{aligned}$$

where $w:=s+2n+1$, $z(w):=(w+1-2(n+1))/2=s/2$; thus we obtain (4.36). (4.35) is an immediate consequence of (4.36) and Lemma 4.13. □

Now we consider the metric (4.16b),

$$dy^2+h_1(y)^2d\theta^2,\quad h_1(y(x))=Nxh(x^{1/N}),$$
$$h(x)=1 \quad \text{for } x \text{ sufficiently large}\,.$$

Since $y(x)=\int_0^x h(t^{1/N})\,dt \cong x+\sum_{j\geqq 1}\frac{h^{(j)}(0)}{j!}\,\frac{x^{1+j/N}}{1+j/N}$ we can write

$$h_1(y) =: Nyh_2(y^{1/N}),\quad h_2\in C^\infty(\mathbb{R}_+),\quad h_2(0)=1\,. \tag{4.37}$$

Assume that

$$h_2^{(l)}(0)=0,\quad 1\leqq l\leqq j-1,\quad b:=\frac{h_2^{(j)}(0)}{Nj!}\neq 0\,, \tag{4.38}$$

i.e. that we have

$$h_1(y)=Ny+by^{1+j/N}+O(y^{1+(j+1)/N})\,.$$

Then the Laplacian is unitarily equivalent to

$$\begin{aligned}\tau &:= -\partial_x^2-h_1^{-2}\partial_\theta^2+\frac{1}{2}\frac{h_1''}{h_1}-\frac{1}{4}\left(\frac{h_1'}{h_1}\right)^2\\ &=: -\partial_x^2+X^{-2}A(x^{1/N})\,,\end{aligned} \tag{4.39}$$

where A satisfies the assumptions (4.30). In fact, introducing

$$l_0(x):=h_2(x)^{-1},\quad l_1(x):=xh_2'(x)h_2(x)^{-1},\quad l_2(x):=x^2h_2''(x)h_2(x)^{-1},$$

we have the explicit formula

$$\begin{aligned}A(x) &= -l_0(x)^2N^{-2}\partial_\theta^2-\frac{1}{4}-l_1(x)(2N)^{-1}-l_2(x)^2(2N)^{-2}\\ &\quad+\frac{1}{2}(N+1)N^{-2}l_1(x)+\frac{1}{2}N^{-2}l_2(x)\,.\end{aligned} \tag{4.40}$$

Moreover, we have

$$\begin{aligned}A(0) &= -\frac{1}{N^2}\partial_\theta^2-\frac{1}{4},\\ A^{(l)}(0) &= 0,\quad 1\leqq l\leqq j-1,\\ \frac{1}{j!}A^{(j)}(0) &= b\left(\frac{2}{N^3}\partial_\theta^2+\frac{j^2}{2N^3}\right)=:-\frac{2b}{N}\left(-\frac{1}{N^2}\partial_\theta^2-\lambda^2\right),\quad \lambda=\frac{j}{2N}.\end{aligned} \tag{4.41}$$

Lemma 4.15. *For $0 \leqq \lambda < \frac{1}{2}$ the function*

$$f(z,\lambda) := \sum_{m=1}^{\infty} \left(\left(\frac{m}{N}\right)^2 - \lambda^2\right) \frac{\Gamma\left(\frac{m}{N}+z+\lambda\right)}{\Gamma\left(\frac{m}{N}-z-\lambda+1\right)}$$

is holomorphic for $\mathrm{Re}\, z < -1$ *and admits a meromorphic extension to the complex plane, regular at* $z=0$.

Using this, we have for the operator τ *in* (4.39), *with* $\lambda = j/2N$,

$$c_j^r = \frac{b}{j}\,\frac{\Gamma(\lambda+n)\Gamma(-\lambda+1/2)}{\sqrt{\pi}(n-1)!}\left[-\lambda^3\frac{\Gamma(\lambda)}{\Gamma(-\lambda+1)} + 2\lambda f(0,\lambda)\right]. \tag{4.42}$$

Proof. For $|w| \leqq w_0$ one has the asymptotic expansion (cf. [BS2], p. 419, and the reference there)

$$\frac{\Gamma(v+w)}{\Gamma(v-w+1)} = \sum_{k=0}^{l} P_k(w)\, v^{2w-1-k} + w R_l(w,v),$$

and

$$|R_l(w,v)| \leqq C_{w_0} v^{2\mathrm{Re}\, w - l - 2}, \quad v \geqq v_0 > 0,$$

with polynomials $P_k \in \mathbb{C}[w]$. Moreover,

$$P_0 = 1, \quad P_1 = 0, \quad P_k(0) = 0 \quad \text{for} \quad k \geqq 1, \tag{4.44}$$
$$P_2(w) = -\frac{1}{3}w^3 + \frac{1}{2}w^2 - \frac{1}{6}w.$$

Denoting by ζ_R the Riemann ζ-function and substituting $1/N = 2\lambda/j$ we infer from (4.43)

$$\begin{aligned} f(z,\lambda) = \sum_{k=0}^{l} P_k(z+\lambda) \Bigg[& \left(\frac{2\lambda}{j}\right)^{2(z+\lambda)+1-k} \zeta_R(k-1-2(z+\lambda)) \\ & -\lambda^2 \left(\frac{2\lambda}{j}\right)^{2(z+\lambda)-1-k} \zeta_R(k+1-2(z+\lambda)) \Bigg] \\ & + (z+\lambda) \sum_{m=1}^{\infty} \left(\left(\frac{2\lambda m}{j}\right)^2 - \lambda^2\right) R_l\left(z+\lambda, \frac{2\lambda m}{j}\right). \end{aligned}$$

This proves the existence of $f(z,\lambda)$ and that it extends meromorphically to $\mathbb{C}$. Moreover, since $0 < \lambda < 1/2$, $f(z,\lambda)$ is regular at $z=0$. Next, putting

$$A(v,\lambda) := \frac{\Gamma(v+\lambda)}{\Gamma(v-\lambda+1)} - v^{2\lambda-1} - P_2(\lambda)\, v^{2\lambda-3}, \tag{4.45}$$

we get from (4.43) the estimate

$$|A(v,\lambda)| \leqq C\lambda v^{2\lambda-4}, \quad v \geqq v_0 > 0, \tag{4.46}$$

and we find

$$\begin{aligned} f(0,\lambda) = {} & \left(\frac{2\lambda}{j}\right)^{2\lambda+1} \zeta(-1-2\lambda) - \lambda^2 \left(\frac{2\lambda}{j}\right)^{2\lambda-1} \zeta(1-2\lambda) \\ & + P_2(\lambda) \left(\frac{2\lambda}{j}\right)^{2\lambda-1} \zeta(1-2\lambda) - P_2(\lambda)\lambda^2 \left(\frac{2\lambda}{j}\right)^{2\lambda-3} \zeta(3-2\lambda) \\ & + \sum_{m=1}^{\infty} \lambda^2 \left(\left(\frac{2m}{j}\right)^2 - 1\right) A\left(\frac{2\lambda m}{j}, \lambda\right). \end{aligned} \tag{4.47}$$

Now we infer from Lemma 4.13 and Proposition 4.14, since $\left(-\frac{1}{N^2}\partial_\theta^2 - \lambda^2\right)$ and $-\frac{1}{N^2}\partial_\theta^2 - \frac{1}{4}$ commute,

$$\begin{aligned} c_j^r &= \frac{2b}{N}\, n\, \widetilde{\mathrm{tr}}_{L^2(\mathbb{R}_+, L^2(S^1))}\left[X^{j/N-2}\left(-\frac{1}{N^2}\partial_\theta^2 - \lambda^2\right) G_0^{n+1}(1)\right] \\ &= \frac{4b\lambda}{j}\, \mathrm{Res}_0|_{s=j/N-2}\, n\, \widetilde{\mathrm{tr}}_{L^2(\mathbb{R}_+, L^2(S^1))}\left[X^s\left(-\frac{1}{N^2}\partial_\theta^2 - \lambda^2\right) G_0^{n+1}(1)\right] \\ &=: \frac{4b\lambda}{j}\, \mathrm{Res}_0|_{s=j/N-2}\, F(s,\lambda), \end{aligned}$$

where again we have used $1/N = 2\lambda/j$. With (4.36) we find

$$\begin{aligned} F(s,\lambda) &= -\lambda^2 n\, \widetilde{\mathrm{tr}}_{L^2(\mathbb{R}_+, L^2(S^1))}[X^s G^{n+1,0}(1)] \\ &\quad + 2n \sum_{m=1}^{\infty} \left(\left(\frac{m}{N}\right)^2 - \lambda^2\right) \widetilde{\mathrm{tr}}_{L^2(\mathbb{R}_+, L^2(S^1))}[X^s G^{n+1,m/N}(1)] \\ &= \frac{\Gamma(s/2+n+1)\Gamma(-(s+1)/2)}{4\sqrt{\pi}(n-1)!}\Bigg[-\lambda^2\frac{\Gamma(s/2+1)}{\Gamma(-s/2)} \\ &\quad + 2\sum_{m=1}^{\infty} ((m/N)^2 - \lambda^2)\frac{\Gamma(s/2+m/N+1)}{\Gamma(m/N-s/2)}\Bigg], \end{aligned}$$

and the lemma is proved. □

Formula (4.42) is rather involved and it is a priori not obvious that $c_j^r \neq 0$ in general. However, we can show the following.

Theorem 4.16. *Under the assumptions* (4.41) *we have*

$$c_j^r = C(\lambda)\frac{b}{j} \tag{4.48}$$

where C is a function independent of the algebraic curve under consideration. Moreover, for fixed j one has

$$\lim_{\lambda\to 0} C(\lambda) = \frac{j}{12}.$$

Proof. (4.48) is an immediate consequence of formula (4.42). (4.42) also shows that we only have to prove

$$\lim_{\lambda\to 0} 2\lambda f(0,\lambda) = \frac{j}{12}.$$

This, in turn, follows from the identity (4.47) and the next lemma. □

Lemma 4.17. *We have with A in* (4.45)

$$\lim_{\lambda\to 0}\lambda^3\sum_{m=1}^{\infty}\left(\left(\frac{2m}{j}\right)^2-1\right)A\left(\frac{2\lambda m}{j},\lambda\right)=0.$$

Proof. Choose $\varepsilon > 0$. The estimate (4.46) holds for any $v_0 > 0$. Thus we first consider the sum with $m > \varepsilon/\lambda$ and estimate for $\lambda < 1/2$

$$\begin{aligned}
&\left|\lambda^3\sum_{m>\varepsilon/\lambda}\left(\left(\frac{2m}{j}\right)^2-1\right)A\left(\frac{2\lambda m}{j},\lambda\right)\right|\\
&\leqq c(\varepsilon)\lambda^3\sum_{m>\varepsilon/\lambda}\lambda\left(\frac{2\lambda m}{j}\right)^{2\lambda-4}m^2\\
&\leqq c_1(\varepsilon)\lambda^{2\lambda}\int_{\varepsilon/\lambda}^{\infty}x^{2\lambda-2}\,dx\\
&\leqq c_2(\varepsilon)\lambda^{2\lambda}\frac{1}{2\lambda-1}\left(\frac{\varepsilon}{\lambda}\right)^{2\lambda-1}\\
&\leqq c_3(\varepsilon)\lambda.
\end{aligned}$$

Next we need to estimate $A(v,\lambda)$ for v small. From (4.45) we obtain for $0<\lambda\leqq cv\leqq v_1$, v_1 small,

$$\begin{aligned}
(v^2-\lambda^2)|A(v,\lambda)| &\leqq \left|\frac{\Gamma(v+\lambda)}{\Gamma(v-\lambda-1)}\right|(v^2-\lambda^2)+d_1v^{2\lambda+1}+d_2|P_2(\lambda)|v^{2\lambda-1}\\
&=\left|\frac{\Gamma(v+\lambda+1)}{\Gamma(v-\lambda-1)}\right|(v-\lambda)+d_1v^{2\lambda+1}+d_2|P_2(\lambda)|v^{2\lambda-1}\\
&\leqq \tilde c_1 v+\tilde c_2 v^{2\lambda+1}+\tilde c_3\lambda v^{2\lambda-1}\leqq Cv^{2\lambda}\leqq 1,
\end{aligned}$$

where we have used that $P_2(0)=0$. This gives

$$\left|\lambda^3\sum_{1\leqq m\leqq\varepsilon/\lambda}\left(\left(\frac{2m}{j}\right)^2-1\right)A\left(\frac{2\lambda m}{j},\lambda\right)\right|\leqq\lambda\sum_{1\leqq m\leqq\varepsilon/\lambda}1\leqq\varepsilon,$$

and we reach the conclusion. □

Corollary 4.18. *Consider* $C^{k,l}\subset\mathbb{CP}^2$ *as in Lemma* 4.9. *Then for the singular point* $p_1=[0,1,0]$ *we have* $c_j^r(p_1)=0, 1\leqq j<2k$, *and*

$$\lim_{l\to\infty}c_{2k}^r(p_1)=\frac{k}{12}.$$

Here we write $c_j(p_1)$ *instead of* $c_j(1,p_1)$ *since we have* $L(p_1)=1$.

Proof. From (4.22) one easily computes that, using the notations of (4.38), we have $j=2k$ and

$$b=b(k,l)=\frac{2kl}{2(2k+l)}\left(\frac{k+l}{l}\right)^2.$$

Thus the assertion follows from Theorem 4.16. □

Summarizing the results of Sections 3 and 4 we now give the

Proof of Theorem 1.2. (1) The existence of the asymptotic expansion follows from (2.22) and Theorem 2.8.

(2) This follows from Corollary 4.2 and (4.13).

(3) This follows from Lemmas 4.4 and 4.11.

(4) This is Corollary 4.10.

(5) This, finally, is the content of Corollary 4.18.

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