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GAUGE-PERIODIC POINT PERTURBATIONS ON THE LOBACHEVSKY PLANE

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We study periodic point perturbations of the Shrödinger operator with a uniform magnetic field on the Lobachevsky plane. We prove that the spectrum gaps of the perturbed operator are labeled by the elements of the K_0 group of a C^* algebra associated with the operator. In particular, if the C^* algebra has the Kadison property, then the operator spectrum has a band structure.

Introduction

Let H be the periodic Shrödinger operator with a magnetic field on the Euclidean plane $\mathbf{R}^2$. As is well known, the spectrum structure of the operator critically depends on the magnetic field flow η across the elementary cell of the potential period lattice. If the value of η is rational, then the spectrum of H has a band structure. Otherwise, some of its sections are Cantor sets [1]. The discrete version of H (the so-called Harper operator) has a purely Cantor spectrum for almost all values of η [2–4]. The situation is different in the case of the periodic Shrödinger operator H with a magnetic field on the Lobachevsky plane. For example, the discrete version of H with the modular group $A\mathbf{SL}_2(\mathbf{Z})$ as the period group has a band structure for all values of η [5]. Sufficient conditions for the spectrum band structure in the case of an operator H with a smooth potential invariant with respect to the cocompact action of a properly discontinuous group were indicated in [6–8]. On the other hand, a very interesting class of periodic Shrödinger operators that give an extensive set of explicitly soluble models can be obtained using perturbation theory [9–11]. We note that point perturbations of the two-dimensional Shrödinger operator with a magnetic field are widely used in theoretical physics to study transport properties of two-dimensional systems (e.g., see [12, 13] and the references therein).

In this paper, we prove that the spectrum gaps of a point perturbation of the Shrödinger equation with a magnetic field on the Lobachevsky plane are labeled by the elements of the K_0 group of a C^* algebra associated with the operator. In particular, if this algebra has the Kadison property, then the spectrum of the corresponding operator has a band structure (a broad class of algebras with the Kadison property is indicated in [7] and [8]). Among other things, this result is interesting in the context of the question of the effect of the curvature of a two-dimensional electron system on its spectral and transport properties [14–16, 8].

1. Preliminaries

In this paper, the Lobachevsky plane X is realized as the upper Poincaré half-plane

$$\{z = x + iy \in \mathbf{C}\colon y > 0\}$$

with the standard metric

$$ds^2 = \frac{a^2}{y^2}\,(dx^2 + dy^2),$$

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for which the curvature of X is equal to $R = -2/a^2$. In this case, the distance between two points $z, z' \in X$ has the form

$$d(z, z') = a \operatorname{arcosh}\left(1 + \frac{|z - z'|^2}{2yy'}\right),$$

and the invariant area is given by the formula

$$d\sigma = \frac{a^2}{y^2}\, dx \wedge dy.$$

By definition, a constant homogeneous magnetic field $\mathbf{B}$ orthogonal to X is the 2-form

$$\mathbf{B} = B\frac{a^2}{y^2}\, dx \wedge dy,$$

where $B \in \mathbf{R}$ is the field intensity. The form $\mathbf{B}$ is exact, i.e., $\mathbf{B} = d\mathbf{A}$, where the 1-form $\mathbf{A}$ is called the vector potential of the field $\mathbf{B}$. We use the so-called Landau gauge

$$\mathbf{A} = Ba^2 y^{-1}\, dx$$

for the vector potential. The Hamiltonian H^0 of a free quantum mechanical particle moving on the plane X in the field $\mathbf{B}$ is the closure of the symmetrical operator

$$\frac{1}{a^2}\left\{-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2iby\,\frac{\partial}{\partial x} + b^2\right\}$$

with the domain $C_0^\infty(X)$ [17]. We use the system of units in which $e = c = \hbar = 1$ and $m = 1/2$, and we write $b = Ba^2$. As is well known, $\mathcal{D}(H^0) \subset C(X)$ (e.g., see [18]). The spectrum of H^0 consists of two parts [17], namely, finitely many eigenvalues (Landau levels)

$$E_n = a^{-2}(|b|(2n+1) - n(n+1)), \quad 0 \le n \le |b| - \frac{1}{2},$$

and the continuous spectrum

$$E(\nu) = \frac{1}{a^2}\left[\frac{1}{4} + b^2 + \nu^2\right], \quad 0 \le \nu < \infty. \tag{1}$$

The resolvent $R^0(\zeta) = (H^0 - \zeta)^{-1}$ of the operator H^0 plays the main role in our investigation. The integral kernel of the resolvent (i.e., the Green's function $G^0(z, z'; \zeta)$ of the operator H^0) was found in [17]. It has the form

$$G^0(z, z'; \zeta) = \frac{\sigma^{-t}}{4\pi}\, e^{ib\varphi}\, \frac{\Gamma(t+b)\,\Gamma(t-b)}{\Gamma(2t)}\, F\left(t+b, t-b; 2t; \frac{1}{\sigma}\right), \tag{2}$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function,

$$\sigma = \cosh^2\left(\frac{d(z,z')}{2a}\right), \qquad \varphi = 2\arctan\left(\frac{x - x'}{y + y'}\right),$$

and, according to (1), the function $t = t(\zeta)$ is uniquely defined for $\zeta \in \mathbf{C} \setminus \sigma(H^0)$ by the condition

$$\zeta = \frac{t(1-t) + b^2}{a^2}, \quad \operatorname{Re} t > 0.$$

The lemma below enumerates the properties of the Green's function G^0 needed in what follows (we omit the corresponding purely technical proofs that are based on formula (2)).

Lemma 1.

1. *For any $z \in X$, the limit*

$$q(\zeta) = \lim_{z' \to z} \left[G^0(z, z'; \zeta) - \frac{1}{2\pi} \log d(z, z') \right]$$

exists. It does not depend on z and is given by the formula

$$q(\zeta) = \frac{1}{4\pi} \left[\psi(t+b) + \psi(t-b) + 2C_E - \log 4a^2 \right],$$

where $\psi(z) = \log \Gamma(z)'$ and C_E is Euler's constant.

2. $\operatorname{Re} q(\zeta) \to -\infty$ *as* $\operatorname{Re} \zeta \to -\infty$.

3. *Let K be a compact subset in X and z_0 be a point belonging to X. Then for any $\varepsilon > 0$ and an arbitrary $\zeta \in \mathbf{C}$ such that $\operatorname{Re} \zeta < 0$, there are constants $c_1(K, z_0, \varepsilon, \zeta) \equiv c_1 > 0$ and $\tilde{c}_1(\varepsilon, \zeta) \equiv \tilde{c}_1 > 0$ such that*

$$\sup\{ |G^0(z, z'; \zeta)| : z' \in K \} \le c_1 e^{-\tilde{c}_1 d(z, z_0)}$$

when $d(z, K) \ge \varepsilon$. Moreover, if K, z_0, and ε are fixed, then $c_1(\zeta) = o(1)$ and $\tilde{c}_1(\zeta) \to +\infty$ as $\operatorname{Re} \zeta \to -\infty$.

4. *The integral*

$$\int_X |G^0(z, z'; \zeta)|^2 \, d\sigma(z')$$

is finite and is independent of z for all $\zeta \in \mathbf{C} \setminus \sigma(H^0)$ and $z \in X$.

5. *If K is a compact subset in X and z_0 is a fixed point belonging to X, then for any $\zeta \in \mathbf{C}$ such that $\operatorname{Re} \zeta < 0$, there are constants $c_2(K, z_0, \zeta) \equiv c_2 > 0$ and $\tilde{c}_2(\zeta) \equiv \tilde{c}_2 > 0$ such that*

$$\left[\int_K |G^0(z, z'; \zeta)|^2 \, d\sigma(z') \right]^{1/2} \le c_2 e^{-\tilde{c}_2 d(z, z_0)}.$$

Moreover, $\tilde{c}_2(\zeta) \to +\infty$ as $\operatorname{Re} \zeta \to -\infty$. If K and z_0 are fixed, then $c_2(\zeta) = O(1)$ as $\operatorname{Re} \zeta \to -\infty$.

2. Point perturbations of the operator H^0

We consider an isometry group Γ of the plane X. Although the field $\mathbf{B}$ is invariant relative to the transformations in Γ, the Hamiltonian H^0 is not Γ invariant in the general case. To obtain the invariance group for H^0, we must extend Γ to the so-called group of magnetic translations [19, 7]. We recall the construction of this group. Let U be the standard unitary representation of Γ in $L^2(X)$, $U_\gamma f(z) = f(\gamma^{-1} z)$ $(\gamma \in \Gamma$ and $f \in L^2(X))$, and let γ^* be the induced map of the space of differential forms. We suppose that $\gamma^* \mathbf{A} \ne \mathbf{A}$. Then $U_\gamma H^0 \ne H^0 U_\gamma$. Nevertheless, we have $d(\gamma^* \mathbf{A} - \mathbf{A}) = 0$ because $\gamma^* \mathbf{B} = \mathbf{B}$. Consequently, there is a function $\omega_\gamma \in C^\infty(X)$ such that $d\omega_\gamma = \gamma^* \mathbf{A} - \mathbf{A}$. We fix one of these functions ω_γ for each $\gamma \in \Gamma$. For $\gamma = 1$, we set $\omega_1 = 0$. Let T^0_γ be the unitary operator in $L^2(X)$ acting according to the formula $T^0_\gamma f = \exp(i\omega_\gamma) U_\gamma f$, $f \in L^2(X)$. Then $T^0_\gamma H^0 = H^0 T^0_\gamma$ for all $\gamma \in \Gamma$. The map $\gamma \mapsto T^0_\gamma$ is a projective representation of Γ in $L^2(X)$. In other words, $T^0_\beta T^0_\gamma = \Theta(\beta, \gamma) T^0_{\beta\gamma}$ for any $\beta, \gamma \in \Gamma$, where $\Theta(\beta, \gamma) \in \mathbf{C}$ and $|\Theta(\beta, \gamma)| = 1$. The family $\Theta(\beta, \gamma)$ possesses the property $\Theta(\gamma_1, \gamma_2) \Theta(\gamma_1 \gamma_2, \gamma_3) = \Theta(\gamma_1, \gamma_2 \gamma_3) \Theta(\gamma_2, \gamma_3)$, i.e., it is a 2-cocycle on Γ with coefficients in $\mathbf{U}(1)$. This cocycle defines the extension $1 \to \mathbf{U}(1) \to M(\Gamma, \Theta) \to \Gamma \to 1$ of Γ using $\mathbf{U}(1)$. The group $M(\Gamma, \Theta)$ is called the group of magnetic translations. Its element is uniquely determined by a pair (γ, ζ), where $\gamma \in \Gamma$ and $\zeta \in \mathbf{U}(1)$. In what follows, we identify it with the pair. Let $[\gamma, \zeta]$ denote the unitary operator ζT^0_γ. Then the correspondence $(\gamma, \zeta) \mapsto [\gamma, \zeta]$ is the exact linear unitary representation of the group $M(\Gamma, \Theta)$ in $L^2(X)$. This representation is denoted by T. Clearly, the operator H^0 is invariant with respect to T.

In what follows, we consider only the groups Γ satisfying the conditions

1. *the action of* Γ *is properly discontinuous on* X *and*
2. *the orbit space* $\Gamma \setminus X$ *is compact.*

We fix a fundamental domain F of Γ once and for all, i.e., a set $F \subset X$ such that (a) $\overline{F} = \overline{\operatorname{Int} F}$, (b) $\overline{F}$ is compact, and (c) the restriction of the canonical map $X \to \Gamma \setminus X$ to F is bijective. To construct a Γ-periodic point perturbation of the operator H^0, we select a finite subset $\mathrm{K} \subset F$ and let $\Lambda = \Gamma \cdot \mathrm{K}$ denote the orbit of this set. Each element $\lambda \in \Lambda$ has a unique representation of the form $\lambda = \gamma\kappa$, where $\gamma \in \Gamma$ and $\kappa \in \mathrm{K}$. In our further considerations, we need the following lemma, which was proved in essence in [20] (in what follows, $\#Y$ denotes the number of elements in the set Y).

Lemma 2. *There are constants* $c_\Lambda > 0$ *and* $\tilde{c}_\Lambda > 0$ *such that*

$$\#\{\lambda \in \Lambda \colon d(\lambda, \lambda_0) \le r\} \le c_\Lambda e^{\tilde{c}_\Lambda r}$$

for all $\lambda_0 \in \Lambda$ *and* $r \in \mathbf{R}$, $r > 0$.

We now present the construction of a point perturbation for the operator H^0. Because $\mathcal{D}(H^0) \subset C(X)$, the set

$$\mathcal{D} = \{f \in \mathcal{D}(H^0) \colon f(\lambda) = 0 \ \forall \lambda \in \Lambda\}$$

is well defined. Let S be the restriction of H^0 to the domain $\mathcal{D}(S) \equiv \mathcal{D}$. It is clear that S is a symmetrical operator in $L^2(X)$. The self-adjoint extension H of S is called a point perturbation of H^0 concentrated on the set Λ if $\mathcal{D}(H) \cap \mathcal{D}(H^0) = \mathcal{D}(S)$. It is convenient to describe point perturbations of the operator H^0 using the Krein resolvent formula (10.11 in [21]). For this, we fix a Hilbert space $\mathcal{G}$ isomorphic to an arbitrary deficiency space of the operator S and define the holomorphic operator-valued functions

$$B \colon \mathbf{C} \setminus \sigma(H^0) \to L(\mathcal{G}, L^2(X)), \qquad Q \colon \mathbf{C} \setminus \sigma(H^0) \to L(\mathcal{G}, \mathcal{G})$$

satisfying the so-called Krein Γ and $\mathcal{Q}$ conditions [21] (as usual, $L(E, P)$ denotes the Banach space of bounded operators from E into F). Accordingly, B and Q are called the Krein Γ and $\mathcal{Q}$ functions. If the Γ and $\mathcal{Q}$ functions are fixed, then there is a one-to-one correspondence between the point perturbations H of H^0 concentrated on Λ and the self-adjoint operators A in the space $\mathcal{G}$. This correspondence is expressed by the Krein formula for resolvents,

$$(H - \zeta)^{-1} = (H^0 - \zeta)^{-1} - B(\zeta)[Q(\zeta) + A]^{-1} B^*(\bar{\zeta}). \tag{3}$$

In what follows, H_A denotes the point perturbation of H that corresponds to A in accordance with formula (3). The resolvent of H_A is denoted by $R_A(\zeta)$.

Below, we give explicit descriptions of the Krein Γ and $\mathcal{Q}$ functions using Theorem 4 and Proposition 4 in [22] (the proof of these assertions in [22] for the space $L^2(X)$, where X is a domain in the Euclidean space, remains valid in the case of a Riemannian manifold X). We let $\mathcal{G}$ denote the space $l^2(\Lambda)$ and $Q(\zeta)$ denote the infinite matrix

$$Q(\lambda, \mu; \zeta) = \begin{cases} G^0(\lambda, \mu; \zeta) & \text{for} \quad \lambda, \mu \in \Lambda, \quad \lambda \ne \mu, \\ q(\zeta) & \text{for} \quad \lambda, \mu \in \Lambda, \quad \lambda = \mu. \end{cases}$$

The lemma below is an immediate consequence of Lemma 1.

Lemma 3.

1. *There are constants $c_3(\zeta) \equiv c_3 > 0$ and $\tilde{c}_3(\zeta) \equiv \tilde{c}_3 > 0$ such that*

$$|Q(\lambda, \mu; \zeta)| \leq c_3(\zeta) e^{-\tilde{c}_3(\zeta) d(\lambda,\mu)}$$

for Re $\zeta < 0$ *and* $\lambda \neq \mu$. *Moreover,* $c_3(\zeta) = o(1)$ *and* $\tilde{c}_3(\zeta) \to +\infty$ *as* $\mathrm{Re}\,\zeta \to -\infty$.

2. $|Q(\lambda, \lambda; \zeta)| \to +\infty$ *as* $\mathrm{Re}\,\zeta \to -\infty$.

For arbitrary $\zeta \in \mathbf{C} \setminus \sigma(H^0)$ and $\lambda \in \Lambda$, we let $g_\lambda(\zeta)$ denote the function $z \mapsto G^0(z, \lambda; \zeta)$ on X. It is clear that $g_\lambda(\zeta) \in L^2(X)$ (see Lemma 1). Lemmas 2, 3, and A.2 (in the appendix) imply the following lemma.

Lemma 4. *There is a number $E_1 \in \mathbf{R}$ such that the matrices $Q(\zeta)$ and $\big(\langle g_\lambda(\zeta) | g_\mu(\zeta)\rangle\big)_{\lambda,\mu\in\Lambda}$ define bounded operators in $l^2(X)$ for all $\zeta \in \mathbf{C}$ and* Re $\zeta < E_1$.

We now state the main result of this section.

Theorem 1.

1. *For any $\zeta \in \mathbf{C} \setminus \sigma(H^0)$, the family $(g_\lambda(\zeta))_{\lambda\in\Lambda}$ forms the Riecz basis in the closure of its linear span. Let the map $B(\zeta) \colon l^2(\Lambda) \to L^2(X)$ be given by the formula*

$$B(\zeta)\varphi = \sum_{\lambda\in\Lambda} \varphi(\lambda) g_\lambda(\zeta), \quad \varphi \in l^2(\Lambda).$$

Then $B(\zeta)$ is the Krein Γ function for the pair of operators (S, H^0).

2. *There exists an $E_0 \in \mathbf{R}$ such that the matrix $Q(\zeta)$ defines the Krein Q function of the pair (S, H^0) for all $\zeta \in \mathbf{C}$ and* $\mathrm{Re}\,\zeta < E_0$. *Therefore,*

$$R_A(\zeta) f = R^0(\zeta) f - \sum_{\lambda\in\Lambda} \left(\sum_{\mu\in\Lambda} [Q(\zeta) + A]^{-1} (\lambda, \mu) \langle g_\mu(\zeta) | f \rangle \right) g_\lambda(\zeta)$$

for all $f \in L^2(X)$.

Proof. The proof of the theorem follows directly from Theorem 4 in [22] with Lemma 4 taken into account.

We are interested only in T-invariant point perturbations H_A. To find a criterion for the T invariance of H_A, we introduce the linear unitary representation T^d of the group $M(\Gamma, \Theta)$ in the space $l^2(\Lambda)$ given by the formula $T^d_{(\gamma,\zeta)}\varphi(\lambda) = \zeta \exp(i\omega_\gamma(\lambda))\varphi(\gamma^{-1}\lambda)$, where $(\gamma, \zeta) \in M(\Gamma, \Theta)$ and $\varphi \in l^2(\Lambda)$. The proposition below (we omit its simple proof) gives a necessary and sufficient condition for the T invariance of H_A.

Proposition 1. *The operator H_A is T invariant if and only if the operator A is T^d invariant.*

In what follows, we consider only operators H_A invariant with respect to the representations T. From the standpoint of physical applications, the most interesting class is formed by the Hamiltonians H_A whose parameterizing operators A have diagonal matrices in the standard basis of the space $l^2(\Lambda)$ [11–13]. In particular, only these operators can be limits of Hamiltonians with short-range local potentials [11]. Therefore, in what follows, we assume that A is determined by a diagonal matrix, $A(\lambda, \mu) = \alpha_\lambda \delta_{\lambda\mu}$. The T^d invariance of A implies that the sequence (α_λ) is completely determined by its elements with indices belonging to K, namely, $\alpha_\kappa = \alpha_{\gamma\kappa}$ for all $\gamma \in \Gamma$ and $\kappa \in$ K. We need the following assertion directly implied by Theorem A.1 in the appendix.

Theorem 2. *There is a number $E_A \in \mathbf{R}$ with the following properties:*

1. *if* $\operatorname{Re}\zeta < E_A$, *then* $Q(\zeta) + A$ *has a bounded inverse operator;*
2. *for any* $\zeta \in \mathbf{C}$, $\operatorname{Re}\zeta < E_A$, *there are constants* $c_4(\zeta) \equiv c_4 > 0$ *and* $\tilde{c}_4(\zeta) \equiv \tilde{c}_4 > 0$ *such that*

$$|[Q(\zeta) + A]^{-1}(\lambda, \mu)| \leq c_4 e^{-\tilde{c}_4\, d(\lambda,\mu)}$$

for all $\lambda, \mu \in \Lambda$. *Moreover,* $c_4(\zeta) = O(1)$, *and* $\tilde{c}_4(\zeta) \to +\infty$ *as* $\operatorname{Re}\zeta \to -\infty$.

Corollary 1. *The operator* H_A *is semibounded from below.*

3. Main results

We first recall the notion of the twisted group algebra $C^*(\Gamma, \Theta)$ for the pair (Γ, Θ) [23, 5]. We define multiplication and involution in the space of finite sequences $C_0(\Gamma)$ by the formulas

$$(a \cdot b)(\gamma) = \sum_{\beta \in \Gamma} \Theta(\gamma\beta^{-1}, \beta)^{-1}\, a(\gamma\beta^{-1})\, b(\beta),$$

$$a^*(\gamma) = \Theta(\gamma^{-1}, \gamma)\, \Theta(1,1)\, \overline{a(\gamma^{-1})}.$$

We let I denote an injective $*$-homomorphism of $C_0(\Gamma)$ into the operator algebra $L(l^2(\Gamma))$ such that it transforms an element $a \in C_0(\Gamma)$ into an operator Ia possessing the property

$$(Ia)\varphi(\gamma) = \sum_{\beta \in \Gamma} \Theta(\gamma\beta^{-1}, \beta)^{-1}\, a(\gamma\beta^{-1})\varphi(\beta)$$

for $\varphi \in l^2(\Gamma)$. The twisted group C^* algebra $C^*(\Gamma, \Theta)$ is defined as the completion of $C_0(\Gamma)$ with respect to the norm $\|a\| = \|Ia\|$. The algebra $C^*(\Gamma, \Theta)$ has the standard trace τ defined as $\tau(a) = a(1)$. We next let ρ_γ ($\gamma \in \Gamma$) denote an operator in $l^2(\Gamma)$ acting according to the rule $(\rho_\gamma\varphi)(\beta) = \Theta(\beta, \gamma)\varphi(\beta\gamma)$. It is easy to verify that $\gamma \mapsto \rho_\gamma$ is a projective unitary representation of Γ in the space $l^2(\Gamma)$.

We now define the "canonical" isomorphism $\Phi\colon L^2(X) \to l^2(\Gamma) \otimes L^2(F) = l^2(\Gamma, L^2(F))$ by the rule $(\Phi f)(\gamma) = r_F \cdot [\gamma, 1]\, f$, where r_F is the restriction operator to F, $r_F f = f|F$ [6]. Using Φ, we extend ρ to the projective unitary representation $\tilde{\rho}$ in $L^2(X)$ by the formula $\tilde{\rho} = \rho \otimes 1$. It can be shown that the operator Φ intertwines the representations T and $\tilde{\rho}$. Let $\mathcal{K}$ be the algebra of compact operators in $L^2(X)$. We let $\mathcal{A}$ denote the tensor product $C^*(\Gamma, \Theta) \otimes \mathcal{K}$. The trace τ in $C^*(\Gamma, \Theta)$ induces the standard trace on $\mathcal{A}$, which, as before, is denoted by τ. The isomorphism Φ determines the canonical embedding $I_{\mathcal{K}}$ of $\mathcal{A}$ in the C^* algebra $L(L^2(X)) = L(l^2(\Gamma) \otimes L^2(F))$. Let $\tilde{\mathcal{A}}$ be the image of $\mathcal{A}$ under the embedding. We write

$$\mathcal{M}(\Gamma, \Theta) = \{A \in L(l^2(\Gamma) \otimes L^2(X))\colon A\tilde{\rho}_\gamma = \tilde{\rho}_\gamma A \ \ \forall \gamma \in \Gamma\}.$$

It is easy to show that $\tilde{\mathcal{A}} \subset \mathcal{M}(\Gamma, \Theta)$ and $R_A(\zeta) \in \mathcal{M}(\Gamma, \Theta)$ for all $\zeta \in \mathbf{C} \setminus \sigma(H_A)$.

Following [6], we now define the Fourier coefficients for the operator $A \in \mathcal{M}(\Gamma, \Theta)$. The Fourier coefficient $\hat{A}(\gamma)$ for an element $\gamma \in \Gamma$ is an operator in $L^2(F)$ acting on the function u in $L^2(F)$ according to the formula $\hat{A}(\gamma)(u) = \tilde{\rho}_\gamma A(\delta_1 \otimes u)(1)$.

Lemma 5 (see [5, 6]). *Let* $A \in \mathcal{M}(\Gamma, \Theta)$. *If the operator* $\hat{A}(\gamma)$ *is compact for all* $\gamma \in \Gamma$ *and satisfies the inequality*

$$\sum_{\gamma \in \Gamma} \|\hat{A}(\gamma)\| < +\infty,$$

then $A \in \tilde{\mathcal{A}}$.

The theorem below is the main result in this paper.

Theorem 3. *Let A be a T^d-invariant self-adjoint operator in the space $l^2(\Lambda)$ with a diagonal matrix in the standard basis of $l^2(\Lambda)$. Then the resolvent $R_A(\zeta)$ belongs to the algebra $\tilde{\mathcal{A}}$ for any $\zeta \in \mathbf{C} \setminus \sigma(H_A)$.*

Proof. We first note that because the resolvent is analytic, it suffices to prove that $R_A(E) \in \tilde{\mathcal{A}}$ for all E belonging to a semi-infinite interval $(-\infty, E_0)$. We next note that $R^0(\zeta) \in \tilde{\mathcal{A}}$ for all $\zeta \in \mathbf{C} \setminus \sigma(H^0)$ [6]; therefore, it is only necessary to prove that $V(E) \equiv R^0(E) - R_A(E) \in \tilde{\mathcal{A}}$ for all E belonging to $(-\infty, E_0)$. By Theorem 2, there is a number $E_0 \in \mathbf{R}$ such that

$$V(\zeta)f = \sum_{\lambda\in\Lambda}\Big(\sum_{\mu\in\Lambda} M(\lambda,\mu;\zeta)\langle g_\mu(\bar\zeta)|f\rangle\Big)\, g_\lambda(\zeta)$$

when $\operatorname{Re}\zeta < E_0$, where $M(\lambda,\mu;\zeta) = [Q(\zeta) + A]^{-1}(\lambda,\mu)$. By Lemmas 2 and 3, there are constants c_0 and $\tilde c_0(\zeta)$ such that

$$|M(\lambda,\mu;\zeta)| \le c_0 e^{-\tilde c_0(\zeta)\, d(\lambda,\mu)}, \quad \tilde c_0(\zeta) > 3\tilde c_\Lambda \tag{4}$$

for $\operatorname{Re}\zeta < E_0$, where $\tilde c_\Lambda$ is the constant in Lemma 3 and c_0 does not depend on ζ.

We define the matrix $M_\beta(\lambda,\mu;\zeta)$ by the relation

$$M_\beta(\lambda,\mu;\zeta) = \begin{cases} M(\lambda,\mu;\zeta) & \text{if } \lambda = \gamma\kappa \text{ and } \mu = \gamma\beta\kappa' \text{ for some } \gamma\in\Gamma \text{ and } \kappa,\kappa'\in\mathrm{K},\\ 0 & \text{otherwise,}\end{cases}$$

where $\beta \in \Gamma$. Because $(g_\lambda(\zeta))_{\lambda\in\Lambda}$ is the Riecz basis, formula (4) and Lemma A.1 imply that the series

$$V_\beta(\zeta)f = \sum_{\lambda\in\Lambda}\Big(\sum_{\mu\in\Lambda} M_\beta(\lambda,\mu;\zeta)\langle g_\mu(\bar\zeta)|f\rangle\Big)\, g_\lambda(\zeta)$$

converges and defines a bounded operator in $L^2(X)$ for any function $f \in L^2(X)$. We prove that

$$\sum_{\beta\in\Gamma} \|V_\beta(\zeta)\| < +\infty \tag{5}$$

for $\operatorname{Re}\zeta < E_0$. Because the family $(g_\lambda(\zeta))_{\lambda\in\Lambda}$ is the Riecz basis in its linear span, we have

$$\Big\|\sum_{\lambda\in\Lambda}\varphi(\lambda)g_\lambda(\zeta)\Big\| \le c_5(\zeta)\|\varphi\|, \qquad \sum_{\lambda\in\Lambda}|\langle g_\lambda(\bar\zeta)|f\rangle|^2 \le c_6^2(\zeta)\|f\|^2 \tag{6}$$

for $\varphi \in l^2(\Lambda)$ and $f \in L^2(X)$. Using (4) and (6), it is easy to derive the inequality

$$\|V_\beta(\zeta)f\|^2 \le (\#\,\mathrm{K})\, c_5^2\, c_6^2 \max\{e^{-2\tilde c_0(\zeta)\, d(\kappa,\beta\kappa')} : \kappa,\kappa'\in\mathrm{K}\}\, \|f\|^2,$$

whence (5) follows in view of Lemma A.1.

We next note that

$$\sum_{\beta\in\Gamma} V_\beta(\zeta) = V(\zeta) \tag{7}$$

for $\operatorname{Re}\zeta < E_0$ because by Lemmas 1, 3, and A.1, the series

$$\sum_{\beta\in\Gamma}\langle f_1 \mid V_\beta(\zeta) f_2\rangle$$

is absolutely convergent for any $f_1, f_2 \in C_0^\infty(X)$.

It remains to prove that $V_\beta(\zeta) \in \tilde{\mathcal{A}}$ for any $\beta \in \Gamma$ whenever $\operatorname{Re}\zeta < E_0$. For the Fourier coefficient $\widehat{V}_\beta(\gamma) \equiv \widehat{V}_\beta(\zeta)(\gamma)$, we obtain

$$\widehat{V}_\beta(\gamma)(u) = \Theta(1,\gamma)\sum_{\alpha\in\Gamma}\sum_{\kappa,\kappa'\in\mathrm{K}} e^{\omega(\gamma\alpha\kappa)} L_{\alpha,\kappa,\kappa'}(u)$$

by direct calculation. Here, $L_{\alpha,\kappa,\kappa'}$ is a one-dimensional operator of the form

$$L_{\alpha,\kappa,\kappa'}(u) = M(\alpha\kappa, \alpha\beta\kappa'; \zeta)\langle g_{\alpha\beta\kappa'}(\bar{\zeta})|\tilde{u}\rangle \tilde{g}_{\gamma\alpha\kappa}(\zeta),$$

where $\tilde{u}$ is the extension of the function $u \in L^2(F)$ throughout the plane X and $\tilde{g}$ is the restriction of $g \in L^2(X)$ to F. It is easy to show that

$$\sum_{\alpha,\kappa,\kappa'} \|L_{\alpha,\kappa,\kappa'}\| < +\infty.$$

Therefore, the operator $\widehat{V}_\beta(\gamma)$ is compact. By Lemma 5, to complete the proof, it remains to verify that

$$\sum_{\gamma\in\Gamma}\|\widehat{V}_\beta(\gamma)\| < +\infty. \tag{8}$$

Let $u \in L^2(F)$ and $\|u\| \le 1$. In this case,

$$\|\widehat{V}_\beta(\gamma)(u)\| \le c_7 \sum_{\kappa,\kappa'\in\mathrm{K}}\sum_{\alpha\in\Gamma} |\langle g_{\alpha\beta\kappa'}(\bar{\zeta})|\tilde{u}\rangle|\, \|\tilde{g}_{\gamma\alpha\kappa}(\zeta)\|.$$

Cumbersome calculations, which we omit, give

$$\sum_{\alpha\in\Gamma_2} |\langle g_{\alpha\nu}(\bar{\zeta})|\tilde{u}\rangle|\, \|\tilde{g}_{\gamma\alpha\kappa}(\zeta)\| \le c_8(\kappa,\zeta)\exp\Big(-\frac{3}{2}\tilde{c}_\Lambda\, d(\kappa,\gamma\kappa)\Big),$$

whence

$$\|\widehat{V}_\beta(\gamma)\| \le c_9(\zeta)\sum_{\kappa\in\mathrm{K}}\exp\Big(-\frac{3}{2}\tilde{c}_\Lambda\, d(\kappa,\gamma\kappa)\Big).$$

It remains to use Lemma A.1 to derive (8).

Corollary 2. *Let $E_1, E_2 \in \mathbf{R}\setminus\sigma(H_A)$ and $E_1 \le E_2$. Then the spectral projection $P_{[E_1,E_2]}$ of H_A belongs to the algebra $\tilde{\mathcal{A}}$.*

We now fix a number $E' \in \mathbf{R}$ such that $E' < \inf\sigma(H_A)$ and consider the function

$$\mathcal{N}(E) = \begin{cases} \tau(P_{[E',E]}) & \text{for } E \ge E', \\ 0 & \text{for } E < E'. \end{cases}$$

Clearly, it does not depend on the choice of the point E'. The values of $\mathcal{N}(E)$ are constant on the spectrum gaps of H_A and thus give a natural parameterization of the gaps [24].

Corollary 3 (the gap parameterization theorem). *The values of the function $\mathcal{N}(E)$ on the spectrum gaps in $\sigma(H_A)$ belong to a denumerable set of real numbers $\tau^*(K_0C^*(\Gamma,\Theta))$, where $K_0\mathcal{B}$ denotes the K_0 group of the C^* algebra $\mathcal{B}$.*

We recall that, by definition, a pair (Γ,Θ) possesses the Kadison property if there is a constant $c_K > 0$ such that $\tau(P) \geq c_K$ for any nonzero projection P in the C^* algebra $C^*(\Gamma,\Theta) \otimes \mathcal{K}$.

Corollary 4. *If the pair (Γ,Θ) has the Kadison property, then the spectrum $\sigma(H_A)$ has a band structure.*

We note that the condition in this corollary holds for a torsion-free Fuchsian group Γ and a magnetic field with a rational flow across the fundamental domain F [7, 8]. (For a detailed discussion of the Kadison property in the context of the Hall quantum effect on the Lobachevsky plane, see [8].)

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Appendix

We consider a discrete metric space Λ with a metric d possessing the property

there are constants $c_\Lambda > 0$ and $\tilde{c}_\Lambda > 0$ such that

$$\#\{\lambda \in \Lambda \colon d(\lambda,\lambda_0) \leq r\} \leq c_\Lambda e^{\tilde{c}_\Lambda r}$$

for all $\lambda_0 \in \Lambda$ and $r \in \mathbf{R}$, $r > 0$.

(For the proof of the lemmas below, see [25].)

Lemma A.1. *Let a function $\varphi\colon \Lambda \to \mathbf{C}$ satisfy the inequality*

$$|\varphi(\lambda)| \leq ce^{-(1+\delta)\tilde{c}_\Lambda d(\lambda,\mu)}$$

for some $\mu \in \Lambda$ and positive c and δ. Then

$$\sum_{\lambda\in\Lambda} |\varphi(\lambda)| \leq cc_\Lambda \delta^{-1}.$$

Lemma A.2 (Schur's test). *Let $(L(\lambda,\mu))_{\lambda,\mu\in\Lambda}$ be an infinite matrix such that*

$$\sup_{\mu\in\Lambda} \sum_{\lambda\in\Lambda} |L(\lambda.\mu)| \leq C, \qquad \sup_{\lambda\in\Lambda} \sum_{\mu\in\Lambda} |L(\lambda,\mu)| \leq C$$

for some $C > 0$. Then $L(\lambda,\mu)$ defines a bounded linear operator L in $l^2(\Lambda)$ such that $\|L\| \leq C$.

Theorem A.1. *Let $(K_n)_{n\geq 0}$ be a sequence of bounded linear operators in $l^2(\Lambda)$ satisfying the following conditions in the standard basis of the matrix $(K_n(\lambda,\mu))_{\lambda,\mu\in\Lambda}$:*

1. *if $\lambda \neq \mu$, then $|K_n(\lambda,\mu)| \leq a\exp(-b_n\, d(\lambda,\mu))$, where a does not depend on n and $b_n \to +\infty$ as $n \to +\infty$;*
2. *$\inf_{\lambda\in\Lambda} |K_n(\lambda,\lambda)| \to +\infty$ as $n \to +\infty$.*

Then for any α, $0 < \alpha < 1$, there is an $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$, the operator K_n has a bounded inverse operator $L_n = K_n^{-1}$ whose matrix satisfies the condition $|L_n(\lambda,\mu)| \leq 2c_n \exp(-\alpha b_n d(\lambda,\mu))$, where $c_n = (\inf_{\lambda\in\Lambda} |K_n(\lambda,\lambda)|)^{-1}$.

Proof. Let $D_n(\lambda,\mu) = K_n(\lambda,\lambda)\delta_{\lambda\mu}$ and $S_n(\lambda,\mu) = K_n(\lambda,\mu) - D_n(\lambda,\mu)$. We can assume without loss of generality that $\inf\{|K_n(\lambda,\lambda)|\colon \lambda \in \Lambda\} > 0$ and $b_n > \bar{c}_\Lambda$ for all $n \in \mathbf{N}$. In this case, we have $K_n = D_n(1 + D_n^{-1}S_n)$, where D_n and S_n are determined by the corresponding matrices. Conditions 1 and 2 in Theorem A.1 and also Lemmas A.1 and A.2 imply that the operator $1 + D_n^{-1}S_n$ has a bounded inverse operator $T_n = (1 + D_n^{-1}S_n)^{-1}$ for sufficiently large n. To prove the theorem, it suffices to show that

$$|T_n(\lambda,\mu)| \leq 2e^{-\alpha b_n d(\lambda,\mu)}. \tag{9}$$

We fix $\alpha \in (0,1)$ and take a sufficiently large n such that

$$(1-\alpha)b_n \geq 2\bar{c}_\Lambda, \qquad ac_\Lambda c_n \leq \frac{1}{2}, \qquad \|D_n^{-1}S_n\| < 1.$$

We estimate the sum

$$T_n(\lambda,\mu) = \sum_{j\geq 0} (-D_n^{-1}S_n)^j(\lambda,\mu).$$

To prove (9), it suffices to show by induction that

$$|(D_n^{-1}S_n)^j(\lambda,\mu)| \leq (ac_\Lambda c_n)^j e^{-\alpha b_n d(\lambda,\mu)}. \tag{10}$$

Inequality (10) is obvious for $j = 0$. We suppose that it holds for some $j \geq 0$. Then

$$\begin{aligned}
|(D_n^{-1}S_n)^{j+1}(\lambda,\mu)| &= \left| K_n(\lambda,\lambda)^{-1} \sum_{\kappa\in\Lambda} S_n(\lambda,\kappa)(D_n^{-1}S_n)^j(\kappa,\mu) \right| \leq \\
&\leq c_n \sum_{\kappa\in\Lambda} ae^{-b_n d(\lambda,\kappa)} (ac_\Lambda c_n)^j e^{-\alpha b_n d(\kappa,\mu)} \leq \\
&\leq ac_n (ac_\Lambda c_n)^j e^{-\alpha b_n d(\lambda,\mu)} \sum_{\kappa\in\Lambda} e^{-2\bar{c}_\Lambda d(\lambda,\kappa)} \leq \\
&\leq (ac_\Lambda c_n)^{j+1} e^{-\alpha b_n d(\lambda,\mu)},
\end{aligned}$$

that is, (10) holds for $j+1$ as well.

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