

# ON THE $\eta$ -INVARIANT OF CERTAIN NONLOCAL BOUNDARY VALUE PROBLEMS

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**1. Introduction.** A very intriguing feature of elliptic operators on compact manifolds without boundary is the locality of their indices. Specifically, if  $M$  denotes a compact Riemannian spin manifold,  $S \rightarrow M$  a spinor bundle,  $E \rightarrow M$  a hermitian coefficient bundle with unitary connection, and  $D^E$  the Dirac operator on  $M$  with coefficients in  $E$ , then, by the Atiyah-Singer theorem,

$$\operatorname{ind} D_+^E = \int_M \hat{A}(M) \wedge \operatorname{ch} E. \quad (1.1)$$

Here  $D_+^E$  arises from splitting  $S \otimes E$  under the involution induced by the complex volume element on  $M$ .

If  $M$  decomposes along a compact hypersurface  $N$  as  $M = M_1 \cup M_2$ , with  $\partial M_i = N$  for  $i = 1, 2$ , then one is led to ask whether the obvious decomposition of the right-hand side in (1.1) corresponds to a decomposition of the (essentially) selfadjoint operator  $D^E$  into selfadjoint operators  $D_i^E$ , defined in  $M_i$  by suitable boundary conditions on  $N$ , such that

$$\operatorname{ind} D_{1,+}^E + \operatorname{ind} D_{2,+}^E = \operatorname{ind} D_+^E. \quad (1.2)$$

This question was answered in the affirmative by Atiyah, Patodi, and Singer [APS] who formulated the correct boundary conditions (cf. Sec. 2 for details). More importantly, the resulting index formula (2.7) displayed a new spectral invariant of selfadjoint elliptic operators (defined on  $N$ ), which they called the  $\eta$ -invariant. It is not locally computable by a formula as in (1.1), as can be seen from its behaviour under coverings. Nevertheless, one can ask how the  $\eta$ -

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invariant behaves under splitting  $N$  as  $N_1 \cup N_2$ , and this is the problem we address in this work.

One motivation for posing this question may be seen in trying to understand the signature theorem on manifolds with corners. From a systematical point of view, splitting formulas for spectral invariants should also be very useful for computational purposes—as illustrated nicely by the analytic torsion (compare with [Ch] and [M1])—and as a possible source of new invariants. Another recent motivation is provided by topological quantum field theory.

The “gluing law” for  $\eta$ -invariants that we prove here (Thms. 3.10, 3.11) is not new; compare with Section 2 for an account of previous work. Our proof, however, attacks the problem directly on the cut manifold  $M^{\text{cut}}$  by analyzing families of “generalized Atiyah-Patodi-Singer boundary value problems.” These new abstract boundary conditions are defined by three simple axioms (see (3.23)–(3.25)), which are designed in such a way that the heat kernel of the model operator is explicitly computable. Incidentally, our formula generalizes a result of Sommerfeld in the scalar case. Moreover, under this class we find the spectral boundary conditions introduced by Atiyah, Patodi, and Singer, as well as the (local) absolute and relative boundary conditions for the Gauss-Bonnet operator. Thus, our method gives a uniform way to derive the asymptotic expansion of the heat trace in both cases, generalizing in particular recent work by Grubb and Seeley [GrSe1] (cf. Thm. 3.4). The family we define interpolates between the “uncut manifold” (the case of smooth transmission) and actual Atiyah-Patodi-Singer boundary value problems. This is similar to Vishik’s approach [V] to the splitting behavior of the analytic torsion, and we hope to explore this further in another publication. The special structure of our family, on the other hand, resembles closely the finite-dimensional variations constructed by Lesch and Wojciechowski [LW]. This allows us to produce explicit variation formulas (Thm. 3.5). We evaluate them using the vanishing of the noncommutative residue on pseudodifferential idempotents and a special symmetry of the cutting problem.

The plan of the paper is as follows. In Section 2, we review some abstract facts on  $\eta$ -invariants and previous work on the gluing law. All results are presented in Section 3, while the details of most proofs are carried out in Section 4.

**2. Generalities.** In this section we briefly review some more or less well-known properties of  $\eta$ -invariants that are needed below, together with some of the previous work leading to the gluing law.

The  $\eta$ -invariant was introduced in the seminal work [APS] by Atiyah, Patodi, and Singer. They considered the signature operator on a smooth oriented Riemannian manifold  $M$  with compact boundary  $\partial M = N$ ,  $\dim M = m = 4k$ . The signature operator is the operator  $D = d + \delta$  restricted to the space of self-dual forms (cf. (2.6)). Assuming that the metric is a product in a neighborhood

$$U \simeq [0, 1) \times N \tag{2.1a}$$

of the boundary, separation of variables leads to the representation

$$D = \gamma \left( \frac{\partial}{\partial x} + A \right). \tag{2.1b}$$

Here, we use the decomposition of a smooth form  $\alpha$  as  $\alpha = dx \wedge \alpha_1(x) + \alpha_2(x)$ . Thus, the operator on the right acts on  $C_0^\infty((0, 1), \Omega(N) \oplus \Omega(N))$ ,  $\Omega(N)$  are the smooth forms on  $N$ , and one has

$$\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I, \quad A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \otimes (d_N + \delta_N). \tag{2.1c}$$

Thus  $A$  is symmetric, and we have the relations

$$\gamma^2 = -I, \quad \gamma^* = -\gamma, \quad \gamma A + A\gamma = 0. \tag{2.2}$$

A symmetric operator of type (2.1b) does not in general admit local boundary conditions that define a selfadjoint extension (cf., however, [GSm] and [Si]), even though local boundary conditions do exist in the special case (2.1c), that is, the absolute and relative boundary conditions. But there is always a nonlocal boundary condition given (essentially) by the Calderón projector (see [C]). Thus we introduce the boundary condition

$$P_{>0}(A)u(0) = 0, \tag{2.3a}$$

where  $P_{>0}(A)$  is the orthogonal projection onto the subspace spanned by eigenvectors of  $A$  with positive eigenvalues. To define a symmetric operator, this needs to be supplemented by

$$P_\sigma u(0) = 0, \tag{2.3b}$$

where  $P_\sigma$  projects onto a Lagrangian subspace of  $\ker A$  with respect to the symplectic form (note that  $\dim \ker A$  is even)

$$\omega(u, v) := \langle \gamma u, v \rangle, \quad u, v \in \ker A.$$

Such a space can always be viewed as the  $+1$ -eigenspace of an involution  $\sigma$  on  $\ker A$ , satisfying

$$\sigma\gamma + \gamma\sigma = 0; \tag{2.4a}$$

then

$$P_\sigma = \frac{1}{2}(I + \sigma). \tag{2.4b}$$

In the case at hand, a convenient choice of  $\sigma$  is (Clifford multiplication by) the complex volume element  $\omega_M$ ; that is, we put  $\sigma_0 := \omega_M|_{\ker A}$ , and observe that it takes the form

$$\omega_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \omega_N,$$

where  $\omega_N$  denotes the complex volume element on  $N$ .

It is not hard to see that these data define a selfadjoint extension of  $D, D_{\sigma_0}$ , which anticommutes with  $\omega_M$ . Then the signature operator  $D_S$  for a manifold with boundary is the closure of

$$D_{\sigma_0} | \mathcal{D}(D_{\sigma_0}) \cap \{u \in \Omega(\bar{M}) \mid \omega_M u = u\},$$

and [APS, Thm. I.3.10] asserts that  $D_S$  is a Fredholm operator with

$$\text{ind } D_S = \int_M L(M) - \frac{1}{2}(\eta(B) + \dim \ker B). \tag{2.5}$$

Here,  $L(M)$  denotes the Hirzebruch  $L$ -form, and the operator  $B$  is defined by a representation of  $D_S$  in  $U$  analogous to (2.1b). In fact, near  $\partial M$  we have

$$\begin{aligned} D_S &= \omega_N(\partial_x + \omega_N(d_N + \delta_N)) \\ &=: \omega_N(\partial_x + B), \end{aligned}$$

and a core is given by the space (with obvious notation)

$$\mathcal{D}(D_S) = \{u \in \Omega(\bar{M}) \mid P_{\geq 0}(B)u(0) = 0, \omega_M u = u\}. \tag{2.6}$$

Rewriting (2.5) in terms of the signature of  $M$  (as a manifold with boundary) gives

$$\text{sign } M = \int_M L(M) - \frac{1}{2}\eta(B) \tag{2.7}$$

(see [APS, Thm. (I.4.14)]), and thus it gives an analytic interpretation of the additivity of the signature under cutting along a separating hypersurface.

The  $\eta$ -invariant figuring in (2.5) and (2.7) is derived from a meromorphic function generalizing the  $\zeta$ -function of an elliptic operator. It is convenient to derive the main properties of these functions in an abstract functional analytic setting. Thus consider a selfadjoint operator  $A$  with dense domain  $\mathcal{D}(A)$  in some Hilbert space  $H$ . If we assume that

$$(A + i)^{-1} \in C_p(H), \quad \text{for some } p > 0 \tag{2.8}$$

(where  $C_p$  denotes the Schatten and von Neumann class of order  $p$ ), then the function

$$\eta(A; s) := \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \text{tr}_H(Ae^{-tA^2}) dt = \sum_{\lambda \in \text{spec } A \setminus \{0\}} (\text{sgn } \lambda) |\lambda|^{-s} \tag{2.9}$$

is holomorphic for large  $\text{Re } s$ . More generally, if  $B : \mathcal{D}(A) \rightarrow H$  is any bounded operator satisfying

$$P_0(A)BP_0(A) = 0, \tag{2.10}$$

where  $P_0(A)$  is the orthogonal projection onto  $\ker A$ , then the same is true of

$$\begin{aligned} \eta(A, B; s) &:= \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \text{tr}_H(Be^{-tA^2}) dt \\ &= \sum_{\lambda \in \text{spec } A \setminus \{0\}} (\text{tr}_{\ker(A-\lambda)} B) |\lambda|^{-s-1}. \end{aligned} \tag{2.11}$$

Here, by slight abuse of notation,  $\text{tr}_{\ker(A-\lambda)} B := \text{tr}(P_\lambda B)$ , where  $P_\lambda$  is the orthogonal projection onto the  $\lambda$ -eigenspace of  $A$ . It is very important to determine conditions on  $A$  and  $B$  that guarantee the existence of a meromorphic extension of (2.11) to the whole complex plane. The standard source of such an extension is an asymptotic expansion

$$\text{tr}_H(Be^{-tA^2}) \sim_{t \rightarrow 0^+} \sum_{\substack{\text{Re } \alpha \rightarrow \infty \\ 0 \leq k \leq k(\alpha)}} a_{\alpha k}(A, B) t^\alpha \log^k t. \tag{2.12}$$

The notation used means, of course, that  $\{\alpha \in \mathbf{C} \mid a_{\alpha k}(A, B) \neq 0 \text{ for some } k \in \mathbf{Z}_+, k \leq k(\alpha)\}$  is a countable subset of  $\mathbf{C}$  whose real parts accumulate at most at  $\infty$ .

Using the notation  $f(s) =: \sum_k \text{Res}_k f(s_0)(s - s_0)^{-k}$ , introduced in [BS2] for Laurent expansions, one has the following lemma.

LEMMA 2.1. *Under the conditions (2.8), (2.10), and (2.12),  $\eta$  extends to a meromorphic function on  $\mathbf{C}$ . The poles are situated at the points  $s_\alpha = -2\alpha - 1$ , and the principal part of  $\eta$  at  $s_\alpha$  is given by*

$$\frac{1}{\Gamma((s+1)/2)} \sum_{k=0}^{k(\alpha)} a_{\alpha k}(A, B) (-1)^k k! 2^{k+1} (s - s_\alpha)^{-k-1}.$$

In particular, the poles are of order

(1)  $k(\alpha) + 1$ , if  $\alpha \notin \mathbf{Z}_+$  and

$$\text{Res}_{k(\alpha)+1} \eta(A, B; s_\alpha) = \frac{(-1)^{k(\alpha)} k(\alpha)! 2^{k(\alpha)+1}}{\Gamma(-\alpha)} a_{\alpha, k(\alpha)}(A, B), \quad (2.13a)$$

and

(2)  $k(\alpha)$ , if  $\alpha \in \mathbf{Z}_+$  and

$$\text{Res}_{k(\alpha)} \eta(A, B; s_\alpha) = (-1)^{k(\alpha)+\alpha} \alpha! k(\alpha)! 2^{k(\alpha)} a_{\alpha, k(\alpha)}(A, B). \quad (2.13b)$$

LEMMA 2.2. Under the conditions (2.8) and (2.10), the following statements are equivalent.

(i)  $\text{tr}_H(Be^{-tA^2})$  has an asymptotic expansion of type (2.12) that can be differentiated; that is, for  $N, K > 0$ , we have

$$\left| \partial_t^N \left( \text{tr}_H(Be^{-tA^2}) - \sum_{\substack{\text{Re } \alpha \leq N+K \\ 0 \leq k \leq k(\alpha)}} a_{\alpha k}(A, B) t^\alpha \log^k t \right) \right| \leq C_{N,K} t^K, \quad t \rightarrow 0. \quad (2.14)$$

(ii)  $\Gamma((s+1)/2)\eta(A, B; s)$  is holomorphic in the half-plane  $\{s \in \mathbf{C} \mid \text{Re } s > p\}$  and extends meromorphically to  $\mathbf{C}$ . Moreover, for  $a, b \in \mathbf{R}$ , there exists  $s_0 = s_0(a, b) > 0$  such that  $\Gamma((s+1)/2)\eta(A, B; s)$  is holomorphic for  $a \leq \text{Re } s \leq b$ ,  $|s| \geq s_0$  with estimate

$$|\Gamma((s+1)/2)\eta(A, B; s)| \leq C(a, b, N) |s|^{-N}, \quad a \leq \text{Re } s \leq b, |s| \geq s_0, \quad (2.15)$$

for any  $N > 0$ .

Proof. (i)  $\Rightarrow$  (ii). In view of (2.8) and (2.10)  $\Gamma((s+1)/2)\eta(A, B; s)$  is holomorphic in the half-plane  $\{s \in \mathbf{C} \mid \text{Re } s > p\}$  and extends meromorphically to  $\mathbf{C}$ , by Lemma 2.1. Integration by parts gives

$$\begin{aligned} & \Gamma((s+1)/2)\eta(A, B; s) \\ &= \frac{(-1)^N 2^N}{(s+1)(s+3) \cdots (s+2N-1)} \int_0^\infty t^{(s-1)/2+N} \partial_t^N \text{tr}_H(Be^{-tA^2}) dt. \end{aligned} \quad (2.16)$$

In view of (2.10), we have for  $a \leq \text{Re } s \leq b$

$$\left| \int_1^\infty t^{(s-1)/2+N} \partial_t^N \text{tr}_H(Be^{-tA^2}) dt \right| \leq C \int_1^\infty t^{(b-1)/2+N} e^{-\epsilon t} dt =: C_{N,b}. \quad (2.17)$$

Furthermore, choosing  $K$  such that  $(a-1)/2 + K + N > -1$ , we may write

$$\begin{aligned} & \int_0^1 t^{(s-1)/2+N} \partial_t^N \text{tr}_H(Be^{-tA^2}) dt \\ &= \int_0^1 t^{(s-1)/2+N} \varphi_{K,N}(t) dt + \sum_{\substack{\text{Re } \alpha \leq N+K \\ 0 \leq k \leq k(\alpha)}} a_{\alpha k}(A, B) \int_0^1 t^{(s-1)/2+N} \partial_t^N t^\alpha \log^k t dt \end{aligned} \quad (2.18)$$

with  $|\varphi_{K,N}(t)| \leq C_{K,N} t^K$ . Hence, we have for  $a \leq \text{Re } s \leq b$

$$\left| \int_0^1 t^{(s-1)/2+N} \varphi_{K,N}(t) dt \right| \leq C_{N,K}. \quad (2.19)$$

Using  $\partial_t^N t^\alpha \log^k t = \sum_{i=0}^k c_i t^{\alpha-N} \log^i t$ , we get

$$\int_0^1 t^{(s-1)/2+N} \partial_t^N t^\alpha \log^k t dt = \sum_{i=0}^k c_i (-1)^i i! ((s+1)/2 + \alpha)^{-i-1}. \quad (2.20)$$

Combining (2.16) through (2.20), we reach the conclusion.

(ii)  $\Rightarrow$  (i). In view of the estimate (2.15), we can apply the inverse Mellin transform to find, for  $c > p$ ,

$$\text{tr}_H(Be^{-tA^2}) = \frac{1}{4\pi i} \int_{\text{Re } s=c} t^{-(s+1)/2} \Gamma((s+1)/2) \eta(A, B; s) ds.$$

Moreover, we can shift the contour of integration to the left and apply the residue theorem to get

$$\text{tr}_H(Be^{-tA^2}) \sim_{t \rightarrow 0^+} \frac{1}{2} \sum_{s \in \mathbf{C}} \text{Res}_1 \left( t^{-(s+1)/2} \Gamma((s+1)/2) \eta(A, B; s) \right).$$

Clearly, this asymptotic expansion can be differentiated. □

Remarks. (1) Of course,  $B := I - P_0(A)$  gives the  $\zeta$ -function of  $A^2$ ,

$$\zeta_{A^2}((s+1)/2) = \eta(A, I - P_0(A); s).$$

In particular, we can read off the regularity at zero of  $\zeta_{A^2}$ , provided that the asymptotic expansion of  $\text{tr}_H(e^{-tA^2})$  exists and does not contain contributions to  $\log^k t, k \in \mathbf{N}$ .

(2) If  $A$  and  $B$  are classical pseudodifferential operators on a compact manifold  $M$ ,  $\dim M =: m$ , and  $A$  is selfadjoint elliptic of positive order, then (2.8)

holds and we have an asymptotic expansion

$$\mathrm{tr}_H(Be^{-tA^2}) \sim_{t \rightarrow 0+} \sum_{j=0}^{\infty} a_j(A, B) t^{(j-m-b)/2a} + \sum_{j=0}^{\infty} (b_j(A, B) \log t + c_j(A, B)) t^j, \tag{2.21}$$

where  $a := \mathrm{ord} A$ ,  $b := \mathrm{ord} B$  (see [GrSe1, Theorem 2.7]). Moreover, this asymptotic expansion can be differentiated in view of the identity

$$\partial_t^N \mathrm{tr}_H(Be^{-tA^2}) = (-1)^N \mathrm{tr}_H(BA^{2N}e^{-tA^2}).$$

If, in addition, (2.10) holds then we can apply Lemma 2.2 to conclude that (2.15) holds for  $A$  and  $B$ .

Note that in view of (2.21) and Lemma 2.1, in this case  $\eta(A, B; s)$  has a meromorphic continuation to  $\mathbf{C}$  with *simple* poles.

The estimate (2.15) suffices to shift the contour of integration and to deduce a short time asymptotic expansion. However, for some classical pseudodifferential operators  $A, B$ , an even stronger result holds—namely, if  $A$  has a scalar principal symbol, then it follows from [DG] that  $\eta(A, B; s)$  is of polynomial growth on finite vertical strips. Since  $\Gamma((s+1)/2)$  decays exponentially on finite vertical strips, this implies the estimate (2.15). However, our method of proving (2.15) is completely elementary while [DG] uses the machinery of Fourier integral operators.

Given these preparations we define, under the assumptions of Lemma 2.1 (actually, a partial expansion in (2.12) would suffice), the  $\eta$ -invariant of  $A$  as

$$\eta(A) := \mathrm{Res}_0 \eta(A; 0) \tag{2.22a}$$

and, in view of the index formula (2.5), the *reduced*  $\eta$ -invariant of  $A$  as

$$\xi(A) := \frac{1}{2}(\eta(A) + \dim \ker A). \tag{2.22b}$$

Generally,  $\eta(A)$  is difficult to compute. It is thus of great importance that suitable 1-parameter variations turn out to be “locally computable” in the sense of asymptotic expansions of the type (2.12).

To deal with variations in the abstract framework above, we now impose the following assumptions. Consider a connected open subset  $J$  of  $\mathbf{R}$  and for  $a \in J$  a family

$$A(a) : \mathcal{D} \rightarrow H \tag{2.23a}$$

of selfadjoint operators with fixed domain  $\mathcal{D}$ , satisfying (2.8). Moreover, assume that this family has kernel of constant rank; that is, for  $P_0(a) := P_0(A(a))$  we have

$$\dim P_0(a) \text{ is constant in } J. \tag{2.23b}$$

Likewise, let

$$B(a) : \mathcal{D} \rightarrow H \tag{2.23c}$$

be another family of bounded operators satisfying (2.10), which, in addition, commutes with  $A(a)^2$  in the sense that

$$[B(a), (A(a)^2 - \zeta)^{-1}] = 0, \quad a \in J, \zeta \notin \mathrm{spec} A(a)^2. \tag{2.23d}$$

Note that (2.10) and (2.23d) imply that

$$B(a) = (I - P_0(a))B(a)(I - P_0(a)).$$

Finally, we assume that

$$\text{the families } (A(a))_{a \in J}, (B(a))_{a \in J} \subset \mathcal{L}(\mathcal{D}, H) \text{ are strongly differentiable in } J, \text{ with strongly continuous derivative.} \tag{2.23e}$$

Under these assumptions, the operator families  $P_0(a)$  and

$$\tilde{A}(a) := (I - P_0(a))A(a) + P_0(a) \tag{2.24}$$

are strongly differentiable, too. Using the representation

$$e^{-tA(a)^2} = \frac{(m-1)! t^{1-m}}{2\pi i} \int_{\Gamma} e^{-t\zeta} (A(a)^2 - \zeta)^{-m} d\zeta,$$

with  $\Gamma$  a suitable contour, one can easily derive the identity

$$\begin{aligned} \frac{\partial}{\partial a} \mathrm{tr}_H [B(a)e^{-tA(a)^2}] &= \mathrm{tr}_H [B'(a)e^{-tA(a)^2}] \\ &+ t \frac{\partial}{\partial t} \mathrm{tr}_H \left[ B(a) \left( \frac{d}{da} A(a)^2 \right) \tilde{A}(a)^{-2} e^{-tA(a)^2} \right]. \end{aligned}$$

Our assumptions imply the absolute and locally uniform convergence of the relevant  $t$ -integrals, and we arrive at the following lemma.

LEMMA 2.3. Under the assumptions (2.8) and (2.23a–e), we have the identity

$$\frac{\partial}{\partial a} \eta(A(a), B(a); s) = \eta(A(a), B'(a); s) - \frac{s+1}{2} \eta\left(A(a), B(a) \left(\frac{d}{da} A(a)^2\right) \tilde{A}(a)^{-2}; s\right). \quad (2.25)$$

If we assume in addition that

$$[B(a), (A(a) - \zeta)^{-1}] = 0 \quad \text{for } a \in J, \zeta \notin \text{spec } A(a), \quad (2.23d')$$

then (2.25) simplifies to

$$\frac{\partial}{\partial a} \eta(A(a), B(a); s) = \eta(A(a), B'(a); s) - (s+1) \eta(A(a), A'BA\tilde{A}(a)^{-2}; s). \quad (2.26)$$

So, if both sides extend meromorphically to  $\mathbb{C}$ , then (2.26) holds in  $\mathbb{C}$ , too. We note in particular that

$$\frac{\partial}{\partial a} \eta(A(a); s) = -s \eta(A(a), A'(a); s). \quad (2.27)$$

Thus we obtain the following well-known corollary.

COROLLARY 2.4. Assume (2.8), (2.23a,b,e), and (2.12) with  $A(a)$  and  $A'(a)$  in place of  $B$ . Then, for  $k = K(-\gamma_2) \in \mathbb{Z}_+$ ,

$$\begin{aligned} \frac{d}{da} \text{Res}_k \eta(A(a); 0) &= -\text{Res}_{k+1} \eta(A(a), A'(a); 0) \\ &= \frac{(-1)^{k+1} k! 2^{k+1}}{\sqrt{\pi}} a_{-1/2,k}(A(a), A'(a)). \end{aligned} \quad (2.28)$$

Condition (2.23b) is not satisfied in interesting situations. One can get rid of it by choosing a real number  $c > 0$  so that  $c \notin \text{spec } A(a)$  for  $a$  near  $a_0 \in J$ . Then we put  $\tilde{P}_{<c}(a) := P_{<c}(a)P_{>-c}(a)$ ,  $\tilde{P}_{>c}(a) := I - \tilde{P}_{<c}(a)$ , and we replace  $A(a)$  by  $A^c(a) := \tilde{P}_{>c}(a)A(a) + \tilde{P}_{<c}(a)$  and  $B(a)$  by  $B^c(a) := \tilde{P}_{>c}(a)B(a)\tilde{P}_{>c}(a) + \tilde{P}_{<c}(a)$ , obtaining the modified  $\eta$ -function  $\eta^c(A(a), B(a); s) := \eta(A^c(a), B^c(a); s)$ .  $\eta^c$  admits, near  $a_0$ , the same analysis as outlined for  $\eta$  with (2.23b), and from (2.11) we obtain

$$(\eta - \eta^c)(A(a), B(a); s) = \sum_{\substack{\lambda \in \text{spec } A(a) \\ 0 < |\lambda| < c}} |\lambda|^{-s-1} \text{tr}_{\ker(A(a)-\lambda)} B(a) - \dim \tilde{P}_{<c}(a). \quad (2.29)$$

This is a smooth function of  $a$  and is holomorphic in  $s \in \mathbb{C}$ . On the other hand, the negative  $t$ -powers in the expansion (2.12) are unaffected if we modify  $A$  and  $B$  by an operator of finite rank. Evaluating (2.29) with  $B(a) := A(a)$ , we obtain

$$\frac{1}{2}(\eta - \eta^c)(A(a); s) + \frac{1}{2} \dim \ker A(a) = \sum_{\substack{\lambda \in \text{spec } A(a) \\ 0 < |\lambda| < c}} \frac{1}{2} (\text{sgn } \lambda |\lambda|^{-s} - 1),$$

and consequently we obtain the following lemma.

LEMMA 2.5. Assume that the family  $A(a)_{a \in J}$  satisfies (2.8), (2.23a,e), and (2.12) with  $A(a)$  and  $A'(a)$  in place of  $B$ . Then, for  $a, a_0 \in J$ ,

$$\xi(A(a)) - \xi(A(a_0)) + \frac{1}{\sqrt{\pi}} \int_{a_0}^a a_{-1/2,0}(A(a), A'(a)) da \in \mathbb{Z}. \quad (2.30)$$

This implies that the function

$$\tau(A(a)) := e^{2mi\xi(A(a))} \quad (2.31)$$

is always smooth in  $a \in J$  under our assumptions; the invariant  $\tau$  was introduced in [DF]. Instead of  $\tau$  we use the reduced  $\eta$ -invariant

$$\bar{\eta}(A(a)) := \xi(A(a)) \bmod \mathbb{Z}, \quad (2.32)$$

that is, the image of the  $\tau$ -invariant under the diffeomorphism  $S^1 \cong \mathbb{R}/\mathbb{Z}$ .

If the asymptotic expansion of  $\text{tr}_H [A'(a)e^{-tA(a)^2}]$  does not contain terms of the form  $t^\alpha \log^k t$  with  $\alpha < 0$  and  $k \in \mathbb{N}$  for  $j = 0, 1$ —as it is the case for (classical) elliptic pseudodifferential operators on compact manifolds (cf. the remarks after Lemma 2.2)—then it follows from Lemmas 2.1 and 2.3 that zero is at most a simple pole of  $\eta$  and that the residue is a homotopy invariant. This is the basis for proving that  $\eta(A; s)$  is, in fact, regular at  $s = 0$  if  $A$  happens to be a (classical) pseudodifferential operator on a compact manifold (cf. [G, Sec. 3.8]). More generally, Wodzicki [Wod1], [Wod2] observed the remarkable fact that, in this class of operators,

$$\text{Res } B := (\text{ord } A) \text{Res}_1 \eta(A, B; -1) = -2(\text{ord } A) a_{0,1}(A, B) \quad (2.33)$$

defines the unique trace (up to a constant) on classical pseudodifferential operators if  $A$  is elliptic of positive order  $\text{ord } A$ . Wodzicki also observed the following result, which is stated without proof in his thesis [Wod1].

LEMMA 2.6. If  $B$  is a classical pseudodifferential operator on a compact manifold and an idempotent, then

$$\text{Res } B = 0.$$

The only proof we know of shows that the statement of this lemma follows from the regularity at zero of the  $\eta$ -function for general, classical, elliptic pseudodifferential operators on a compact manifold. For completeness, we indicate that these facts are actually equivalent.

LEMMA 2.7. *The assertion of Lemma 2.6 is equivalent to the following. Let  $P$  be a selfadjoint, classical, elliptic pseudodifferential operator of positive order on the compact manifold  $M$ . Then*

$$\text{Res}_1 \eta(P; 0) = 0.$$

*Proof.* (1) First we assume Lemma 2.6. Let  $P$  be a selfadjoint, classical, elliptic pseudodifferential operator of order  $d > 0$  on a compact manifold  $M$ . We consider the pseudodifferential operator

$$\text{sgn } P := P|P|^{-1} : x \mapsto \begin{cases} |P|^{-1}Px, & x \in \ker P^\perp, \\ 0, & x \in \ker P. \end{cases}$$

We find

$$\eta(P^2, \text{sgn } P; s) = \sum_{\lambda \in \text{spec } P} (\text{sgn } \lambda) |\lambda|^{-s-1} = \eta(P; s+1),$$

an hence in view of (2.33) and Lemma 2.6

$$0 = \text{Res sgn } P = (\text{ord } P) \text{Res}_1 \eta(P^2, \text{sgn } P; -1) = (\text{ord } P) \text{Res}_1 \eta(P; 0). \quad (2.34)$$

(2) To prove the converse, we consider a classical pseudodifferential idempotent  $B$  on a compact manifold  $M$ .  $B$  is similar to a selfadjoint idempotent, and it is not difficult to see that the similarity can be effected through a pseudodifferential operator. Since the residue is a trace, similar operators have the same residue. Hence we may assume  $B$  to be an orthogonal projection. The assertion will follow from (2.34) if we can show that there exists an invertible selfadjoint classical pseudodifferential operator  $P$  of order 1 with

$$B = \frac{1}{2}(\text{sgn } P + I).$$

We choose a first-order, selfadjoint, classical pseudodifferential operator  $Q$  with scalar principal symbol  $\sigma_Q(\xi) = |\xi|$ . Furthermore, we may choose  $Q$  to be positive. Then put

$$\tilde{P} := BQB - (I - B)Q(I - B).$$

$\tilde{P}$  is elliptic and commutes with  $B$ . To make it invertible, we put

$$Px := \begin{cases} \tilde{P}x, & x \in \ker \tilde{P}^\perp, \\ x, & x \in \ker \tilde{P} \cap \text{Im } B, \\ -x, & x \in \ker \tilde{P} \cap \ker B. \end{cases}$$

By construction, we have  $B = (\text{sgn } P + I)/2$ , and hence we reach the conclusion. □

We emphasize that the regularity at 0 of  $\eta$ -functions is not essential if one wants to establish index theorems [BS1] or the gluing law below; the definition (2.22a) is perfectly sufficient for these purposes.

If one wants to widen the class of operators that admit reasonable  $\eta$ -invariants, then it is most natural to consider elliptic boundary value problems. As illustrated by the gluing question, one may also expect further insight in the compact case. The first work in this direction seems to be [GSm], which deals with local boundary conditions leading to (mildly) nonselfadjoint operators that do, however, admit reasonable  $\eta$ -invariants. This was used by Singer [Si] who showed, among other things, that the difference of  $\eta$ -invariants associated to two natural boundary value problems of this kind is an interesting spectral invariant of the boundary, at least asymptotically. More precisely, let  $M$  be an odd-dimensional Riemannian spin manifold with spinor bundle  $S(M)$ , and assume again that the metric is a product near  $N$  (this assumption is kept from now on). Thus, a neighborhood of  $N$  in  $M$  is isometric to the cylinder  $N_R = [0, R) \times N$  for some  $R > 0$ , therefore we write  $M_R$  to make the dependence on  $R$  more transparent. Then we have again a representation of type (2.1b) for the Dirac operator  $D^{M_R}$  on  $S(M_R)$ , where  $A = D^N$  becomes the Dirac operator on  $S(N) = S(M_R)|_N$ . Under  $\gamma$ ,  $S(N)$  splits into  $S^+(N) \oplus S^-(N)$  with projections  $Q_\pm : L^2(S(N)) \rightarrow L^2(S^\pm(N))$ . Then  $D_\pm^{M_R} := (D^{M_R}, Q_\pm)$  are well-posed boundary value problems to which the analysis of [GSm] applies. Singer proves that by stretching  $N_R$  the difference of  $\eta$ -invariants localizes, that is,

$$\lim_{R \rightarrow \infty} (\eta(D_+^{M_R}) - \eta(D_-^{M_R})) = \frac{1}{4\pi i} \log \det(D^N)^2. \quad (2.35)$$

Singer's investigation was motivated by Witten's identification of the covariant anomaly with the so-called adiabatic limit of an  $\eta$ -invariant (see [Wi]), but his work, in turn, stimulated greatly the interest in  $\eta$ -invariants for manifolds with boundary.

Douglas and Wojciechowski [DW] then studied systematically the properties of  $\eta$ -invariants for generalized Dirac operators on odd-dimensional manifolds with boundary. They assumed (2.1b) with the additional hypothesis

$$\ker A = 0, \quad (2.36)$$

and they chose the boundary condition (2.3a). In this situation, they established Lemmas 2.1 and 2.3, and for suitable families of such operators they proved (2.28) for  $k = 0$ . Moreover, they showed that stretching the cylinder  $N_R$  produces an “adiabatic limit” in the sense that

$$\lim_{R \rightarrow \infty} \eta(D_R) =: \eta_\infty \tag{2.37}$$

exists. Then the challenge was to identify  $\eta_\infty$  and to extend the results to  $\ker A \neq 0$ . In this case, there is considerable freedom of choice for the “supplementary” boundary condition (2.4a,b), and its variation ought to be allowed, too, in a suitable generalization of (2.28). Note that the analysis of Lemma 2.3 does not apply to this situation right away since the operators under consideration do not have constant domain, so one has to search for a suitable transformation of the family. This was done by Lesch and Wojciechowski [LW]. Since their method also served as a basic motivation for this paper, we present a suitable version of their argument. Theorem 3.5 generalizes considerably the original construction and is the main analytic tool of our present work.

The result of [LW] was obtained independently by Müller [M2]. In addition, Müller presented a thorough analysis of the operators  $D_\sigma$  in the general case. In particular, he showed that  $\eta_\infty$  exists and can be interpreted as the suitably defined  $\eta$ -invariant for an operator on the manifold  $\tilde{M} := M \cup N_\infty$ . Moreover, he proved that

$$\eta_\infty = \eta(D_{\sigma_1}) \tag{2.38}$$

for a suitable  $\sigma_1$ , obtained from scattering theory on  $\tilde{M}$ . He also obtained the regularity of the  $\eta$ -function of  $D_\sigma$  if  $D$  is assumed to be of Dirac type.

In the context of Melrose’s “b-calculus,” Hassell, Mazzeo, and Melrose [MM], [HMM] defined an  $\eta$ -invariant on manifolds with boundary, and they proved a gluing law in this situation. This  $\eta$ -invariant coincides again with  $\eta_\infty$ .

Equation (2.38) can be taken as the starting point to prove the gluing law for  $\eta$ -invariants as done by Müller [M3] and Wojciechowski [W1], [W2]. Bunke [B] gave a complete proof of the gluing law based on cutting the manifold in question three times and reassembling the pieces into a cylinder (carrying both boundary conditions) and a compact manifold, where one can do essentially only “interior” analysis, in view of the finite propagation speed enjoyed by all  $D_\sigma$ . This reduces the analysis to the explicit computation on the cylinder carried out in [LW]. Bunke’s result is, at least theoretically, more precise than ours since he gives a formula for the unknown integer in (2.30). This is possible since his deformation induces a relatively compact perturbation. By contrast, our construction is more direct and more general but less rigid with regard to compactness.

Bunke’s argument, in turn, was generalized and simplified in a substantial paper by Dai and Freed [DF]. They interpreted the invariant (2.31) as a section

of the determinant line if one considers families of operators  $D_\sigma$  fibered over a compact Riemannian manifold. This allows a natural interpretation of Witten’s anomaly formula and also illustrates nicely the philosophy developed in Singer’s paper [Si].

Our proof of the gluing law (Theorem 3.9) arises as a byproduct of an extension of the variation formula to a wider class of boundary conditions, thus furnishing a proof of a rather different nature than those described before.

**3. Expansion theorems and the gluing law.** Our approach to the proof of the gluing law was originally inspired by Vishik’s proof [V] of the Cheeger-Müller theorem. Working out the details, we discovered, however, that we were led to a very natural generalization of the approach in [LW], designed to determine the variation of  $\eta(D_\sigma)$  under a change of  $\sigma$ .

At any rate, the analysis we present here deals with operators of type (2.1b) but with more general boundary conditions than in (2.3). We now explain how this class arises naturally from the gluing problem, define it in general, and outline the proof of the gluing law. Most details are deferred to Section 4.

Now let  $M$  be a compact Riemannian manifold,  $\dim M = m$ , and let

$$D_0 : C_0^\infty(S) \rightarrow C_0^\infty(S) \tag{3.1}$$

be a first-order, symmetric, elliptic differential operator on the hermitian vector bundle  $S \rightarrow M$ . The main examples are, of course, Dirac operators associated to a Dirac bundle  $(S, \nabla)$ , but we work in a more general context, allowing, for example, Dirac operators with potential.

Let  $N \subset M$  be a compact hypersurface. We assume that  $N$  has a tubular neighborhood  $U$  isometric to  $(-1, 1) \times N$  and such that the hermitian structure of  $S$  is a product, too. Moreover, we assume that on  $U$  the operator  $D_0$  has the form

$$D_0 = \gamma \left( \frac{\partial}{\partial x} + A \right), \tag{3.2}$$

where  $\gamma \in C^\infty(\text{End}(S_N))$  is a unitary bundle automorphism and  $A$  is a first-order, selfadjoint, elliptic differential operator on  $S_N := S|_N$ . If  $D_0$  is a compatible Dirac operator, then  $\gamma$  is Clifford multiplication by the inward normal vector and  $A$  is (essentially) a Dirac operator on  $N$ . We assume, furthermore, that  $\gamma$  and  $A$  satisfy (2.2).

Let  $D$  be the restriction of  $D_0$  to  $C_0^\infty(S|M \setminus N)$ . This operator is no longer essentially selfadjoint. In order to obtain selfadjoint extensions, one has to impose boundary conditions. The natural boundary condition inherited from  $M$  is the continuous transmission boundary condition. Interpreting sections of  $S$  with support in  $U$  as functions  $[-1, 1] \rightarrow L^2(S_N)$  in the obvious way, this boundary



condition reads

$$f(0-) = f(0+). \tag{3.3}$$

It is fairly clear that the resulting selfadjoint operator is unitarily equivalent to the closure of  $D$  in  $L^2(S)$ . On the other hand,  $D$  lives naturally on

$$M^{\text{cut}} := (M \setminus U) \cup_{\partial(M \setminus U)} ((-1, 0] \times N \cup [0, 1) \times N), \tag{3.4}$$

obtained by cutting  $M$  along  $N$ . (We adopt here the notation from [DF, p. 5164 and Sec. 4].) Thus,  $M^{\text{cut}}$  is obtained from  $M$  by artificially introducing two copies of  $N$  as boundary.

On  $M^{\text{cut}}$  we can introduce spectral boundary conditions as in Section 2. The natural interpolation between the continuous transmission and the Atiyah-Patodi-Singer boundary condition is furnished by the boundary conditions

$$\cos \theta P_{>0}(A)f(0+) = \sin \theta P_{>0}(A)f(0-), \tag{3.5a}$$

$$\sin \theta P_{<0}(A)f(0+) = \cos \theta P_{<0}(A)f(0-),$$

$$P_0(A)f(0+) = P_0(A)f(0-), \tag{3.5b}$$

where  $|\theta| < \pi/2$ . To render this more transparent, we employ the isomorphism (with  $H := L^2(S_N)$ )

$$\Phi : L^2(S|U) \simeq L^2([-1, 1], H) \rightarrow L^2([0, 1], H \oplus H), \tag{3.6a}$$

which sends  $f \in L^2([-1, 1], H)$  to  $\Phi f$ ,

$$\Phi f(x) = f(x) \oplus f(-x), \quad x \in [0, 1]. \tag{3.6b}$$

It is easy to see that, under  $\Phi$ ,  $D$  is transformed to

$$\tilde{D} := \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \left( \frac{\partial}{\partial x} + \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \right) =: \tilde{\gamma} \left( \frac{\partial}{\partial x} + \tilde{A} \right), \tag{3.7}$$

and the boundary condition is transformed to

$$\cos \theta P_{>0}(\tilde{A})u(0) = \sin \theta \tau P_{<0}(\tilde{A})u(0), \tag{3.8a}$$

where

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_H, \tag{3.8b}$$

supplemented on  $\ker \tilde{A}$  by

$$P_\sigma u(0) = 0, \tag{3.8c}$$

with

$$\sigma := - \begin{pmatrix} 0 & P_0(A) \\ P_0(A) & 0 \end{pmatrix}. \tag{3.8d}$$

Note that

$$\tau \tilde{\gamma} + \tilde{\gamma} \tau = 0 = \tau \tilde{A} + \tilde{A} \tau, \quad \tau^2 = 1, \quad \tau = \tau^*. \tag{3.9}$$

Next we observe that this boundary condition can be written as

$$\tilde{P}(\theta)u(0) = 0, \tag{3.10a}$$

if we introduce the projection

$$\tilde{P}(\theta) = \cos^2 \theta P_{>0}(\tilde{A}) + \sin^2 \theta P_{<0}(\tilde{A}) - \frac{1}{2} (\sin 2\theta) \tau (P_{>0}(\tilde{A}) + P_{<0}(\tilde{A})) + P_\sigma. \tag{3.10b}$$

It is useful to note the following properties of this family of projections, all of which are easily verified.

First, we see that

$$\tilde{\gamma} \tilde{P}(\theta) = (I - \tilde{P}(\theta)) \tilde{\gamma} \tag{3.11}$$

and that  $\tilde{P}(\theta)$  commutes with  $\tilde{A}^2$  in

$$[\tilde{P}(\theta), \tilde{A}^2] = 0. \tag{3.12}$$

We do not have commutativity with  $\tilde{A}$ , however. Instead we find

$$\tilde{P}(\theta) \tilde{A} \tilde{P}(\theta) = \cos 2\theta |\tilde{A}| \tilde{P}(\theta). \tag{3.13}$$

Remembering the argument of Lesch and Wojciechowski [LW], we are led to ask for a natural "parametrization" of the family  $(\tilde{P}(\theta))_{|\theta| < \pi/2}$ . It is easy to verify that with

$$U(\theta) := (\cos \theta (P_{>0}(\tilde{A}) + P_{<0}(\tilde{A})) + \sin \theta (P_{>0}(\tilde{A}) - P_{<0}(\tilde{A})) \tau) \oplus I_{\ker \tilde{A}} \tag{3.14}$$

and

$$\text{sgn } \tilde{A} := P_{>0}(\tilde{A}) - P_{<0}(\tilde{A}), \tag{3.15}$$

we have

$$\tilde{P}(\theta) = U(\theta)\tilde{P}(0)U(\theta)^*, \tag{3.16}$$

where

$$U(\theta) = e^{(\text{sgn } \tilde{A}\tau)\theta}. \tag{3.17}$$

Thus we obtain a family of generalized Atiyah-Patodi-Singer boundary conditions, and the gluing law becomes just the variational formula for this class of operators in the sense of Section 2.

In fact, we generalize the situation further. Thus from now on we consider the following setting.  $M$  is a Riemannian manifold of dimension  $m$ ,  $S \rightarrow M$  is a smooth hermitian vector bundle over  $M$ , and  $D$  is a first-order, symmetric, elliptic differential operator on  $C_0^\infty(S)$ . We assume that  $M$  can be decomposed as

$$M = U \cup M_1, \tag{3.18}$$

where  $M_1$  is a compact manifold with boundary  $N = \partial M_1 = \partial U$  and  $U$  is open. Moreover, we assume an isometry of Hilbert spaces,

$$\Phi : L^2(S|U) \rightarrow L^2([0, 1], H), \tag{3.19}$$

where  $S_N$  is a smooth hermitian bundle over  $N$  and  $H = L^2(S_N)$  as before. This isometry maps smooth sections to smooth sections in the sense that

$$\Phi(C^\infty(S|U) \cap L^2(S|U)) \subset C^\infty((0, 1), C^\infty(S_N)) \cap L^2([0, 1], H). \tag{3.20}$$

Thus we can transform  $D$  on  $U$ , and we require that

$$\Phi D \Phi^* = \gamma(\partial_x + A) =: \tilde{D}, \tag{3.21}$$

with  $A$  a symmetric elliptic operator of first order on  $S_N$ , which we identify with its selfadjoint closure, and  $\gamma$  a bounded operator on  $H$ . We assume, moreover, that  $\gamma$  and  $A$  satisfy the relations (2.2) and (2.8).

Finally, we require that for  $\phi \in C_0^\infty(-1, 1)$  we can find  $\psi_\phi \in C^\infty(M)$  with the following properties:

$$\Phi(\psi_\phi u) = \phi \Phi u, \quad u \in L^2(S|U); \tag{3.22a}$$

and  $\psi_\phi = 0$  near  $\partial M_1$ , and

$$\phi = 1 \text{ near zero implies } 1 - \psi_\phi \in C_0^\infty(M). \tag{3.22b}$$

As usual, we extend  $\tilde{D}$  to  $L^2(\mathbf{R}_+, H) =: \mathcal{H}$  to obtain the model operator. To define a family of boundary conditions, we proceed as in the above analysis of the cutting problem. We consider a family  $P(\theta)_{|\theta| < \pi/2}$  of orthogonal projections with the following properties:

$$\gamma P(\theta) = (I - P(\theta))\gamma, \tag{3.23}$$

$$[P(\theta), A^2] = 0, \tag{3.24}$$

$$A P(\theta) = P(\theta) A P(\theta) = a(\theta) |A| P(\theta) \quad \text{for some } a \in C^\infty(-\pi/2, \pi/2) \text{ with } a > -1. \tag{3.25}$$

These projections are again assumed to be conjugate to  $P(0)$  under a family of isometries  $U(\theta)$ ,

$$P(\theta) = U(\theta)P(0)U(\theta)^*. \tag{3.26}$$

We assume, moreover, a representation

$$U(\theta) = e^{iT(\theta)}, \tag{3.27}$$

with  $T(\theta)$  bounded and selfadjoint in  $H$ , smooth in  $(-\pi/2, \pi/2)$ , and such that

$$[\gamma, T(\theta)] = 0, \tag{3.28a}$$

$$AT(\theta) + T(\theta)A = 0. \tag{3.28b}$$

With these data we define boundary conditions for  $D$  and  $\tilde{D}$  via

$$\tilde{\mathcal{D}}_\theta = \left\{ u \in C(\mathbf{R}_+, H) \cap \mathcal{H} \mid u \in \mathcal{D}(\tilde{D}^*), P(\theta)u(0) = 0 \right\}, \tag{3.29a}$$

$$\mathcal{D}_\theta = \left\{ u \in L^2(S) \mid \begin{array}{l} u \in \mathcal{D}(D^*), \Phi(\psi_\phi u) \in \tilde{\mathcal{D}}_\theta \\ \text{for some } \phi \in C_0^\infty(-1, 1) \text{ with } \phi = 1 \text{ near zero} \end{array} \right\}, \tag{3.29b}$$

and

$$D_\theta := D|_{\mathcal{D}_\theta}, \quad \tilde{D}_\theta := \tilde{D}|_{\tilde{\mathcal{D}}_\theta}. \tag{3.30}$$

A good part of the subsequent analysis rests on these assumptions. For the asymptotic expansions to exist, it is convenient to require in addition that

$$P(\theta), T(\theta) \text{ are classical pseudodifferential operators of order zero on } N, \text{ for } |\theta| < \pi/2. \tag{3.31}$$

This assumption is clearly satisfied in the gluing case (3.10a,b).

We refer to the family  $(D_\theta)_{|\theta| < \pi/2}$  with the properties listed above as a *deformation of Atiyah-Patodi-Singer (APS) type*. Then we have seen that cutting along a compact hypersurface leads naturally to such a family. In this case, we do have a bit more structure since, in (3.25), we have  $a(\theta) = \cos 2\theta$ , in view of (3.13), and we have the additional symmetry  $\tau$  with the properties (3.9).

We note that a single projection  $P$  with the properties (3.23), (3.24), and (3.25) defines a selfadjoint extension of  $D$ ,  $D_P$ , to which the analysis of Section 2 applies. This we call a *generalized APS operator* since, clearly,  $P = P_{>0}(A) + P_\theta$  falls in this class.

We proceed to the spectral analysis of  $D_\theta$ . The proofs are given in Section 4.

**PROPOSITION 3.1.** *The operators  $D_\theta$  and  $\tilde{D}_\theta$  are essentially selfadjoint.*

We identify  $D_\theta$  and  $\tilde{D}_\theta$  with their respective closures in the following.

**PROPOSITION 3.2.**  *$D_\theta$  satisfies (2.8); that is,*

$$(D_\theta + i)^{-1} \in C_p(L^2(\hat{S})) \text{ for every } p > m.$$

We want to apply Lemma 2.5 to the family  $(D_\theta)_{|\theta| < \pi/2}$ , which requires that we first apply a transformation to satisfy (2.23a,e). This we do as in [LW], and this is the motivation for the assumptions (3.26), (3.27), and (3.28a,b).

Thus we choose  $\phi \in C_0^\infty(-1, 1)$  with  $\phi = 1$  near zero, and we introduce the unitary transformation

$$\begin{aligned} \Psi_\theta : L^2([0, 1], H) &\rightarrow L^2([0, 1], H), \\ \Psi_\theta u(x) &:= e^{i\phi(x)T(\theta)}(u(x)). \end{aligned} \tag{3.32}$$

Then  $P(0)u(0) = 0$  implies  $P(\theta)\Psi_\theta u(0) = 0$  in view of (3.26). Hence, extending  $\Psi_\theta$  to  $L^2([0, 1], H) \oplus L^2(S|M_1)$  as the identity on  $L^2(S|M_1)$  and similarly  $\Phi$  in (3.19), we obtain an isometry

$$\Phi_\theta := \Phi^* \Psi_\theta \Phi$$

of  $L^2(S)$  mapping  $\mathcal{D}_0$  to  $\mathcal{D}_\theta$ . Consequently, the family

$$\tilde{D}_\theta := \Phi_\theta^* D_\theta \Phi_\theta \tag{3.33}$$

has constant domain  $\mathcal{D}_0$  and the same spectral invariants as  $D_\theta$ . It is easy to see that  $(\tilde{D}_\theta)_{|\theta| < \pi/2}$  satisfies (2.23a,e). It remains to establish the asymptotic expansions (2.12), with  $D_\theta, dD_\theta/d\theta$  in place of  $B$ .

Our expansion results are expressed in terms of the Mellin transform of a certain meromorphic function  $F_a$ , which we have to introduce first.

**Lemma 3.3.** *Consider for  $a \in (-1, 1]$  and  $x > 0$  the function*

$$F_a(x) := x \int_0^\infty \operatorname{erfc}(z) e^{-2axz - x^2} dz,$$

where

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2} du$$

denotes the complementary error function. Then the Mellin transform of  $F_a$  is, for  $0 < |w| < 1$ ,

$$\begin{aligned} \mathcal{M}F_a(w) &= \frac{1}{4a} \left[ (1 - (1 - a^2)^{-w/2}) \Gamma(w/2) \right. \\ &\quad \left. + \frac{2}{\sqrt{\pi}} (1 - a^2)^{-w/2} \int_0^a (1 - t^2)^{(w/2)-1} dt \Gamma((w + 1)/2) \right], \end{aligned} \tag{3.34}$$

whereas

$$\mathcal{M}F_0(w) = \frac{1}{2\sqrt{\pi}} \Gamma((w + 1)/2) \tag{3.35}$$

and

$$\mathcal{M}F_1(w) = \frac{1}{4} \left[ \Gamma(w/2) - \frac{2}{w\sqrt{\pi}} \Gamma((w + 1)/2) \right]. \tag{3.36}$$

Since  $\mathcal{M}F_a(w)$  is meromorphic in  $\mathbb{C}$  with simple poles at the points  $-k, k \in \mathbb{Z}_+$ . For  $|a| < 1$ , the residues are

$$\operatorname{Res}_{-2l} \mathcal{M}F_a(-2l) = \frac{(-1)^l}{\Gamma(2a)} (1 - (1 - a^2)^l), \quad l \in \mathbb{Z}_+, \tag{3.37}$$

$$\operatorname{Res}_{-2l-1} \mathcal{M}F_a(-2l-1) = \frac{(-1)^l}{\Gamma(2a)} (1 - a^2)^{l+1/2} \int_0^a (1 - t^2)^{-l-3/2} dt, \quad l \in \mathbb{Z}_+.$$

For  $a = 0, 1$ , one has to take the corresponding limit in (3.37). More precisely,

$$\begin{aligned} \text{Res}_1 \mathcal{M}F_0(-2l) &= 0, \quad l \in \mathbf{Z}_+, \\ \text{Res}_1 \mathcal{M}F_0(-2l-1) &= \frac{(-1)^l}{\sqrt{\pi}l!}, \quad l \in \mathbf{Z}_+, \\ \text{Res}_1 \mathcal{M}F_1(-2l) &= \begin{cases} 0, & l = 0, \\ \frac{(-1)^l}{l!2}, & l \in \mathbf{N}, \end{cases} \\ \text{Res}_1 \mathcal{M}F_1(-2l-1) &= \frac{(-1)^l}{l!\sqrt{\pi}(2l+1)}, \quad l \in \mathbf{Z}_+. \end{aligned} \tag{3.38}$$

Now we present our first expansion result.

**THEOREM 3.4.** *Assume that (3.18) through (3.31) hold. For  $l = 0, 1$ , we have an asymptotic expansion of the form*

$$\begin{aligned} \text{tr}_{L^2(S)} [D_\theta^l e^{-tD_\theta^2}] &\sim_{t \rightarrow 0+} \sum_{j=0}^{\infty} a_j(\theta, l) t^{j-m/2} + \sum_{j=0}^{\infty} b_j(\theta, l) t^{j/2} \log t \\ &+ \sum_{j=0}^{\infty} c_j(\theta, l) t^{(j-n-l)/2} + \sum_{j=0}^{\infty} d_j(\theta, l) t^{j/2}. \end{aligned} \tag{3.39}$$

Here, the coefficients  $a_j$  are integrals of local densities on the metric double  $\tilde{M}$  of  $M$ ,  $b_j$  and  $c_j$  are integrals of local densities on  $N$ , and  $d_j$  are nonlocal invariants of  $N$ . They are given explicitly in the formulas (4.15), (4.21a), (4.21b), (4.30a), and (4.30b).

For  $l = 0$ , the leading term is

$$a_0(\theta, 0) = \Gamma(m/2 + 1) \text{vol}(T_1^*M), \tag{3.40}$$

where  $T_1^*M = \{\xi \in T^*M \mid \sigma_{D_\theta}(\xi) \leq 1\}$ .

The logarithmic terms vanish if  $l = 0$  and  $m$  is odd. If  $l = 0$  and  $m$  is even, then  $b_{2j}(\theta, 0) = 0$ . However, the logarithmic terms are present in general.

For  $l = 1$ , the expansion (3.39) implies that  $\eta(D_\theta; s)$  has a meromorphic extension to  $\mathbf{C}$  with at most double poles. Zero is a simple pole, and for the residue at zero we find

$$\text{Res}_1 \eta(D_\theta; 0) = \frac{2}{\sqrt{\pi}} a_{n/2}(\theta, 1) + \left( \frac{2a(\theta)}{\sqrt{\pi}} \mathcal{M}F_{a(\theta)}(1) - \frac{1}{2} \right) \text{Res}(\gamma(\text{sgn } A)P(\theta)). \tag{3.41}$$

For the APS boundary condition, this result was obtained by Grubb and Seeley [GrSe1]. By contrast, our approach is simply based on the spectral theo-

rem and the explicit formula (4.1). Nevertheless, we can handle boundary conditions that are significantly different from the APS condition.

The expansion result for the APS condition is sketched in Müller [M2, Lemma 1.17], overlooking, however, the coefficients that are not local in  $\tilde{M}$  in the case  $l = 0$ . In the case  $l = 1$  and for APS boundary conditions, these nonlocal terms are actually not present. To explain this, let for the moment  $\tilde{D}_\sigma$  be the operator with APS boundary condition. Then a simple symmetry argument shows that for any cut-off function  $\phi \in C_0^\infty(\mathbf{R})$ ,

$$\text{tr}_{L^2(S)} [\phi \tilde{D}_\sigma e^{-t\tilde{D}_\sigma^2}] = 0, \quad t > 0, \tag{3.42}$$

and hence  $b_j(\sigma, 1) = c_j(\sigma, 1) = d_j(\sigma, 1) = 0$  (cf. [L3, Lemma 5.2.4]). For general  $P(\theta)$ , we cannot expect (3.42) to hold.

In the next step, we evaluate the formula for the variation of the  $\bar{\eta}$ -invariant in Lemma 2.5 via the asymptotic expansion of  $\text{tr}((d\tilde{D}_\theta/d\theta)e^{-t\tilde{D}_\theta^2})$ .

**THEOREM 3.5.** *Under the assumptions of Theorem 3.4, we have the following variation formulas:*

$$\frac{d}{d\theta} \text{Res}_1 \eta(D_\theta; 0) = \frac{1}{\sqrt{\pi}} \text{Res}(\gamma i T'(\theta)), \tag{3.43}$$

$$\begin{aligned} \frac{d}{d\theta} \bar{\eta}(D_\theta) &= \frac{1}{2\pi} a_{00}(A, \gamma i T'(\theta)) \\ &+ \left( \frac{2a(\theta)}{\sqrt{\pi}} \mathcal{M}F_{a(\theta)}(1) - \frac{1}{2} \right) \text{Res}(\gamma i T'(\theta)(\text{sgn } A)P(\theta)). \end{aligned} \tag{3.44}$$

**COROLLARY 3.6.** *In the situation of Theorem 3.5, assume in addition that*

$$T'(\theta)P(\theta) = (I - P(\theta))T'(\theta). \tag{3.45}$$

Then

$$\text{Res}(\gamma i T'(\theta)(\text{sgn } A)P(\theta)) = \frac{a(\theta)}{2} \text{Res}(\gamma i T'(\theta)).$$

In particular, if  $\text{Res}(\gamma i T'(\theta)) = 0$ , then  $\text{Res}_1 \eta(D_\theta; 0)$  is independent of  $\theta$  and

$$\frac{d}{d\theta} \bar{\eta}(D_\theta) = \frac{1}{2\pi} a_{00}(A, \gamma i T'(\theta)).$$

*Proof.* We use (3.45), (3.23), (3.25), and the trace property of the noncommutative residue to compute

$$\begin{aligned} \text{Res}(\gamma i T'(\theta)(\text{sgn } A)P(\theta)) &= \text{Res}(\gamma i T'(\theta)P(\theta)(\text{sgn } A)P(\theta)) \\ &= a(\theta) \text{Res}(\gamma i T'(\theta)P(\theta)). \end{aligned}$$

Here we have used that  $\text{Res}$  vanishes on smoothing operators. Furthermore, in view of (3.28a),

$$\begin{aligned} \text{Res}(\gamma i T'(\theta)P(\theta)) &= \text{Res}(\gamma i(I - P(\theta))T'(\theta)) \\ &= \text{Res}(i(I - P(\theta))T'(\theta)\gamma) \\ &= \text{Res}(i(I - P(\theta))\gamma T'(\theta)) \\ &= \text{Res}(\gamma i T'(\theta)(I - P(\theta))), \end{aligned}$$

and we reach the conclusion.  $\square$

Next we introduce a special class of deformations of APS type, which is still slightly more general than the gluing situation (3.5a)–(3.17). We consider again the framework (3.18)–(3.22b). Furthermore, let  $\tau : C^\infty(S_N) \rightarrow C^\infty(S_N)$  be a unitary classical pseudodifferential operator satisfying (cf. (3.9))

$$\tau\gamma + \gamma\tau = 0 = \tau A + A\tau, \quad \tau^2 = I, \quad \tau = \tau^*. \tag{3.46}$$

We abbreviate

$$K^\pm := (\ker A) \cap \ker(\gamma \mp i). \tag{3.47}$$

The relations (3.46) immediately imply

$$\dim K^+ = \dim K^-. \tag{3.48}$$

However, the presence of  $\tau$  is not really necessary for this equality. Equation (3.48) follows already from (3.18)–(3.22b). If  $D$  is a Dirac operator, then this is the well-known cobordism theorem for Dirac operators (see [P, Chap. XVII]). For general  $D$ , this is due to Lesch [L1, Thm. 6.2], [L3, Chap. IV]. It was also proved independently by W. Müller [M2, Prop. 4.26].

In view of (3.48), we can choose an isometry

$$U : K^+ \rightarrow K^- \tag{3.49}$$

and put

$$\sigma = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} : \ker A \rightarrow \ker A. \tag{3.50}$$

With these data we can introduce the projection (cf. (3.10b))

$$P(\theta) := \cos^2 \theta P_{>0}(A) + \sin^2 \theta P_{<0}(A) - \frac{1}{2}(\sin 2\theta)\tau(P_{>0}(A) + P_{<0}(A)) + P_\sigma \tag{3.51}$$

and the unitary family (cf. (3.14))

$$U(\theta) = (\cos \theta(P_{>0}(A) + P_{<0}(A)) + \sin \theta(\text{sgn } A)\tau) \oplus I_{\ker A} = e^{(\text{sgn } A)\tau\theta}. \tag{3.52}$$

One immediately checks the relations (3.11)–(3.13) and (3.16), and hence we are led to a deformation of APS type. We denote the corresponding family of operators by  $D_{\theta,\sigma}$ , indicating explicitly the dependence on the choice of  $\sigma$ . If we fix  $\theta$  and consider a 1-parameter family of reflections  $\sigma_u$ , we obtain another deformation of APS type. In this way we recover the main result of Lesch and Wajsbachowski [LW] as a special case of our present work.

**PROPOSITION 3.7** (cf. [LW], [M2], [DF]). *Let  $\cos \theta \neq 0$ , and let  $U_u : K^+ \rightarrow K^-$  be a smooth family of unitary operators. Put*

$$\sigma_u := \begin{pmatrix} 0 & U_u^* \\ U_u & 0 \end{pmatrix}.$$

*Then  $(D_{\theta,\sigma_u})_u$  is a deformation of APS type,  $\text{Res}_1 \eta(D_{\theta,\sigma_u}; 0)$  is independent of  $u$ , and*

$$\frac{d}{du} \bar{\eta}(D_{\theta,\sigma_u}) = \frac{1}{2\pi i} \text{tr}_{K^+} \left[ U_u^{-1} \frac{d}{du} U_u \right].$$

*Proof.* We put

$$P_u(\theta) := \cos^2 \theta P_{>0}(A) + \sin^2 \theta P_{<0}(A) - \frac{1}{2}(\sin 2\theta)\tau(P_{>0}(A) + P_{<0}(A)) + P_{\sigma_u}.$$

Furthermore, we fix  $u_0$  and define the unitary operator  $V_u \in \mathcal{L}(H)$  by

$$V_u|K^+ := U_u^* U_{u_0}, \quad V_u|K^- \oplus (\ker A)^\perp := I. \tag{3.53}$$

Then we choose a smooth family of selfadjoint operators  $T_u$ , such that

$$V_u = e^{iT_u}, \quad T_{u_0} = 0, \quad T_u|_{K^- \oplus (\ker A)^\perp} = 0. \quad (3.54)$$

It follows that

$$V_u P_{u_0}(\theta) V_u^* = P_u(\theta),$$

and one checks that  $(D_{\theta, \sigma_u})_u$  is a deformation of APS type. Since  $T'_u$  is an operator of finite rank, we have

$$\text{Res}(\gamma i T'_u) = \text{Res}(\gamma i T'_u (\text{sgn } A) P_u(\theta)) = 0.$$

We deduce from Theorem 3.5 that

$$\frac{d}{du} \text{Res}_1 \eta(D_{\theta, \sigma_u}; 0) = 0$$

and

$$\begin{aligned} \frac{d}{du} \bar{\eta}(D_{\theta, \sigma_u}) &= \frac{1}{2\pi} a_{00}(A, \gamma i T'_u) \\ &= \frac{i}{2\pi} \lim_{t \rightarrow 0} \text{tr}_H [\gamma T'_u e^{-tA^2}] \\ &= \frac{i}{2\pi} \text{tr}_{K^+} [\gamma T'_u] \\ &= \frac{1}{2\pi i} \text{tr}_{K^+} \left[ U_u^{-1} \frac{d}{du} U_u \right]. \end{aligned}$$

Next we deal with the deformation  $(D_{\theta, \sigma})_{|\theta| < \pi/2}$ .

**PROPOSITION 3.8.**  $\text{Res}_1 \eta(D_{\theta, \sigma}; 0)$  is independent of  $\theta$  and

$$\begin{aligned} \frac{d}{d\theta} \bar{\eta}(D_{\theta, \sigma}) &= \frac{1}{2\pi} a_{00}(A, \gamma (\text{sgn } A) \tau) \\ &= \frac{1}{2\pi} \text{LIM}_{t \rightarrow 0} \text{tr}_H [\gamma (\text{sgn } A) \tau e^{-tA^2}]. \end{aligned}$$

Here  $\text{LIM}_{t \rightarrow 0}$  denotes the constant term in the asymptotic expansion as  $t \rightarrow 0$ .

*Proof.* In view of (3.52), we put

$$T(\theta) := -i(\text{sgn } A) \tau \theta.$$

Then one checks that (3.23)–(3.28b) and (3.45) are satisfied. We want to apply **Corollary 3.6** to compute  $d\bar{\eta}(D_{\theta, \sigma})/d\theta$ . Since  $\text{Res}$  vanishes on operators of finite rank, we may replace

$$\gamma i T'(\theta) = \gamma (\text{sgn } A) \tau$$

by

$$\gamma((\text{sgn } A) + \sigma) \tau$$

in the assumptions of Corollary 3.6. Since

$$(\gamma((\text{sgn } A) + \sigma) \tau)^2 = I,$$

we infer from Lemma 2.6 that  $\text{Res}(\gamma((\text{sgn } A) + \sigma) \tau) = 0$ . Thus  $\text{Res}_1 \eta(D_{\theta, \sigma}; 0)$  is independent of  $\theta$  and

$$\begin{aligned} \frac{d}{d\theta} \bar{\eta}(D_{\theta, \sigma}) &= \frac{1}{2\pi} a_{00}(A, \gamma i T'(\theta)) \\ &= \frac{1}{2\pi} a_{00}(A, \gamma (\text{sgn } A) \tau) \\ &= \frac{1}{2\pi} \text{LIM}_{t \rightarrow 0} \text{tr}_H [\gamma (\text{sgn } A) \tau e^{-tA^2}]. \quad \square \end{aligned}$$

Finally, we present the gluing law. In this situation (see (3.5a)–(3.17)), we have yet another structure; namely, introducing (with the same notation as in (3.7) and (3.8))

$$\mu := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we see that

$$\mu^2 = -I, \quad \mu \tau + \tau \mu = \mu \tilde{\gamma} + \tilde{\gamma} \mu = \mu \tilde{A} + \tilde{A} \mu = 0.$$

This observation leads to the following theorem.

**THEOREM 3.9 (Gluing law).** Consider the deformation  $(D_{\theta, \sigma})_{|\theta| < \pi/2}$  of APS type introduced in (3.46)–(3.52). If there exists a unitary, classical pseudodiffer-

ential operator  $\mu : C^\infty(S_N) \rightarrow C^\infty(S_N)$  satisfying

$$\mu^2 = -I, \quad \mu\tau + \tau\mu = \mu\gamma + \gamma\mu = \mu A + A\mu = 0, \tag{3.55}$$

then

$$\frac{d}{d\theta} \bar{\eta}(D_{\theta, \sigma}) = 0.$$

*Proof.* In view of (3.55), we have

$$\mu\gamma(\text{sgn } A)\tau + \gamma(\text{sgn } A)\tau\mu = 0,$$

hence

$$\text{tr}_H[\gamma(\text{sgn } A)\tau e^{-tA^2}] = 0.$$

In particular  $a_{00}(A, \gamma(\text{sgn } A)\tau) = 0$ , and by Proposition 3.8 we reach the conclusion.  $\square$

Naming Theorem 3.9 the “gluing law” calls for an explanation. We briefly explain how the usual gluing law for the  $\eta$ -invariant follows from Theorem 3.9. We consider again the situation (3.5a)–(3.17). Then we have

$$\begin{aligned} \tilde{K}^\pm &:= K^\pm(\tilde{A}) = \ker \tilde{A} \cap \ker(\tilde{\gamma} \mp i) \\ &= K^\pm(A) \oplus K^\mp(A) \\ &=: K^\pm \oplus K^\mp, \end{aligned} \tag{3.56}$$

that is,  $\tilde{K}^\pm$  is canonically isomorphic to  $K^+ \oplus K^-$ , and we use this identification in the following. As in (3.49) and (3.50), we write involutions  $\sigma$  of  $\ker \tilde{A}$  with  $\tilde{\gamma}\sigma + \sigma\tilde{\gamma} = 0$  in the form

$$\sigma(T) = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}, \tag{3.57}$$

where  $T : \tilde{K}^+ \rightarrow \tilde{K}^-$  is an isometry. The isometry corresponding to the distinguished involution

$$\sigma := -\tau|_{\ker \tilde{A}} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_H \tag{3.58}$$

in (3.8d) therefore corresponds to the isometry  $-I : K^+ \oplus K^- \rightarrow K^+ \oplus K^-$ .

Now let  $\eta(D, M^{\text{cut}}, T)$  be the  $\bar{\eta}$ -invariant of the operator  $\tilde{D}$  with boundary condition given by

$$\tilde{P}_T := P_{>}(\tilde{A}) \oplus P_\sigma(T). \tag{3.59}$$

Putting

$$\tilde{P}_T(\theta) := \cos^2 \theta P_{>0}(\tilde{A}) + \sin^2 \theta P_{<0}(\tilde{A}) - \frac{1}{2}(\sin 2\theta)\tau(P_{>0}(\tilde{A}) + P_{<0}(\tilde{A})) + P_\sigma(T) \tag{3.60}$$

as in (3.10b), we obtain a deformation of APS type  $(\tilde{D}_{\theta, \sigma(T)})_{|\theta| < \pi/2}$ . Then,  $\bar{\eta}(\tilde{D}_{\theta, \sigma(T)})$  is independent of  $\theta$  by Theorem 3.9.

For  $T = -I$  and  $\theta = \pi/4$ , the boundary condition is the continuous transmission boundary condition, and hence  $\bar{\eta}(\tilde{D}_{\pi/4, \sigma(-I)}) \equiv \bar{\eta}(D, M) \pmod{\mathbf{Z}}$ , the  $\bar{\eta}$ -invariant of the closure of  $D$  on  $L^2(S)$ . Furthermore, for  $\theta = 0$  we obtain

$$\bar{\eta}(\tilde{D}_{0, \sigma(T)}) \equiv \bar{\eta}(D, M^{\text{cut}}, T) \pmod{\mathbf{Z}}. \tag{3.61}$$

Thus, for  $T = -I$  we have proved

$$\bar{\eta}(D, M^{\text{cut}}, -I) \equiv \bar{\eta}(D, M) \pmod{\mathbf{Z}}. \tag{3.62}$$

For an arbitrary isometry  $T : K^+ \oplus K^- \rightarrow K^+ \oplus K^-$ , we choose a smooth path of isometries  $(T_u)_{0 \leq u \leq 1}$  with  $T_0 = -I$  and  $T_1 = T$ , and we apply Proposition 3.7 to  $\tilde{D}_{0, \sigma(T_u)}$ . Then

$$\frac{d}{du} \bar{\eta}(D, M^{\text{cut}}, T_u) = \frac{1}{2\pi i} \text{tr}_{K^+ \oplus K^-} \left( T_u^{-1} \frac{d}{du} T_u \right) = \frac{1}{2\pi i} \frac{d}{du} \log \det(T_u), \tag{3.63}$$

and hence

$$\bar{\eta}(D, M^{\text{cut}}, T) \equiv \bar{\eta}(D, M^{\text{cut}}, -I) + \frac{1}{2\pi i} \log \det(T) - \frac{1}{2}(\dim K^+ + \dim K^-) \pmod{\mathbf{Z}}. \tag{3.64}$$

This can be written more nicely in terms of the  $\tau$ -invariant (2.31). Namely,

$$\tau(D, M^{\text{cut}}, T) = (-1)^{\dim K^+ + \dim K^-} \det(T) \tau(D, M). \tag{3.65}$$

Note that  $A_+ := A|_{\ker(\gamma - i)}$  is a Fredholm operator between  $\ker(\gamma - i)$  and  $\ker(\gamma + i)$  with  $\text{ind } A_+ = \dim K^+ - \dim K^-$ , and hence we end up with the gluing law in the version of Dai and Freed [DF, Prop. 4.5].

**THEOREM 3.10.** *We have  $\tau(D, M^{\text{cut}}, T) = (-1)^{\text{ind } A_+} \det(T)\tau(D, M)$ .*

Actually, this result is slightly more general than loc. cit. since Dai and Freed deal with Dirac operators on spin manifolds only.

In the special case where the hypersurface  $N$  separates  $M$  into two components  $M_{\pm}$  such that  $M = M_- \cup_N M_+$ , the index of  $A_+$  vanishes by the cobordism theorem (cf. the discussion after (3.48)). Hence we can choose isometries  $T_+ : K^+ \rightarrow K^-$ ,  $T_- : K^- \rightarrow K^+$  and put

$$T := \begin{pmatrix} 0 & T_+ \\ T_- & 0 \end{pmatrix}. \tag{3.66}$$

Then

$$\tilde{D}_{0,\sigma(T)} = D_{0,\sigma(T_+)}^+ \oplus D_{0,\sigma(T_-)}^-, \tag{3.67}$$

where  $D_{0,\sigma(T_{\pm})}^{\pm}$  is the operator  $D$  on the manifold  $M_{\pm}$  with boundary condition given by  $P_{>}(A) \oplus P_{\sigma(T_+)}$  (resp.,  $P_{>}(-A) \oplus P_{\sigma(T_-)}$ ). Denoting their respective  $\eta$ -invariants by  $\bar{\eta}(D, M_{\pm}, T_{\pm})$ , we obtain the gluing formula for the  $\eta$ -invariant:

$$\bar{\eta}(D, M) \equiv \bar{\eta}(D, M_+, T_+) + \bar{\eta}(D, M_-, T_-) + \frac{1}{2\pi i} \log \det(-T_1 T_2) \pmod{\mathbf{Z}} \tag{3.68}$$

or, in multiplicative notation,

$$\tau(D, M) = \det(-T_1 T_2) \tau(D, M_+) \tau(D, M_-). \tag{3.69}$$

As explained in [B, Sec. 1],  $\det(-T_1 T_2)$  is related to the Maslov index of the corresponding Lagrangian subspaces defined by  $L_j := \ker \sigma(T_j)$ ,  $j = 1, 2$ . Namely, putting

$$m(L_1, L_2) = -\frac{1}{\pi} \sum_{\substack{\beta \in (-\pi, \pi) \\ e^{i\beta} \in \text{spec}(-T_1 T_2)}} \beta$$

(see [LW, Thm. 2.1], [B, Def. 1.3]), then

$$m(L_1, L_2) = \int_K \tau(kL, L_1, L_2) dk.$$

Here,  $K$  is the stabilizer of  $\tilde{\gamma}$  in the symplectic group,  $L$  is an arbitrary Lagrangian subspace, and  $\tau$  is the Maslov triple index (cf. [B, Sec. 2] for details).

Summing up, we can state the gluing law as follows.

**THEOREM 3.11.** *We have*

$$\bar{\eta}(D, M) \equiv \bar{\eta}(D, M_+, T_+) + \bar{\eta}(D, M_-, T_-) + \frac{1}{2} m(\ker \sigma(T_1), \ker \sigma(T_2)) \pmod{\mathbf{Z}}.$$

Our last comment concerns the residue at zero of the  $\eta$ -function. We expect that in general the residue in (3.41) will not vanish. In the cutting case, however, there is no pole.

**THEOREM 3.12.** *If  $(D_{\theta})_{|\theta| < \pi/2}$  arises from cutting  $M$  along a compact hypersurface (as explained in (3.5a)–(3.17)), then  $\eta(D_{\theta}; s)$  is regular at  $s = 0$  for all  $\theta$ .*

*Proof.* By Proposition 3.8,  $\text{Res}_1 \eta(D_{\theta}; 0)$  is independent of  $\theta$ , and hence

$$\text{Res}_1 \eta(D_{\theta}; 0) = \text{Res}_1 \eta(D_{\pi/4}; 0) = 0,$$

since the  $\eta$ -function of a selfadjoint elliptic differential operator on a compact manifold is regular at zero (see [G, Sec. 3.8]).  $\square$

**4. Proofs.** We now prove the statements presented in the previous section.

*Proof of Proposition 3.1.* We consider  $\tilde{D}_{\theta}$  first. Let  $u \in \mathcal{D}(\tilde{D}_{\theta}^*)$  satisfy

$$\tilde{D}_{\theta}^* u = \pm \sqrt{-1} u.$$

This implies, for  $v \in \tilde{\mathcal{D}}_{\theta}$ , that

$$(\tilde{D}_{\theta} v, u) = \mp \sqrt{-1} (v, u).$$

Then a standard regularity argument shows that  $u \in C(\mathbf{R}_+, L^2(S_N))$  with

$$P(\theta)u(0) = 0$$

by (3.23). Choosing  $\phi \in C_0^{\infty}(\mathbf{R})$  with  $\phi = 1$  near zero, we put  $\phi_N(x) := \phi(x/N)$  and obtain  $\phi_N^2 u \in \mathcal{D}_{\theta}$ . Consequently, we find that

$$\pm \sqrt{-1} \|u\|^2 = \lim_{N \rightarrow \infty} (\tilde{D}_{\theta} \phi_N^2 u, u) = \lim_{N \rightarrow \infty} (u, \tilde{D}_{\theta} \phi_N^2 u) \in \mathbf{R},$$

and hence  $u = 0$ .

For  $D_{\theta}$ , we appeal to the localization principle for deficiency indices derived in [L1, Thm. 2.1] (cf. also [L3, Chap. IV]).  $\square$

In what follows, it is crucial that we can give an explicit formula for the operator heat kernel of  $\tilde{D}_{\theta}$ . It is the operator analogue of a formula derived by Sommerfeld [So, p. 61].



THEOREM 4.1. We have for  $t, x, y > 0$

$$e^{-t\tilde{D}_\theta^2}(x, y) = (4\pi t)^{-1/2} \left( e^{-(x-y)^2/4t} + (I - 2P(\theta))e^{-(x+y)^2/4t} \right) e^{-tA^2} \\ + (\pi t)^{-1/2} (I - P(\theta)) \int_0^\infty e^{-(x+y+z)^2/4t} \tilde{A}(\theta) e^{\tilde{A}(\theta)z-tA^2} dz, \quad (4.1)$$

where  $\tilde{A}(\theta) := (I - P(\theta))A(I - P(\theta))$ .

*Proof.* The point is the convergence of the integral in (4.1). Note that  $P(\theta)$  commutes with  $|A|$  by (3.24) and the discreteness of  $A$ . Thus from (3.23), (2.2), and (3.25)

$$\tilde{A}(\theta) = \gamma P(\theta) \gamma^* A \gamma P(\theta) \gamma^* = -\gamma P(\theta) A P(\theta) \gamma^* \\ = -a(\theta) \gamma |A| P(\theta) \gamma^* \\ = -a(\theta) |A| (I - P(\theta)).$$

In particular,  $\tilde{A}(\theta)$  commutes with  $(I - P(\theta))$ , so

$$\tilde{A}(\theta) e^{\tilde{A}(\theta)z-tA^2} = -a(\theta) |A| (I - P(\theta)) e^{-a(\theta)|A|z-tA^2}.$$

Introducing  $a_-(\theta) := -\min\{0, a(\theta)\} \in [0, 1]$ , we find

$$-a(\theta) |A| z \leq a_-(\theta) \left( \frac{z^2}{4t} + A^2 t \right)$$

and

$$0 \leq |A| (I - P(\theta)) e^{\tilde{A}(\theta)z-tA^2} \leq |A| (I - P(\theta)) e^{a_-(\theta)z^2/4t} e^{-(1-a_-(\theta))tA^2}. \quad (4.2)$$

This implies that the integral converges in the trace norm of  $L^2(S_N)$ .

Now pick  $u \in C_0^\infty((0, \infty), L^2(S_N))$  and form

$$Q_t u(x) := \int_0^\infty Q_t(x, y) u(y) dy,$$

where  $Q_t$  denotes the right-hand side of (4.1). Then it is a routine matter to check that we have

$$Q_t u \in C^1((0, \infty), \mathcal{D}(\tilde{D}^*)) \cap C(\mathbf{R}_+, \mathcal{H}), \\ (\partial_t + (\tilde{D}^*)^2) Q_t u(x) = 0, \quad t, x > 0, \quad (4.3) \\ \lim_{t \rightarrow 0^+} Q_t u(x) = u(x).$$

Hence it remains to verify the boundary conditions. Clearly,

$$P(\theta) Q_t(x, y) = (4\pi t)^{-1/2} \left( e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t} \right) P(\theta) e^{-tA^2} \xrightarrow{x \rightarrow 0^+} 0,$$

and the same holds for  $P(\theta) Q_t u(x)$  and  $AP(\theta) Q_t u(x)$  by dominated convergence. This implies

$$Q_t u \in \mathcal{D}(\tilde{D}_\theta).$$

We finally have to show that

$$0 = \lim_{x \rightarrow 0^+} P(\theta) \gamma (\partial_x + A) Q_t u(x) \\ = \lim_{x \rightarrow 0^+} \gamma (I - P(\theta)) (\partial_x + A) Q_t u(x) \\ = \lim_{x \rightarrow 0^+} \left\{ \gamma (\partial_x + \tilde{A}(\theta)) (I - P(\theta)) Q_t u(x) + \gamma (I - P(\theta)) A P(\theta) Q_t u(x) \right\} \\ = \lim_{x \rightarrow 0^+} \gamma (\partial_x + \tilde{A}(\theta)) (I - P(\theta)) Q_t u(x).$$

An easy calculation shows that

$$(\partial_x + \tilde{A}(\theta)) (I - P(\theta)) Q_t(x, y) \\ = (4\pi t)^{-1/2} \left\{ e^{-(x-y)^2/4t} \left( \frac{y-x}{2t} + \tilde{A}(\theta) \right) \right. \\ \left. + e^{-(x+y)^2/4t} \left( -\frac{y+x}{2t} + \tilde{A}(\theta) \right) \right\} (I - P(\theta)) e^{-tA^2} \\ - (\pi t)^{-1/2} e^{-(x+y)^2/4t} \tilde{A}(\theta) (I - P(\theta)) e^{-tA^2} \xrightarrow{x \rightarrow 0^+} 0.$$

Then the proof is completed using dominated convergence as above.  $\square$

*Proof of Proposition 3.2.* We propose to show that, for  $u \in \mathcal{D}(D_\theta^k)$  with  $k > m/2$ , we have the estimate

$$|u(p)| \leq C(1 - a_-(\theta))^{-k-1/2} (\|u\|_{L^2(S)} + \|D_\theta^k u\|_{L^2(S)}). \quad (4.4)$$

As explained in [L2] (cf. also [L3, Sec. 1.4]), this estimate implies the Hilbert-Schmidt property of suitable functions of  $D_\theta$  and, in particular, the assertion of the proposition.

To prove (4.4), it is clearly enough to assume that  $\text{supp } u \subset U$ , and we are reduced to proving the analogue of (4.4) for  $\tilde{D}_\theta$  if  $\text{supp } u \subset [0, 1)$ . To do so, we write for  $q \in N$

$$u(x)(q) = (\tilde{D}_\theta^2 + 1)^{-j} (\tilde{D}_\theta^2 + 1)^j u(x)(q) \\ = \frac{1}{\Gamma(j)} \int_0^\infty e^{-t} t^{j-1} \int_0^\infty e^{-t\tilde{D}_\theta^2}(x, y) (\tilde{D}_\theta^2 + 1)^j u(y) dy dt(q). \quad (4.5)$$

From the ellipticity of  $A$  we get, for  $k > (m - 1)/2$ ,

$$|u(x)(q)| \leq C_k \|(A^2 + 1)^k u(x)\|_{L^2(S_N)},$$

hence, with  $j = k + 1/2 + \varepsilon$ ,  $\varepsilon > 0$ ,

$$|u(x)(q)| \leq C_k \int_0^\infty e^{-t} t^{k-1/2+\varepsilon} \int_0^\infty \|(A^2 + 1)^k e^{-t\tilde{D}_\theta^2}(x, y) \cdot ((\tilde{D}_\theta^2 + 1)^{k+1/2+\varepsilon} u(y))\|_{L^2(S_N)} dy dt. \quad (4.6)$$

From (4.1) and (4.2) we derive the norm estimate

$$\|(A^2 + 1)^k e^{-t\tilde{D}_\theta^2}(x, y)\|_{\mathcal{L}(L^2(S_N))} \\ \leq C_k (1 - a_-(\theta))^{-k-1} t^{-k-1/2} (e^{-(x-y)^2/4t} + e^{-(x+y)^2/4t}). \quad (4.7)$$

Using (4.7) and the Cauchy-Schwarz inequality in (4.6), we obtain the result.  $\square$

*Proof of Lemma 3.3.* An integration by parts gives

$$F_a(x) = -\frac{1}{2a} \int_0^\infty \text{erfc}(z) \frac{\partial}{\partial z} (e^{-2axz-x^2}) dz \\ = \frac{1}{2a} e^{-x^2} - \frac{1}{a\sqrt{\pi}} \int_0^\infty e^{-(2axz+x^2+z^2)} dz \\ =: \frac{1}{2a} (G(x) - \tilde{F}_a(x)). \quad (4.8)$$

Clearly,

$$\mathcal{M}G(w) = \frac{1}{2} \Gamma(w/2). \quad (4.9)$$

To determine  $\mathcal{M}\tilde{F}_a$ , we observe that

$$\tilde{F}_a(x) = e^{-(1-a^2)x^2} \text{erfc}(ax),$$

and we derive a differential equation in  $a$ . In fact, for  $\text{Re } w > 0$ ,  $0 < |a| < 1$ ,

$$\frac{\partial}{\partial a} (1 - a^2)^{w/2} \mathcal{M}\tilde{F}_a(w) = \frac{\partial}{\partial a} \int_0^\infty x^{w-1} e^{-x^2} \text{erfc}\left(\frac{a}{\sqrt{1-a^2}} x\right) dx \\ = -\frac{2}{\sqrt{\pi}} \int_0^\infty x^w e^{-x^2/(1-a^2)} dx (1 - a^2)^{-3/2} \\ = -\frac{1}{\sqrt{\pi}} \Gamma((w+1)/2) (1 - a^2)^{w/2-1}. \quad (4.10)$$

The initial condition at  $a = 0$  is

$$\mathcal{M}\tilde{F}_0(w) = \frac{1}{2} \Gamma(w/2). \quad (4.11)$$

The solution of this initial value problem is, for  $|a| < 1$ ,

$$\mathcal{M}\tilde{F}_a(w) = (1 - a^2)^{-w/2} \left( \frac{1}{2} \Gamma(w/2) - \frac{1}{\sqrt{\pi}} \Gamma((w+1)/2) \int_0^a (1 - t^2)^{w/2-1} dt \right),$$

hence

$$\mathcal{M}F_a(w) = \frac{1}{2a} \left[ (1 - (1 - a^2)^{-w/2}) \frac{1}{2} \Gamma(w/2) \right. \\ \left. + \frac{1}{\sqrt{\pi}} (1 - a^2)^{-w/2} \int_0^a (1 - t^2)^{w/2-1} dt \Gamma((w+1)/2) \right]. \quad (4.12)$$

Furthermore,

$$\mathcal{M}\tilde{F}_1(w) = \int_0^\infty x^{w-1} \text{erfc}(x) dx \\ = \frac{2}{w\sqrt{\pi}} \int_0^\infty x^w e^{-x^2} dx = \frac{1}{w\sqrt{\pi}} \Gamma((w+1)/2), \quad (4.13)$$

thus

$$\mathcal{M}F_1(w) = \frac{1}{4} \left[ \Gamma(w/2) - \frac{2}{w\sqrt{\pi}} \Gamma((w+1)/2) \right]. \tag{4.14}$$

The poles and residues of  $\mathcal{M}F_a$  can now easily be calculated in terms of the poles and residues of the  $\Gamma$ -function.  $\square$

*Proof of Theorem 3.4.* We choose  $\phi \in C_0^\infty(-1, 1)$  with  $\phi = 1$  near zero. Then, from [G, Lemma 1.9.1] (cf. Remark (2) after Lemma 2.2), we obtain the asymptotic expansion, for  $l = 0, 1$ ,

$$\text{tr}_{L^2(S)} [(1 - \psi_\phi) D_\theta^l e^{-tD_\theta^2}] \sim_{t \rightarrow 0+} \sum_{j=0}^\infty a_j(\phi; \theta, l) t^{j-m/2}. \tag{4.15}$$

The coefficients can be computed locally in terms of the natural extension of  $D$  to the metric double  $\tilde{M}$  of  $M$ , and  $\psi_\phi$ . Thus, since  $e^{-t\tilde{D}^2}$  can serve as a parametrix for  $D_\theta^2$ , we obtain from [L2, Thm. 2.10 and Prop. 3.4] (what Lesch called the “singular elliptic estimate” was proved in (4.4)) that

$$\text{tr}_{L^2(S)} [\psi_\phi D_\theta^l e^{-tD_\theta^2}] \sim_{t \rightarrow 0+} \text{tr}_{\mathcal{H}} [\phi \tilde{D}_\theta^l e^{-t\tilde{D}_\theta^2}], \tag{4.16}$$

and it is enough to expand the right-hand side of (4.16) for  $l = 0, 1$ .

Consider  $l = 0$  first. We obtain from the explicit formula (4.1) and the trace lemma [BS1, Appendix] that

$$\begin{aligned} \text{tr}_{\mathcal{H}} [\phi e^{-t\tilde{D}_\theta^2}] &= \int_0^\infty \phi(x) (4\pi t)^{-1/2} \text{tr}_H [e^{-tA^2}] dx \\ &+ \int_0^\infty \phi(x) e^{-x^2/t} (4\pi t)^{-1/2} \text{tr}_H [(I - 2P(\theta)) e^{-tA^2}] dx \\ &- a(\theta) \int_0^\infty \int_0^\infty \phi(x) e^{-(2x+z)^2/4t} (\pi t)^{-1/2} \text{tr}_H [P(\theta) |A| e^{-a(\theta)|A|z-tA^2}] dz dx \\ &=: \text{I}(t) + \text{II}(t) + \text{III}(t). \end{aligned} \tag{4.17}$$

Since  $A$  is elliptic on  $S_N$ , we have for the first term

$$\text{I}(t) \sim_{t \rightarrow 0+} (4\pi t)^{-1/2} \int_0^\infty \phi(x) dx \sum_{j=0}^\infty b_j(A^2) t^{j-(m-1)/2}. \tag{4.18}$$

Next, as an easy consequence of (3.23) we see that

$$\text{II}(t) \equiv 0. \tag{4.19}$$

For  $\text{III}(t)$ , we write, with

$$c(\lambda) := \dim \ker(|A| - \lambda) = 2 \text{tr}_{\ker(|A|-\lambda)} (P(\theta)),$$

$$\begin{aligned} \text{III}(t) &= -a(\theta) \int_0^\infty \int_0^\infty \phi(x\sqrt{t}) \frac{1}{\sqrt{\pi}} e^{-(x+z)^2} \sum_{\lambda \in \text{spec}|A| \setminus \{0\}} c(\lambda) \sqrt{t\lambda} e^{-2a(\theta)\sqrt{t\lambda}z-t\lambda^2} dz dx \\ &\sim_{t \rightarrow 0+} -\frac{a(\theta)}{2} \int_0^\infty \text{erfc}(z) \sum_{\lambda \in \text{spec}|A| \setminus \{0\}} c(\lambda) \sqrt{t\lambda} e^{-2a(\theta)\sqrt{t\lambda}z-t\lambda^2} dz \\ &= -\frac{a(\theta)}{2} \sum_{\lambda \in \text{spec}|A| \setminus \{0\}} c(\lambda) F_{a(\theta)}(\sqrt{t\lambda}) \\ &= -\frac{a(\theta)}{2} \sum_{\lambda \in \text{spec}|A| \setminus \{0\}} c(\lambda) \frac{1}{2\pi i} \int_{\text{Re } w=c} t^{-w/2} \lambda^{-w} \mathcal{M}F_{a(\theta)}(w) dw \\ &= -\frac{a(\theta)}{4\pi i} \int_{\text{Re } w=c} t^{-w/2} \zeta_{A^2}(w/2) \mathcal{M}F_{a(\theta)}(w) dw. \end{aligned} \tag{4.20}$$

We now collect the various contributions. First, replacing  $\phi$  by  $\phi_\varepsilon$ ,  $\phi_\varepsilon(x) := \phi(x/\varepsilon)$ , and letting  $\varepsilon \rightarrow 0$ , we obtain from (4.15) and (4.18) a contribution

$$\tilde{\text{I}}(t) \sim_{t \rightarrow 0+} \sum_{j=0}^\infty a_j(\theta, 0) t^{j-m/2}, \tag{4.21a}$$

where

$$a_j(\theta, 0) = \int_M \tilde{u}_j(\theta, 0),$$

with  $\tilde{u}_j$  a local density computed for the natural extension of  $D$  to the double  $\tilde{M}$  of  $M$ . The remaining contribution  $\text{III}(t)$  can be evaluated by the residue theorem, since the integrand decays in vertical strips with bounded real part (by Lemma 3.3, Lemma 2.2, and (2.21)). Thus we find (using, e.g., the description of

the singularities of  $\zeta_{A^2}$  in [BL, Lemma 2.1])

$$\begin{aligned} \text{III}(t) &= -\frac{a(\theta)}{2} \sum_{w \in \mathbb{C}} \text{Res}_1(t^{-w/2} \zeta_{A^2}(w/2) \mathcal{M}F_{a(\theta)}(w)) \\ &\sim_{t \rightarrow 0+} \frac{a(\theta)}{2} \sum_{j=0}^{\infty} t^{j-n/2} \left\{ \log t \text{Res}_1 \zeta_{A^2}(n/2 - j) \text{Res}_1 \mathcal{M}F_{a(\theta)}(n - 2j) \right. \\ &\quad \left. - 2 \text{Res}_1 \zeta_{A^2}(n/2 - j) \text{Res}_0 \mathcal{M}F_{a(\theta)}(n - 2j) \right\} \\ &\quad - \frac{a(\theta)}{2} \sum_{j=0}^{\infty} t^{j/2} \text{Res}_0 \zeta_{A^2}(-j/2) \text{Res}_1 \mathcal{M}F_{a(\theta)}(-j). \end{aligned} \tag{4.21b}$$

From this, we can read off our assertions on the structure of the coefficients. First of all, the leading contribution comes from (4.21a) only, as  $a_0 t^{-m/2}$ , and so it is computed as in the compact case. Next, we observe that  $\zeta_{A^2}$  has no poles at the points  $n/2 - j$  for  $j \geq n/2$  if  $n$  is even. If  $n$  is odd, however, the log terms occur, as can be seen from Lemma 3.3. The coefficients of the terms in the first sum in (4.21b) are computed from local densities on  $N$ , whereas those in the second sum are, in general, nonlocal.

Next we consider the case  $l = 1$ . In view of (4.15) and the previous analysis, it is enough to expand

$$\int_0^{\infty} \phi(x) \text{tr}_H[\gamma(\partial_x + A)e^{-t\bar{D}_0^2}(x, x)] dx =: \tilde{\text{I}}(t) + \tilde{\text{II}}(t) + \tilde{\text{III}}(t), \tag{4.22}$$

numbering again the contributions according to the three terms in (4.1). In view of (3.23), (3.24), (2.2), and (3.48), we find

$$\text{tr}_H[\gamma e^{-tA^2}] = \text{tr}_H[\gamma P(\theta)e^{-tA^2}] = 0, \tag{4.23}$$

$$\text{tr}_H[\gamma P(\theta)|A|e^{-a(\theta)|A|z-tA^2}] = 0,$$

and thus

$$\text{tr}_H[\gamma \partial_x e^{-t\bar{D}_0^2}(x, x)] = 0. \tag{4.24}$$

Again from (2.2) we conclude that

$$\text{tr}_H[\gamma A e^{-tA^2}] = 0, \tag{4.25}$$

which implies

$$\tilde{\text{I}}(t) \equiv 0. \tag{4.26}$$

Furthermore,

$$\begin{aligned} \tilde{\text{II}}(t) &= (4\pi t)^{-1/2} \int_0^{\infty} \phi(x) e^{-x^2/t} \text{tr}_H[\gamma A(I - 2P(\theta))e^{-tA^2}] dx \\ &= (4\pi)^{-1/2} \int_0^{\infty} \phi(x\sqrt{t}) e^{-x^2} dx \text{tr}_H[\gamma A(I - 2P(\theta))e^{-tA^2}] \\ &\sim_{t \rightarrow 0+} \frac{1}{4} \text{tr}_H[\gamma A(I - 2P(\theta))e^{-tA^2}] \\ &= -\frac{1}{2} \text{tr}_H[\gamma AP(\theta)e^{-tA^2}]. \end{aligned} \tag{4.27}$$

Finally, we note that, using again (3.23) and (2.2),

$$\text{tr}_H[\gamma A(I - P(\theta))\tilde{A}(\theta)e^{\tilde{A}(\theta)z-tA^2}] = a(\theta) \text{tr}_H[\gamma AP(\theta)|A|e^{-a(\theta)|A|z-tA^2}], \tag{4.28}$$

and so, as in (4.20), with  $d(\lambda) := \text{tr}_{\ker(|A|-\lambda)}[\gamma AP(\theta)]$ ,

$$\begin{aligned} \tilde{\text{III}}(t) &= a(\theta) \int_0^{\infty} \int_0^{\infty} \phi(x) e^{-(2x+z)^2/4t} (\pi t)^{-1/2} \text{tr}_H[\gamma AP(\theta)|A|e^{-a(\theta)|A|z-tA^2}] dz dx \\ &= a(\theta) \int_0^{\infty} \int_0^{\infty} \phi(x\sqrt{t}) \frac{2}{\sqrt{\pi}} e^{-(x+z)^2} \sum_{\lambda \in \text{spec}|A| \setminus \{0\}} d(\lambda) \sqrt{t} \lambda e^{-2a(\theta)\sqrt{t}\lambda z-t\lambda^2} dz dx \\ &\sim_{t \rightarrow 0+} a(\theta) \sum_{\lambda \in \text{spec}|A| \setminus \{0\}} d(\lambda) F_{a(\theta)}(\sqrt{t}\lambda) \\ &= \frac{a(\theta)}{2\pi i} \int_{\text{Re } w=c} t^{-w/2} \eta(A, \gamma AP(\theta); w-1) \mathcal{M}F_{a(\theta)}(w) dw. \end{aligned} \tag{4.29}$$

Combining our computations, we see that the terms local on  $\tilde{M}$  protrude from (4.15) as before.

We obtain the second contribution from (4.27). However, since  $P(\theta)$  is a pseudodifferential operator, we now have to employ the general expansion theorem for pseudodifferential operators (2.21) (see [GrSe1, Thm. 2.7]). Namely,

$$\begin{aligned} \tilde{\text{II}}(t) &\sim_{t \rightarrow 0+} -\frac{1}{2} \text{tr}_H[\gamma AP(\theta)e^{-tA^2}] \\ &\sim_{t \rightarrow 0+} \sum_{j=0}^{\infty} c_j^1(\theta, 1) t^{(j-m)/2} + \sum_{j=0}^{\infty} (b_j^1(\theta, 1) t^j \log t + d_j^1(\theta, 1) t^j). \end{aligned} \tag{4.30a}$$

Here,  $b_j^1, c_j^1$  are integrals of local densities over  $N$ , whereas the  $d_j^1(\theta, 1)$  are, in general, nonlocal spectral invariants on  $N$ .

For the third contribution, we use again the estimate (2.15) with  $B = \gamma AP(\theta)$  (stemming from the fact that  $P(\theta)$  is a pseudodifferential operator) to obtain

$$\widetilde{\text{III}}(t) \sim_{t \rightarrow 0^+} a(\theta) \sum_{w \in \mathbb{C}} \text{Res}_1 \left( t^{-w/2} \eta(A, \gamma AP(\theta); w - 1) \mathcal{M}F_{a(\theta)}(w) \right).$$

From the expansion (4.30a) and Lemma 2.1, one derives that  $\eta(A, \gamma AP(\theta); w)$  is meromorphic in  $\mathbb{C}$  with simple poles at the points  $n - k, k \in \mathbb{Z}_+$ . Furthermore, the residues of the poles are integrals of local densities over  $N$ . Thus

$$\begin{aligned} \widetilde{\text{III}}(t) \sim_{t \rightarrow 0^+} & -\frac{a(\theta)}{2} \sum_{j=0}^{\infty} t^{j/2} \log t \text{Res}_1(\eta(A, \gamma AP(\theta); -j - 1)) \text{Res}_1 \mathcal{M}F_{a(\theta)}(-j) \\ & + a(\theta) \sum_{j=0}^{\infty} t^{(j-m)/2} \text{Res}_1(\eta(A, \gamma AP(\theta); m - j - 1)) \text{Res}_0 \mathcal{M}F_{a(\theta)}(m - j) \\ & + a(\theta) \sum_{j=0}^{\infty} t^{j/2} \text{Res}_0(\eta(A, \gamma AP(\theta); -j - 1)) \text{Res}_1 \mathcal{M}F_{a(\theta)}(-j). \end{aligned} \quad (4.30b)$$

The coefficients in the first and second sum are again local, like  $c_j^1$  in (4.30a), whereas those in the third sum are not.

It remains to compute the contribution to  $t^{-1/2}$  from (4.30a,b). Using Lemma 2.1, it turns out to be equal to

$$\begin{aligned} & -\frac{1}{2} a_{-1/2,0}(A, \gamma AP(\theta)) + a(\theta) \text{Res}_0 \mathcal{M}F_{a(\theta)}(1) \text{Res}_1 \eta(A, \gamma AP(\theta); 0) \\ & = \left( -\frac{\sqrt{\pi}}{4} + a(\theta) \mathcal{M}F_{a(\theta)}(1) \right) \text{Res}_1 \eta(A, \gamma AP(\theta); 0) \\ & = \left( -\frac{\sqrt{\pi}}{4} + a(\theta) \mathcal{M}F_{a(\theta)}(1) \right) \text{Res}_1 \eta(A, \gamma(\text{sgn } A)P(\theta); -1) \\ & = \left( -\frac{\sqrt{\pi}}{4} + a(\theta) \mathcal{M}F_{a(\theta)}(1) \right) \text{Res}(\gamma(\text{sgn } A)P(\theta)), \end{aligned}$$

using (2.33) in the last step. □

*Proof of Theorem 3.5.* We choose  $\tilde{\phi} \in C_0^\infty(-1, 1)$  with  $\tilde{\phi} = 1$  in a neighborhood of  $\text{supp } \phi$ , with  $\phi$  from (3.32). Then, for  $u \in \mathcal{D}_0$ , one easily computes (writing

$$\tilde{\psi} = \psi_{\tilde{\phi}})$$

$$\check{D}_\theta \tilde{\psi} u = \Phi_\theta^* \Phi^* \gamma(\partial_x + A) \Phi \Phi_\theta \tilde{\psi} u \quad (4.31)$$

$$=: \Phi^* i \gamma \phi' T(\theta) \Phi \tilde{\psi} u + \Phi^* \Phi_\theta^* \gamma A \Phi_\theta \Phi \tilde{\psi} u + v, \quad (4.32)$$

with  $v$  independent of  $\theta$ . Hence,

$$\frac{d}{d\theta} \check{D}_\theta \tilde{\psi} u = \Phi_\theta^* \Phi^* i \gamma(\phi' T'(\theta) - 2\phi T'(\theta)A) \Phi \Phi_\theta \tilde{\psi} u$$

and

$$\text{tr}_{L^2(S)} \left[ \frac{d}{d\theta} \check{D}_\theta e^{-t\check{D}_\theta^2} \right] = i \text{tr}_{L^2(S)} [\gamma(\phi' T'(\theta) - 2\phi T'(\theta)A) e^{-tD_\theta^2} \tilde{\psi}]. \quad (4.33)$$

We can argue as in the proof of Theorem 3.4 to replace  $e^{-tD_\theta^2}$  by  $e^{-t\check{D}_\theta^2}$ , that is,

$$\begin{aligned} & i \text{tr}_{L^2(S)} [\gamma(\phi' T'(\theta) - 2\phi T'(\theta)A) e^{-tD_\theta^2} \tilde{\psi}] \\ & \sim_{t \rightarrow 0^+} i \text{tr}_{L^2(S)} [\gamma(\phi' T'(\theta) - 2\phi T'(\theta)A) e^{-t\check{D}_\theta^2} \tilde{\psi}]. \end{aligned} \quad (4.34)$$

Again as in the proof of Theorem 3.4, we obtain three terms twice from plugging the kernel (4.1) in (4.34).

We start with

$$\begin{aligned} i \text{tr}_{L^2(S)} [\gamma \phi' T'(\theta) e^{-t\check{D}_\theta^2}] & = i \int_0^\infty \phi'(x) \text{tr}_H [\gamma T'(\theta) e^{-t\check{D}_\theta^2}(x, x)] dx \\ & =: \text{I}(t) + \text{II}(t) + \text{III}(t). \end{aligned} \quad (4.35)$$

We find

$$\begin{aligned} \text{I}(t) & = i(4\pi t)^{-1/2} \int_0^\infty \phi'(x) dx \text{tr}_H [\gamma T'(\theta) e^{-tA^2}] \\ & = -i(4\pi t)^{-1/2} \text{tr}_H [\gamma T'(\theta) e^{-tA^2}]. \end{aligned} \quad (4.36)$$

Since  $\phi'$  is supported away from zero, it is easy to see that

$$\text{II}(t) \sim_{t \rightarrow 0^+} \text{III}(t) \sim_{t \rightarrow 0^+} 0. \quad (4.37)$$

The second contribution is

$$\begin{aligned}
 -2i \operatorname{tr}_{L^2(S)}[\gamma \phi T'(\theta) A e^{-t\tilde{D}_\theta^2}] &= -2i \int_0^\infty \phi(x) \operatorname{tr}_H[\gamma T'(\theta) A e^{-t\tilde{D}_\theta^2}(x, x)] dx \\
 &=: \tilde{\text{I}}(t) + \tilde{\text{II}}(t) + \tilde{\text{III}}(t).
 \end{aligned}
 \tag{4.38}$$

We compute

$$\tilde{\text{I}}(t) = -2i(4\pi t)^{-1/2} \int_0^\infty \phi(x) \operatorname{tr}_H[\gamma T'(\theta) A e^{-tA^2}] dx = 0,
 \tag{4.39}$$

since  $\gamma$  commutes with  $T'(\theta)$  but anticommutes with  $A$ . Next we see that

$$\begin{aligned}
 \tilde{\text{II}}(t) &= -2i(4\pi t)^{-1/2} \int_0^\infty \phi(x) e^{-x^2/t} \operatorname{tr}[\gamma T'(\theta) A(I - 2P(\theta))e^{-tA^2}] dx \\
 &\sim_{t \rightarrow 0^+} i \operatorname{tr}[\gamma T'(\theta) AP(\theta)e^{-tA^2}].
 \end{aligned}
 \tag{4.40}$$

Finally, with  $d(\lambda) = \operatorname{tr}_{\ker(|A|-\lambda)}[\gamma T'(\theta) AP(\theta)e^{-tA^2}]$ ,

$$\begin{aligned}
 \tilde{\text{III}}(t) &\sim_{t \rightarrow 0^+} -2ia(\theta) \sum_{\lambda \in \operatorname{spec}|A| \setminus \{0\}} d(\lambda) \sqrt{t\lambda} \int_0^\infty e^{-2a(\theta)\lambda\sqrt{tz-t\lambda^2}} \operatorname{erfc}(z) dz \\
 &= -2ia(\theta) \sum_{\lambda \in \operatorname{spec}|A| \setminus \{0\}} d(\lambda) F_{a(\theta)}(\sqrt{t\lambda}) \\
 &= -\frac{a(\theta)}{\pi} \int_{\operatorname{Re} w=c} t^{-w/2} \eta(A, \gamma T'(\theta) AP(\theta); w-1) \mathcal{M}F_{a(\theta)}(w) dw.
 \end{aligned}
 \tag{4.41}$$

The existence of the asymptotic expansion hence follows from our assumptions, Lemma 3.3, (4.36), (4.40), and (4.41). Consequently, with (2.13a), (2.11), (2.33), and (2.21), we obtain

$$\begin{aligned}
 a_{-1/2,1} \left( \dot{D}_\theta, \frac{d}{d\theta} \dot{D}_\theta \right) &= -(4\pi)^{-1/2} a_{0,1}(A, \gamma i T'(\theta)) \\
 &= \frac{1}{4\sqrt{\pi}} \operatorname{Res}(\gamma i T'(\theta)),
 \end{aligned}$$

$$\begin{aligned}
 a_{-1/2,0} \left( \dot{D}_\theta, \frac{d}{d\theta} \dot{D}_\theta \right) &= -(4\pi)^{-1/2} a_{00}(A, \gamma i T'(\theta)) + a_{-1/2,0}(A, \gamma i T'(\theta) AP(\theta)) \\
 &\quad - 2a(\theta) \mathcal{M}F_{a(\theta)}(1) \operatorname{Res}_1 \eta(A, \gamma i T'(\theta) AP(\theta); 0) \\
 &= -(4\pi)^{-1/2} a_{00}(A, \gamma i T'(\theta)) \\
 &\quad + \left( \frac{\sqrt{\pi}}{2} - 2a(\theta) \mathcal{M}F_{a(\theta)}(1) \right) \operatorname{Res}(\gamma i T'(\theta) (\operatorname{sgn} A) P(\theta)).
 \end{aligned}$$

In view of (2.28), we reach the conclusion. □

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