THE RESOLVENT EXPANSION ON SINGULAR SPACES

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ABSTRACT. We discuss the resolvent trace expansion for singular spaces. We prove the existence of such an expansion for a certain class of isolated singularities.

INTRODUCTION

The resolvent expansion is a well established tool of geometric analysis. To recall what is involved, consider a compact Riemannian manifold M, of dimension m, and a Hermitian vector bundle E over M. On the smooth sections of E we consider, in addition, a "Laplace type" operator, Δ, by which we mean a second order formally symmetric differential operator with principal symbol given by the metric tensor. Typically, such operators arise as square of a generalized Dirac operator plus a potential; an example of particular importance is provided by the Laplacian on differential forms.

Then it is well known that Δ is essentially self-adjoint in $L^2(E)$ with domain $C_0^\infty(E)$ and that its closure, also to be denoted by Δ, has a discrete spectrum. But much more is true: the resolvent powers,

$$R^l(z) := (\Delta + z^2)^{-l},$$

are trace class if $2l > m$, and as $z \to \infty$ we even have a full asymptotic expansion

$$\mathrm{tr}_{L^2(E)}[R^l(z)] \sim \sum_{j=0}^{\infty} a_j z^{m-2l-j}. \tag{0.1}$$

Moreover, the coefficients are local, that is

$$a_j = \int_M a_j(p)\,\mathrm{vol}_M(p), \tag{0.2}$$

where the numbers $a_j(p)$ are given by universal formulas in the germ of the symbol of Δ at p.

It seems that Carleman [Ca] was the first to observe that already the leading term in (0.1) implies Weyl's law for the eigenvalue distribution. Carleman treated only a special case; the result was established in great generality by Gårding [Ga] who constructed a good parametrix for $(\Delta + z^2)^l$ (and actually for scalar elliptic operators of any order).

On the other hand, Hadamard [H] constructed a very precise parametrix for the wave and the heat equation which was adapted by Minakshisundaram and Pleijel [MPl] to the Riemannian case, thus obtaining a full asymptotic expansion for $\mathrm{tr}_{L^2(E)}\, e^{-t\Delta}$. This expansion gives the same coefficients, modulo universal constants, as (0.1). And it is also equivalent to the knowledge of the poles of the ζ-function which is given by

$$\zeta_\Delta(s) := \mathrm{tr}_{L^2(E)}\, \Delta^{-s},$$

for $\mathrm{Re}\, s$ large. The ζ-function was Seeley's point of departure for his thorough analysis of the resolvent of general elliptic pseudodifferential operators [S] which proved in particular the analogue of (0.1) in this generality.

The resolvent (or heat) expansion turned out to be very useful not only for the investigation of the eigenvalue distribution but also for Getzler's direct proof of the Atiyah-Singer Index Theorem [Ge]. Moreover, following the programmatic articles by Gelfand [G] and Kac [K] the field of *spectral geometry* evolved, investigating the amount of geometric information encoded in the resolvent expansion; cf. [Gi] for a good survey of many aspects. It is fair to say, however, that no satisfying picture has emerged yet in spite of a lot of good work in this area.

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In the singular case, on the other hand, we should expect that the coefficients supported in the singular set contain interesting information on the nature of the singularities. For example, we may hope to extract the most important numerical invariants. That this is a realistic hope is shown already by the Dirichlet problem in a bounded domain M in $\mathbb{R}^2$: the natural extension of (0.1) to this case determines both $\operatorname{vol} M$ and $\operatorname{vol} \partial M$ (so we can spectrally recognize the circle, in view of the isoperimetric inequality!) and possibly even M itself, if we assume enough regularity for the boundary. The case of elliptic boundary value problems has been dealt with thoroughly in the work of Seeley and Grubb [GS] deriving the resolvent expansion in great generality.

Other types of singularities occurred in the work of Cheeger [Ch] who was the first to extend the relevant analysis to conic singularities and opened up a large new field of investigation. Despite the considerably more difficult techniques to be mastered in the presence of singularities, one may suspect that singularities are spectrally more rigid than smooth structures. That this suspicion is substantial has been confirmed in the case of algebraic curves by the author and M. Lesch [BL1]; in this case it is possible to distinguish smooth curves from those with singularities and, quite often, to determine all multiplicities, simply from the knowledge of the resolvent expansion.

Such results provide a strong motivation for further efforts to extend this kind of asymptotic analysis to larger classes of singularities. In this note, we develop an axiomatic approach which proves the resolvent expansion for a certain class of isolated singularities which may be called *hyperconical* since it excludes cones. It is not quite clear yet which e.g. algebraic singularities do satisfy these assumptions; this will be the object of further investigation.

Our approach is based on a systematic use of self-adjoint "model operators" which are attached locally to the various strata of the given singular space such that their asymptotic analysis synthesizes to produce the asymptotic spectral data for the elliptic (Laplace type) operator in question. Thus we start with a somewhat new look at the classical case of compact manifolds and the simple model operators involved (Sec. 2), after some general remarks on self-adjointness in singular situations (Sec. 1). It turns out that an operator valued version of the analysis in the smooth case is the key to our result; this is done in Sec. 3. For the singular asymptotic expansions we need a generalization of the Singular Asymptotics Lemma from [BS1], which is quoted in Sec. 4; the proof can be found in [B2]. This reflects the fact that the "conic" scaling is vital also in non-conic situations. The main result is given in Sec. 5, and Sec. 6 provides a sketch of possible extensions to more general stratified spaces.

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1. Self-adjoint extensions

We now keep the setting of the Introduction but we do not assume any more that the Riemannian manifold M under consideration is compact. In what follows we think of it as the smooth part of a compact stratified space, $\bar{M}$, equipped with a metric which makes it a length space and induces the given metric on M. In this section we describe some useful consequences of general results on self-adjointness which have been obtained jointly with Henri Moscovici; a detailed account will be given elsewhere. Here we will concentrate on the case of essential self-adjointness as indicated in the introduction. What we are going to describe applies to symmetric elliptic operators of any order, in particular to the generalized Dirac and Laplace operators. For the special case of the latter, we can always restrict attention to the Friedrichs extension, in view of semiboundedness, even if the operator in question is not essentially self-adjoint. This will be done in the rest of the paper, leaving a more thorough discussion for a later occasion.

The obvious idea is that we use different model spaces near the different strata, and we make this precise through the following set of assumptions.

(SA1) *M admits an open cover, $(U_i)_{i=1}^N$, and partial isometries*

$$r_i : L^2(E|U_i) \mapsto L^2(E_i),$$

with the property that

$$r_i(C_0^\infty(E|U_i)) \subset C_0^\infty(E_i), \quad r_i^*(C_0^\infty(E_i)) \subset C_0^\infty(E|U_i);$$

here E_i is a Hermitian vector bundle over a Riemannian manifold M_i, the "model space". Moreover, on $C_0^\infty(E_i)$ we are given a Laplace type operator Δ_i, the "model operator".

The model data and the given operator are linked by the following functional relations holding on smooth sections with compact support:

$$r_i\Delta = \Delta_i r_i \quad \text{and} \quad r_i\psi = \tau_i(\psi) r_i, \quad \psi \in C^\infty(U_i),$$

for some map $\tau_i : C^\infty(U_i) \mapsto C^\infty(M_i)$.

It is easy to see that this axiom holds if the partial isometries are derived from smooth bundle maps, with appropriate choice of Δ_i.

In addition, we need an axiom to patch together the various model operators.

(SA2) *There are functions* $\phi_i \in C^\infty(M)$ *with* supp $d\phi_i$ *compact in* U_i *and such that*

$$\sum_{i=1}^N \phi_i^2 = 1.$$

With these data we now obtain the following easy result.

Lemma 1.1. *The map* $\Phi : L^2(E) \mapsto \oplus_{i=1}^N L^2(E_i)$ *given by*

$$\Phi u := (r_i\phi_i u)_{i=1}^N, \tag{1.1}$$

is a partial isometry, inducing a unitary equivalence between the operator Δ *with domain* $C_0^\infty(E)$ *in* $L^2(E)$ *and* $P\tilde{\Delta}P$ *with domain* $\oplus_{i=1}^N C_0^\infty(E_i)$ *in* $\oplus_{i=1}^N L^2(E_i)$.

Here we have used the notation

$$P := \Phi\Phi^*,$$

and $\tilde{\Delta}$ *is a differential operator of the form*

$$\tilde{\Delta} := \oplus_{i=1}^N \Delta_i + \tilde{\Delta}_1 \tag{1.2}$$

$$=: \tilde{\Delta}_0 + \tilde{\Delta}_1. \tag{1.3}$$

Here $\tilde{\Delta}_1$ *is a differential operator of lower order if e.g.* Δ *has scalar principal symbol.*

It is obvious that Lemma 1.1 provides a basis for constructing and classifying self-adjoint extensions of Δ provided we can deal effectively with $\tilde{\Delta}_1$. Since we concentrate on essential self-adjointness in this discussion, we single out here only the relevant consequence of Lemma 1.1; it boils down to an application of the Kato-Rellich Theorem.

Lemma 1.2. *If all* Δ_i *are essentially self-adjoint on* $C_0^\infty(E_i)$ *and if, with some* $b < 1$, *the apriori estimates*

$$\|(P\tilde{\Delta}(I-P) + (I-P)\tilde{\Delta}P)u\| \le b\|\tilde{\Delta}u\| + C\|u\|, \tag{1.4}$$

$$\|\tilde{\Delta}_1 u\| \le b\|\tilde{\Delta}u\| + C\|u\|. \tag{1.5}$$

hold for $u \in \oplus_{i=1}^N C_0^\infty(E_i)$, *then* Δ *is essentially self-adjoint on* $C_0^\infty(E)$.

The crucial estimate necessary to check the assumptions of Lemma 1.2 can then be formulated in terms of the model operators as follows.

Corollary 1.3. *The conditions of Lemma 1.2 are satisfied if the model operators satisfy the apriori estimates*

$$\|[\Delta_j, \tau_j(\psi\phi_j)]u_j\| \le \varepsilon\|\Delta_j u_j\| + C_\varepsilon\|u_j\|, \quad u_j \in C_0^\infty(E_i),$$

for all $\varepsilon \in (0,1)$ *and all* j, *and all* $\psi \in C^\infty(U_j)$ *with* supp $d\psi$ *compact in* U_j.

We remark in conclusion that the results described here provide a very simple approach to the standard theory of elliptic operators on compact manifolds since in this case the model operators Δ_i can be chosen as small perturbations of constant coefficient operators. Thus we obtain the mapping properties between suitable function spaces, arising naturally as domains of self-adjoint operators, and hence regularity and Fredholm properties without building an elaborate calculus first. This approach seems to be very useful, in view of its abstractness and flexibility, also in the singular case. We will return to these questions in a future publication.

2. The smooth scalar case

We begin by considering an arbitrary Riemannian manifold, M, of dimension m, a Hermitian bundle, $E \to M$ of rank k, and a Laplacian, Δ, acting on the smooth sections of E, as described in the Introduction. We fix a self-adjoint extension of Δ which we denote by the same symbol; note that such extensions do always exist since Δ is semibounded. The resolvent will be denoted by

$$R(z) := (\Delta + z^2)^{-1},$$

where we will always assume that $z \geq 1$. We will also write $R^l(z) := (R(z))^l$, for $l \in \mathbb{Z}_+$. We also fix a function $\phi \in C_0^\infty(M)$. Next we introduce a "continuous partition of unity" as in [B3]. By this we mean a family of functions, $\chi_{p\varepsilon} \in C_0^\infty(B_\varepsilon(p))$, where $p \in M$ and ε is any sufficiently small positive number, with the following properties:

$$\chi_{p\varepsilon} \geq 0,$$
$$\text{the map } p \to \chi_{p\varepsilon} \text{ is continuous},$$
$$\int_M \chi_{p\varepsilon}(q) \operatorname{vol}_M(p) = 1 \text{ for all } q \in M,$$
$$\lim_{\varepsilon \to 0} \int_M \chi_{p\varepsilon}(q) \operatorname{vol}_M(q) = 1 \text{ for all } p \in M.$$

With these preparations, we can state our first expansion result which is, in principle, well known. Our derivation leads to a different description of the expansion coefficients, though, which will be important later on.

Theorem 2.1. *For $2l > m$, $\phi R^l(z)$ is a trace class operator in $L^2(E)$. As $z \to \infty$, we have an asymptotic expansion*

$$\operatorname{tr}_{L^2(E)}[\phi R^l(z)] \sim \sum_{j \geq 0} z^{m-2l-j} \int_M \phi a_j(p) \operatorname{vol}_M(p).$$

The coefficients a_j arise from the two-parameter asymptotic expansion of the trace $\operatorname{tr}_{L^2(E)}[\chi_{p\varepsilon} R^l(z)]$, as $\varepsilon \to 0$ and $z \to \infty$, as precisely those terms which are independent of ε.

The proof of this theorem will occupy the rest of this section. The first step consists in showing that $\phi R^l(z)$ is actually of trace class. By the compactness of $\operatorname{supp} \phi$, it is enough to consider the situation when ϕ is supported in an arbitrarily small neighborhood of a given point $p \in M$.

Thus we are in the position to compare Δ near $p \in M$ with a suitable model, Δ_p, in $\mathbb{R}^m$. To do so we determine ε_0 such that the exponential map is a diffeomorphism in $B_\varepsilon^m(0)$ and we choose a local orthonormal frame for E in $B_{\varepsilon_0}(p)$. These data determine an isometry

$$\Phi_p : L^2(E|B_{\varepsilon_0}(p)) \to L^2(\mathbb{R}^m, \mathbb{C}^k).$$

For $\phi \in C_0^\infty(B_{\varepsilon_0}(p))$ and $u \in L^2(E|B_{\varepsilon_0}(p))$ we obtain

$$\Phi_p(\phi u) =: \tilde{\phi} \Phi_p(u),$$

with a suitable $\tilde{\phi} \in C_0^\infty(B_{\varepsilon_0}^m(0)$. In particular, we get a coordinate representation of Δ in $B_{\varepsilon_0}^m(0)$; note that this construction can be carried out continuously for $p \in B_{\varepsilon_0/2}(p_0)$, for any $p_0 \in M$. We extend the constructed differential operator on $B_{\varepsilon_0}^m(0)$ smoothly to a symmetric differential operator on all of $\mathbb{R}^m$ by making it equal to its principal plus zero order part with coefficients evaluated at 0 – this is the "model operator" – outside $B_{\varepsilon_0/2}^m(0)$. The resulting operator will be denoted by Δ_p; it can be thought of as a small perturbation of a constant coefficient operator. In the chosen coordinates and frames we write

$$\Delta_p = \sum_{|\alpha| \leq 2} A_p^\alpha(x) D_x^\alpha, \tag{2.1}$$

$$\Delta_{p0} := \sum_{|\alpha|=2} A_p^\alpha(0) D_x^\alpha + \operatorname{Re} A_p^0(0) = -\sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} + \operatorname{Re} A_p^0(0), \tag{2.2}$$

$$\tilde{\Delta}_{pj} := \Delta_p^j - \Delta_{p0}^j \quad \text{for } j \in \mathbb{N}. \tag{2.3}$$

The last identity can be extended to $j = 0$ by setting $\tilde{\Delta}_{p0} = 0$.

We may assume that Re $A_p^0(0) \geq 0$ so that we can write

$$\operatorname{Re} A_p^0(0) =: (\tilde{A}_p)^2; \tag{2.4}$$

the reason for this choice of the model operator will become clear below when we treat operator coefficients. Since supp ϕ is compact, we can make ε_0 as small as we want. Thus, denoting by $\|\cdot\|$ the norm in $L^2(\mathbb{R}^m, \mathbb{C}^k)$, we may assume that we have the apriori estimates

$$\|\tilde{\Delta}_{pj} u\| \leq b\,\|\Delta_{p0}^j u\| + C_b\|u\|, \; u \in C_0^\infty(\mathbb{R}^m, \mathbb{C}^k), \tag{2.5}$$

for any fixed $b < 1$, with some positive constant C_b and uniformly in $p \in \operatorname{supp}\phi$ and $j \in \mathbb{Z}_+, j \leq l_0$. In view of the Kato-Rellich Theorem we conclude that both Δ_p and Δ_{p0} are essentially self-adjoint in $L^2(\mathbb{R}^m, \mathbb{C}^k)$ with domain $C_0^\infty(\mathbb{R}^m, \mathbb{C}^k)$. Thus we can introduce the localized resolvents,

$$R_p^j(z) := (\Delta_p + z^2)^{-j}, \tag{2.6}$$

$$R_{p0}^j(z) := (\Delta_{p0} + z^2)^{-j}. \tag{2.7}$$

We calculate for $l \leq l_0$:

$$\begin{aligned}(\Delta_p + z^2)^l R_{p0}^l(z) &= \sum_{j=0}^{l} \binom{l}{j} z^{2(l-j)} (\Delta_{p0}^j + \tilde{\Delta}_{pj}) R_{p0}^l(z)\\ &= I + \sum_{j=1}^{l} \binom{l}{j} z^{2(l-j)} \tilde{\Delta}_{pj} R_{p0}^l(z)\\ &=: I - Q_p^l(z).\end{aligned}$$

A straightforward estimate using (2.5) shows that

$$\|Q_p^l(z)\| \leq 2^l(b + C_b z^{-2}), \tag{2.8}$$

and we deduce the following lemma.

Lemma 2.2. *If* $2^l(b + C_b z^{-2}) < 1$, *then*

$$\begin{aligned}R_p^l(z) &= R_{p0}^l(z)(I - Q_p^l(z))^{-1}\\ &= \sum_{j\geq 0} R_{p0}^l(z)(Q_p^l(z))^j.\end{aligned}$$

We wish to replace $R^l(z)$ by $R_p^l(z)$ and then to use the Neumann series just derived. To do so we want to apply the abstract framework of [B3, Sec. 4] to $R_p(z)$, wherefrom we adopt the notation $\phi < \psi$ iff ϕ and ψ are in $C_0(\mathbb{R}^m)$ and $\psi = 1$ in a neighbourhood of supp ϕ. This makes it necessary to establish some relevant (and probably well known) Schatten estimates. We use the following notation: the Schatten–v.Neumann class of order $q > 0$ in a Hilbert space H will be denoted by $\mathcal{C}_q(H)$, with norm $\|\cdot\|_q$; $\|\cdot\|_\infty$ then denotes the norm in H. To establish the needed estimates we work with a simple class of homogeneous vector valued symbols in $\mathbb{R}^m$: for a complex vector space V, a function $p \in C^\infty(\mathbb{R}^m \times \mathbb{R}_+ \setminus \{0\}, \operatorname{End} V)$ is said to be in the symbol space $\mathcal{S}^{-\sigma}(\mathbb{R}^m \times \mathbb{R}_+, \operatorname{End} V) =: \mathcal{S}^{-\sigma}(V)$, $\sigma \geq 0$, if

$$p(t\xi, tz) = t^{-\sigma} p(\xi, z) \;\text{ for all } t > 0. \tag{2.9}$$

With such a symbol we define the usual pseudodifferential operator

$$(\operatorname{Op} p(z))u(x) = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{i\langle x,\xi\rangle} p(\xi, z)(\hat{u}(\xi))d\xi.$$

Then we have the following estimates.

Lemma 2.3. *Let* $p \in \mathcal{S}^{-\sigma}(\mathbb{C}^k)$ *with* $\sigma \geq 0$; *assume also that* $z \geq 1$.

1) $\|\operatorname{Op} p(z)\|_\infty \leq C z^{-\sigma}$.
2) *For* $\phi, \psi \in C_0^\infty(\mathbb{R}^m)$ *with* $\phi < \psi$ *we have*

$$\phi[\operatorname{Op} p(z)](1-\psi) \in \mathcal{C}_1(L^2(\mathbb{R}^m, \mathbb{C}^k)) \;\; \textit{and}$$
$$\|\phi[\operatorname{Op} p(z)](1-\psi)\|_1 \leq C_N z^{-N},$$

for all $N \in \mathbb{N}$.

3) *If* $\sigma > m/q$, *then* $\phi[\operatorname{Op} p] \in \mathcal{C}_q(L^2(\mathbb{R}^m, \mathbb{C}^k))$ *and*

$$\|\phi \operatorname{Op} p(z)\|_q \le C z^{m/q-\sigma}.$$

Proof. 1) is obvious. For the proof of 2) we fix ϕ and choose $R > 0$ such that $\operatorname{supp}\phi$ is contained in $B_R(0)$. If Δ_R denotes the Dirichlet Laplacian on $B_R(0)$ then it is clearly enough to show that for $2l > m$ and all $N \in \mathbb{N}$ the operator $Q(z) := (\Delta_R + z^2)^l \phi \operatorname{Op} p\,(1-\psi)$ is bounded with norm bound $C_N z^{-N}$. To see this we observe that the kernel of $Q(z)$ becomes a convergent oscillatory integral after multiplication by $(x-y)^{2N}$, implying the estimate

$$|Q(x,y;z)| \le C_N(1+|x-y|+z)^{-N}\chi(x)(1-\psi(y)),$$

for all $N \in \mathbb{N}$ and some $\chi \in C_0^\infty(\mathbb{R}^m)$ satisfying $\chi < \psi$. Then the claimed norm bound follows e.g. from Schur's test.
We turn to the proof of 3). In view of 1), we may assume that

$$p(\xi, z) = (|\xi|^2 + z^2)^{-\sigma} =: p_0(\xi, z)^{-\sigma}.$$

Assume first that our assumption holds with $q = 1$. Then, since $\operatorname{Op} p_0$ is nonnegative we see that $\phi \operatorname{Op} p_0 \phi$ has a finite trace which equals its trace norm. Hence $\phi \operatorname{Op} p_0 \phi$ is trace class with

$$\|\phi \operatorname{Op} p_0 \phi\|_1 = \operatorname{tr} \phi \operatorname{Op} p_0 \phi.$$

Thus, with $\psi > \phi$ we may write

$$\phi \operatorname{Op} p_0 = \phi[\operatorname{Op} p_0]\phi\psi + \phi[\operatorname{Op} p_0](1-\psi),$$

so the assertion follows from 2) in this case. Next we see by an easy induction that the assertion holds for $q = 2^k$, for any $k \in \mathbb{Z}_+$. Finally, for arbitrary q we choose $q_0 < q$ such that q_0 has a finite dyadic expansion,

$$q_0 = \sum_{i=1}^N 2^{-k_i},$$

with $k_i < k_{i+1}$. Now we choose $\sigma_i > k_i$ such that $\sum \sigma_i = \sigma$ and a sequence $\phi_i \in C_0^\infty(\mathbb{R}^m)$ with $\phi_1 = \phi$, $\phi_i < \phi_{i+1}$. Then we write

$$\phi \operatorname{Op} p_0^{-\sigma} =: \prod_{i=1}^N \phi_i \operatorname{Op} p_0^{-\sigma_i} + R.$$

By 2), $R \in \mathcal{C}_1(H)$ with rapid norm decay in z, and the ith factor of the operator product is in $\mathcal{C}_{2^{k_i}}(H)$, with the right norm estimate. But then we obtain the assertion for $\phi \operatorname{Op} p_0$ and q_0 from the "Hölder inequality" for Schatten norms. Using the simple interpolation inequality

$$\|A\|_q \le \|A\|^{1-p/q}\|A\|_p^{p/q},$$

valid for all $A \in \mathcal{C}_p(H)$ and all $q > p > 0$, we complete the proof. □

The estimates just derived imply that we can replace locally $R^l(z)$ by $R^l_p(z)$ as intended.

Lemma 2.4. *Let* $2l > m$. *Then for all* $N \in \mathbb{N}$, *all* $p \in M$, *and all* $\phi \in C_0^\infty(B_{\varepsilon_0}(p))$, *we have, locally uniformly in p,*

$$|\operatorname{tr}_{L^2(E)}[\phi R^l(z)] - \operatorname{tr}_{L^2(\mathbb{R}^m, \mathbb{C}^k)}[\tilde\phi R^l_p(z)]| \le C_N z^{-N}.$$

Hence $\psi R^l(z)$ *is trace class for any* $\psi \in C_0^\infty(M)$.

Proof. We pick functions ϕ, ψ, χ in $C_0^\infty(B_{\varepsilon_0}(p))$ with $\psi > \phi > \chi$ and introduce the operator

$$B_p(z) := \Phi_p^{-1}\tilde\phi R^l_p(z)\Phi_p\chi.$$

Then, clearly, $B_p(z)(C_0^\infty(E)) \subset C_0^\infty(E)$ and we compute with Lemma 2.2

$$\begin{aligned}(\Delta + z^2)^l B_p(z) &= \chi + \Phi_p^{-1}[(\Delta_p + z^2)^l, \tilde\phi]R^l_p(z)\Phi_p\chi \\ &=: \chi + B^1_p(z) \\ &= \chi + \phi B^1_p(z).\end{aligned}$$

This implies the identity

$$B_p(z) = R^l(z)\chi + R^l(z)\phi B^1_p(z). \tag{2.10}$$

Next we observe that

$$[(\Delta_p + z^2)^l, \tilde{\phi}]\, R^l_{p0}(z) = \sum_{j=0}^{l} \sum_{|\alpha|\leq 2j-1} B_p^{\alpha j}(x) D_x^\alpha R^l_{p0}(z),$$

with certain $B_p^{\alpha j} \in C_0^\infty(B^m_{\varepsilon_0}(0), \operatorname{End}\mathbb{C}^k)$. Now $D_x^\alpha R^l_{p0}(z) = \operatorname{Op}[\xi^\alpha p_0(\xi,z)^{-l}]$, and all symbols involved are in $\mathcal{S}^{-\sigma}$ for some $\sigma \geq 1$. Hence Lemma 2.3 applies and gives

$$B_p^1(z) \in \mathcal{C}_{m+1}, \quad \|B_p^1(z)\|_{m+1} \leq C z^{-1/(m+1)}.$$

Using this in (2.10) gives

$$\|R^l(z)\chi - B_p(z)\|_{m+1} \leq C z^{-1/(m+1)}.$$

Iterating this argument, using the "Hölder inequality" for Schatten norms, we find for $1 \leq j \leq m+1$

$$(R^l(z)\chi - B_p(z)) \in \mathcal{C}_{(m+1)/j}, \quad \|R^l(z)\chi - B_p(z)\|_{(m+1)/j} \leq C z^{-2l+jm/(m+1)}.$$

In particular, we see that for any $\chi \in C_0^\infty(M)$

$$R^l(z)\chi \in \mathcal{C}_1(L^2(E)), \quad \|R^l(z)\chi\|_1 \leq C z^{m-2l}.$$

To prove the asserted rapid decay of $R^l(z)\phi B_p^1(z)$ we use Lemma 2.2, plugging in the Neumann series in the definition of $B_p^1(z)$. Since supp $d\tilde{\phi}$ is disjoint from supp χ, by construction, we can apply Lemma 2.3, 2) to each term in the series. Hence, given $N \in \mathbb{N}$ we can split $B_p^1(z)$ in a term with rapid decay in $\mathcal{C}_1$ and a term with operator norm $O(z^{-N})$; thus the assertion follows from (2.10). □

Now we bring in the continuous partition of unity introduced above; an easy calculation shows that

$$\operatorname{tr}_{L^2(E)}[\phi R^l(z)] = \int_M \operatorname{tr}_{L^2(E)}[\phi \chi_{p\varepsilon} R^l(z)] \operatorname{vol}_M(p). \tag{2.11}$$

By Lemma 2.4 and the local uniformity in p we then immediately obtain the following consequence:

Corollary 2.5. *If $\tilde{\chi}_{p\varepsilon}$ denotes the function in $C_0^\infty(\mathbb{R}^m)$ induced by $\chi_{p\varepsilon}$ and the chosen coordinates, then for $\varepsilon \leq \varepsilon_0$ and all $N \in \mathbb{N}$ we have*

$$\operatorname{tr}_{L^2(E)}[\phi R^l(z)] = \int_M \operatorname{tr}_{L^2(\mathbb{R}^m,\mathbb{C}^k)}[\tilde{\phi}\tilde{\chi}_{p\varepsilon} R^l_p(z)] \operatorname{vol}_M(p) + O_N(\varepsilon^{-m} z^{-N}).$$

For the remainder estimate in Corollary 2.5 we have used the estimate

$$|\chi_{p\varepsilon}(q)| \leq C_p\, \varepsilon^{-m}, \quad \text{for all } q \in M, \quad \text{and locally uniformly in } p.$$

To derive the desired asymptotic expansion from Corollary 2.5 we want to plug in the Neumann series for $R^l_p(z)$. This series is *not* asymptotic in z but it will be asymptotic in the sense of a two-variable expansion in ε and z, i.e. if we admit remainder estimates of the form $C_N(\varepsilon + z^{-1})^N$. To substantiate this, we have to elaborate on our estimates in Lemma 2.3. Thus we choose a function $\tilde{\chi} \in C_0^\infty(B_1(0))$ and define, for $\varepsilon > 0$, $\tilde{\chi}_\varepsilon(x) := \tilde{\chi}(x/\varepsilon)$. We also consider $A_i \in C_b^\infty(\mathbb{R}^m, \operatorname{End}\mathbb{C}^k)$, the space of smooth sections with uniformly bounded derivatives, $p_i \in \mathcal{S}^{-\sigma_i}$, and multiindices $\alpha_i \in \mathbb{Z}_+^m$. With these data, we introduce the operator

$$B(\varepsilon, z) := \tilde{\chi}_\varepsilon\, A_1 x^{\alpha_1} \operatorname{Op} p_1(z) \cdots A_L x^{\alpha_L} \operatorname{Op} p_L(z) \tag{2.12}$$

Then we obtain the following result.

Lemma 2.6. *Put $\sigma := \sum_{i=1}^L \sigma_i$, $a := \sum_{i=1}^L |\alpha_i|$. Then the operator $B(\varepsilon,z)$ is in $\mathcal{C}_q(H)$ if $\sigma > m/q$, with norm estimate*

$$\|B(\varepsilon,z)\|_q \leq C \prod_{i=1}^L \sup_{x\in\mathbb{R}^m} \|A_i(x)\| (\varepsilon + z^{-1})^a \varepsilon^{-m} z^{m/q-\sigma},$$

where C is independent of ε and z. This estimate also holds for $q = \infty$.

Proof. We put

$$b := \sum_{i=1}^{L} i|\alpha_i|$$

and do the proof by induction on $b \geq a$. For $b = a$ we have $\alpha_i = 0$ if $i > 1$, so the assertion follows from Lemma 2.3. Assume next that the assertion holds if $b \leq b_0$ and consider an operator B with $b = b_0 + 1$. We assume that $|\alpha_L| > 0$ and observe the commutation relation

$$[\operatorname{Op} p_{L-1}, x_j] = \operatorname{Op} D_j p_{L-1}.$$

This splits B into two parts to which the induction hypothesis applies, and the proof is complete. □

To see that the estimate in Lemma 2.6 actually implies the asymptotic character of the Neumann series after left multiplication with $\phi\tilde{\chi}_{p\varepsilon}$, we only have to observe that the jth term in the Neumann series, $R_j(z)$ say, produces a finite sum of operators of the type just considered, with $L = j$ and $\sigma_i = 2l - b_i$, for certain numbers $0 \leq b_i \leq 2l$, such that $b_1 = 0$ and, for $i \geq 2$, $|\alpha_i| \geq 1$ if $b_i = 2l$. Hence, Lemma 2.6 gives the estimate

$$\|R\|_1 \leq C_j\, \varepsilon^{-m}(\varepsilon + z^{-1})^j.$$

It remains to recall that ε can be chosen as an arbitrary number not larger than ε_0, so the asymptotic character of the series follows if we put $\varepsilon := z^{-1}$.

By the same token, we obtain an asymptotic expansion upon using the Taylor expansion around zero for all arising coefficients in the Neumann series

Next we have to take a closer look at the structure of the expansion coefficients arising in this way. By our discussion so far, we know that the coefficients are given by operators of the form dealt with in Lemma 2.6, where all endomorphisms A_i are independent of x. In this case, the pseudodifferential calculus is exact so we get explicit formulas, at least in principle. It is well known that important geometric information can be derived from good knowledge of these coefficients. In this paper, however, we restrict attention to the existence of the expansion; the (closely connected) discussion of computation will be pursued elsewhere. For our purposes here it is thus enough to have the following description of the expansion coefficients, which follows directly from what we have done.

Lemma 2.7. *Consider the summand with i factors in the Neumann series for $\operatorname{tr}_{L^2(\mathbb{R}^m,\mathbb{C}^k)}[\tilde{\chi}_{p\varepsilon} R_p^l(z)]$. Every term arising from it by expanding the coefficients in Taylor series around 0 can be written as*

$$\tilde{\chi}_{p\varepsilon} \sum_{|\alpha|\leq a,\ 0\leq j\leq il} z^{2il-2j} x^\alpha \operatorname{Op} p_\alpha,$$

where a is the total number of $x-$powers involved, and $p_\alpha \in S^{-\sigma_\alpha}(\mathbb{C}^k)$ with $|\sigma_\alpha| \geq 2l(i+1) - 2j + a - |\alpha|$.

Moreover, each p_α is a product with i factors of powers of ξ and $\xi-$derivatives of $p_0(\xi, \tilde{z})$, where $\tilde{z} := (z^2 + (\tilde{A}_p)^2)^{1/2}$.

Finally, we observe that our expansion argument actually gives the expansion of the resolvent kernel on the diagonal.

Lemma 2.8. *If $2l > m+1$, then $R^l(z)$ has a C^1-kernel near $\operatorname{supp}\phi \times \operatorname{supp}\phi$ such that*

$$\operatorname{tr}_{E_p} R^l(p,p;z) = \sum_{0\leq j\leq L+m} a_j(p) z^{m-2l-j} + O_L(z^{-L}). \tag{2.13}$$

The functions a_j are given by universal polynomials in the derivatives of the coefficients A_p^α of the chosen trivialization, evaluated at 0.

Proof. It follows from our discussion that, uniformly in $p \in \operatorname{supp}\phi$,

$$\operatorname{tr}_{L^2(E)}[\chi_{p\varepsilon} R^l(z)] = \sum_{0\leq j\leq N+m} a_j(p) z^{m-2l-j} + O_{pN}(\varepsilon^{-m}(\varepsilon + z^{-1})^N + \varepsilon).$$

On the other hand, by elliptic regularity we see that the kernel of $R^l(z)$ is C^1 near $\operatorname{supp}\phi \times \operatorname{supp}\phi$ if $2l > m+1$, hence

$$\operatorname{tr}_{E_p} R^l(p,p;z) = \operatorname{tr}_{L^2(E)}[\chi_{p\varepsilon} R^l(z)] + O_p(\varepsilon).$$

Choosing $N := (m+1)L$ and $\varepsilon := z^{-L}$ gives the desired expansion. □

3. The case of operator coefficients

As we have already pointed out, our method focuses on the self-adjoint operators appearing naturally in the discussion. This makes it necessary to consider model operators with operator coefficients. The analytic properties we have to develop follow closely the outline of the previous section, however, so we will only point out where we have to use different arguments. To begin our approach we consider the same situation as before but assume that the operator under consideration is a second order partial differential operator, Δ, acting on the sections of a Hilbert bundle, $\mathcal{E}$, over M. Instead of describing its properties we *assume* now the existence of local trivializations which transform the situation locally isometrically to differential operators acting on Hilbert space valued functions defined on $\mathbb{R}^m$. Precisely, we assume that, on smooth sections of $\mathcal{E}|B_{\varepsilon_0}(p), \Delta$ is unitarily equivalent, under an isometry

$$\Phi_p : L^2(\mathcal{E}) \mapsto L^2(\mathbb{R}^m, H) =: \mathcal{H}$$

to

$$\Delta_p := \sum_{|\alpha|\leq 2} A_p^\alpha(x) D_x^\alpha, \tag{3.1}$$

acting on $C_0^\infty(B_{\varepsilon_0}(0), H_1)$ where H is a separable Hilbert space and H_1 is a dense subspace. Concerning the coefficients, we assume that each A_p^α is a closed operator in H with domain $\mathcal{D}(A_p^\alpha) \supset H_1$, and that there is a selfadjoint operator $\tilde{A}_p \geq 0$ with domain H_1, for all α. In addition, we assume that

$$(\tilde{A}_p)^2 = \operatorname{Re} A_p^0(0);$$
$$A_p^\alpha \text{ is scalar for } |\alpha| = 2, \text{ with } A_p^{ij}(0) = \delta_{ij}.$$

We recall the notion of *order with respect to* $\tilde{A}_p$, as introduced e.g. in [BL2, Sec. 2.A]. We then also assume that A_p^α is of order $2-|\alpha|$, for $|\alpha|>0$, and that $\operatorname{Im} A_p^0$ is of order 1, together with all derivatives.

As before, we construct a model operator, Δ_{p0}, with constant coefficients and extend Δ_p symmetrically to all of $\mathbb{R}^m$ such that it equals the model operator outside a small neighborhood of 0; the resulting operator will still be denoted by Δ_p. Then we have as before

$$\Delta_{p0} := \sum_{|\alpha|=2} A_p^\alpha(0) D_x^\alpha + \operatorname{Re} A_p^0(0) = -\sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} + \tilde{A}_p^2, \tag{3.2}$$

$$\tilde{\Delta}_{pj} := \Delta_p^j - \Delta_{p0}^j \quad \text{for } j \in \mathbb{Z}_+. \tag{3.3}$$

We assume, of course, that this construction depends again locally continuously on $p \in M$.

Now we follow the outline of the previous section step by step. First, we observe that we arrive at the apriori estimates (2.5) and (2.8) exactly as before. Next we introduce the appropriate operator valued symbol space $\mathcal{S}^{-\sigma}(\mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}_+, \mathcal{L}(H)) =: \mathcal{S}^{-\sigma}(H)$ as the set of functions in $C^\infty(\mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{0\}, \mathcal{L}(H))$ satisfying

$$p(t\xi, t\lambda, tz) = t^{-\sigma} p(\xi, \lambda, z), \quad \text{for all } t > 0. \tag{3.4}$$

Then it is clear that we can form the operator symbol $p(\xi, A, z)$, for any self-adjoint and nonnegative operator A. We denote by $\operatorname{Op} p(A, z)$ the corresponding pseudodifferential operator acting on $C_0^\infty(\mathbb{R}^m, H)$, as before. Then we obtain easily the following analogue of Lemma 2.3.

Lemma 3.1. *Let $p \in \mathcal{S}^{-\sigma}(H)$ with $\sigma \geq 0$, and assume $z \geq 1$. Assume, moreover, that $(A+I)^{-1} \in \mathcal{C}_{q_0}(H)$, for some $q_0 > 0$.*

1) $\|\operatorname{Op} p(A,z)\|_\infty \leq C z^{-\sigma}$.
2) *For $\phi, \psi \in C_0^\infty(\mathbb{R}^m)$ with $\phi < \psi$ we have*

$$\phi \operatorname{Op} p(A,z)(1-\psi) \in \mathcal{C}_1(L^2(\mathbb{R}^m, H)) \quad \text{and}$$
$$\|\phi \operatorname{Op} p(A,z)(1-\psi)\|_1 \leq C_N z^{-N},$$

for all $N \in \mathbb{N}$.

3) *If $\sigma > (m+q_0)/q$, then $\phi \operatorname{Op} p(A,z) \in \mathcal{C}_q(L^2(\mathbb{R}^m, H))$ and*

$$\|\phi \operatorname{Op} p(A,z)\|_q \leq C z^{(m+q_0)/q-\sigma}.$$

Proof. 1) is again obvious, in view of the homogeneity properties of p.
For 2) we proceed as before, using now as comparison operator $\Delta_R + A^2 + z^2$, i.e. the Friedrichs extension of this operator with domain $C_0^\infty(B_R(0), H_1)$ in $L^2(B_R(0), H)$. An easy argument shows that the inverse of this operator raised to the power l is in C_q for $2q > (m + q_0)/l$. Then we conclude the proof as in Lemma 2.3.
3) is again proved as in the scalar case, now using for comparison the symbol $p_0(\xi, \lambda, z) := (|\xi|^2 + A^2 + z^2)^{-1}$. □

With these preparations we are able to establish the analogue of Corollary 2.5 since the necessary estimates needed in the argument are now provided by Lemma 3.1.

Lemma 3.2. *With $\chi_{p\varepsilon}$ and $\tilde{\chi}_{p\varepsilon}$ as before, we have for sufficiently small ε_0 and for all $\varepsilon \leq \varepsilon_0, N \in \mathbb{N}$*

$$\operatorname{tr}_{L^2(\mathcal{E})}[\phi R^l(z)] = \int_M \operatorname{tr}_{L^2(\mathcal{E})}[\phi \tilde{\chi}_{p\varepsilon} R_p^l(z)] \operatorname{vol}_M(p) + O_N(\varepsilon^{-m} z^{-N}). \tag{3.5}$$

The desired asymptotic expansion is now, again, derived as before: in (3.5), we plug in the Neumann series for $R_p^l(z)$, and then we use on all coefficients arising in the jth term their Taylor expansion around zero. This leads to two kinds of remainder terms with similar structure, which are both estimated by the following analogue of Lemma 2.6. Thus we consider operators of type (2.12),

$$B(\tilde{A}_p, \varepsilon, z) := \tilde{\chi}_\varepsilon A_1 x^{\alpha_1} \operatorname{Op} p_1(\tilde{A}_p, z) \cdots A_L x^{\alpha_L} \operatorname{Op} p_L(\tilde{A}_p, z), \tag{3.6}$$

where $\alpha_i \in \mathbb{Z}_+^m$, $A_i \in C_b^\infty(\mathbb{R}^m, \mathcal{L}(H))$, and $p_i(\tilde{A}_p, z) \in \mathcal{S}^{-\sigma}(H)$, for $1 \leq i \leq L$. The proof of Lemma 2.6 now carries over verbatim to give the necessary estimate.

Lemma 3.3. *Put $\sigma := \sum_{i=1}^L \sigma_i$, $a := \sum_{i=1}^L |\alpha_i|$. Then the operator $B(\tilde{A}_p, \varepsilon, z)$ is in $C_q(H)$ if $\sigma > (m + q_0)/q$, with norm estimate*

$$\|B(\tilde{A}_p, \varepsilon, z)\|_q \leq C \prod_{i=1}^L \sup_{x \in \mathbb{R}^m} \|A_i(x)\| (\varepsilon + z^{-1})^a \varepsilon^{-m} z^{(m+q_0)/q - \sigma},$$

where C is independent of ε and z. This estimate also holds for $q = \infty$.

These results lead to the resolvent expansion in the operator valued case as before.

Lemma 3.4. *Consider the summand in the Neumann series for $\operatorname{tr}_{\mathcal{H}}[\tilde{\chi}_{p\varepsilon} R_p^l(z)]$ with i factors. Every term arising from it by expanding the coefficients in Taylor series around 0 can be written as*

$$\tilde{\chi}_{p\varepsilon} \sum_{|\alpha| \leq a,\ 0 \leq j \leq il} z^{2il - 2j} x^\alpha \operatorname{Op} p_{\alpha j}(\tilde{A}_p, z)$$

where a is the total number of x–powers involved, and $p_{\alpha j} \in \mathcal{S}^{-\sigma_{\alpha j}}(H)$ with $|\sigma_{\alpha j}| \geq 2l(i+1) - 2j + a - |\alpha|$.

Moreover, each $p_{\alpha j}(\tilde{A}_p, z)$ is a noncommutative product with i factors where each factor is of the form

$$D_x^\beta A_p^\alpha(0) \xi^\gamma D_\xi^\delta p_0(\xi, \tilde{A}_p, z),$$

for certain multiindices $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_+^m$.

Thus, $\operatorname{tr}_{L^2(\mathcal{E})}[\phi R^l(z)]$ admits an asymptotic expansion as $(\varepsilon + z^{-1}) \to 0$ with coefficients derived from

$$\sum_{|\alpha| \leq a,\ 0 \leq j \leq il} \int_M \operatorname{tr}_{\mathcal{H}}[\tilde{\phi} \tilde{\chi}_{p\varepsilon} z^{2il - 2j} x^\alpha \operatorname{Op} p_{\alpha j}]. \tag{3.7}$$

If $2l > m + 1$, then $R^l(z)$ has a C^1–kernel near $\operatorname{supp} \phi \times \operatorname{supp} \phi$ such that

$$\operatorname{tr}_{\mathcal{E}_p} R^l(p, p; z) = \sum_{0 \leq j \leq L+m} a_j(p) z^{m - 2l - j} + O_L(z^{-L}). \tag{3.8}$$

The functions a_j arise from the expansion (3.7) as the coefficients which are independent of ε.

4. The Singular Asymptotics Lemma

In the singular case, new tools have to be used to derive the asymptotic expansion. The main technical device, introduced already in the groundbreaking work of Jeff Cheeger [Ch], is scaling along the "cone axis", which reduces the expansion problem essentially to a regular problem with parameters. The price one has to pay, at least in the conic case, is that we can no longer restrict the analysis to large negative values of the resolvent parameter; in our terminology, we have to go down to $z = 0$ where the resolvent becomes singular. To deal with this problem and the resulting special form of the integrals to be expanded, Brüning and Seeley derived the Singular Asymptotics Lemma [BS1]. This result has several variants and generalizations; for a thorough discussion, cf. [GG]. The version in [BS1] was designed to handle the case of asymptotic cones, and the remainder estimate made crucial use of a smoothness assumption. Even though this formulation leads to very precise results for cones, it is not sufficient to deal with more general situations like metric horns (cf. [B1] and [LP] for some relevant material). Therefore, we have given in [B2] a generalization which was shown to be sufficient to handle metric horns; it will also be the basic tool for the present discussion. Thus, we quickly review some terminology and the relevant results here from [B2].

By Γ we denote a discrete closed subset of $\mathbb{C}$ with

$$\underline{\gamma} := \inf \operatorname{Re} \gamma > -\infty, \quad \text{and} \tag{4.1}$$

$$\Gamma_N := \{\gamma \in \Gamma;\ \operatorname{Re}\gamma \leq N\} \tag{4.2}$$

finite for all $N \in N$, and equipped with a "weight function" $\nu : \Gamma \to \mathbb{Z}_+$. In addition, we write $\Gamma_N^* := \Gamma_N \cap \{\operatorname{Re} z > 0\}$. Then we consider the following types of asymptotic expansions.

(i) For $f \in L^1_{\rm loc}(0,1]$, we want the expansion

$$f(x) = \sum_{\substack{\alpha\in\Gamma_N \\ k\leq\nu(\alpha)}} f^0_{\alpha k} x^\alpha \log^k x + R^0_N(f;x), \tag{4.3}$$

with certain numbers $f^0_{\alpha k} \in \mathbb{C}$ and remainder estimate

$$|R^0_N(f;x)| \leq C_{N\delta}\, x^{N-\delta}, \tag{4.4}$$

uniformly in $x \in (0,1]$, for every $\delta > 0$. This class of functions we denote by $\mathcal{F}^0_{\Gamma,\nu}$.

(ii) For $f \in L^1_{\rm loc}[1,\infty)$, we want the expansion

$$f(x) = \sum_{\substack{\beta\in\Gamma_N \\ l\leq\nu(\beta)}} f^\infty_{\beta l} x^{-\beta} \log^l x + R^\infty_N(f;x), \tag{4.5}$$

with certain numbers $f^\infty_{\beta l} \in \mathbb{C}$ and remainder estimate

$$|R^\infty_N(f;x)| \leq C_{N,\delta}\, x^{\delta-N}, \tag{4.6}$$

uniformly in $x \in [1,\infty)$, for every $\delta > 0$. This class of functions we denote by $\mathcal{F}^\infty_{\Gamma,\nu}$.

Then we put

$$\mathcal{F}_{\Gamma,\nu} := \{f \in L^1_{\rm loc}(0,\infty);\ f_0 := f|_{(0,1]} \in \mathcal{F}^0_{\Gamma,\nu},\ f_\infty := f|_{[1,\infty)} \in \mathcal{F}^\infty_{\Gamma,\nu}\}; \tag{4.7}$$

this is the function space we are interested in. Occasionally, we will refer to an expansion as in (4.3) or (4.5) as *an expansion of type* Γ, ν.

We can define the Mellin transform on $\mathcal{F}_{\Gamma,\nu}$ by

$$\mathcal{M}f(z) := \mathcal{M}f_0(z) + \mathcal{M}f_\infty(z). \tag{4.8}$$

$\mathcal{M}f$ is meromorphic in $\mathbb{C}$, hence we can define the "regularized integral" by

$$\int_0^\infty f(x)dx := \operatorname{Res}_0 \mathcal{M}f(1), \tag{4.9}$$

where "Res_0" denotes the constant term in the Laurent expansion. This is the appropriate notion for our purposes. We also employ the fairly obvious notion of "two-variable asymptotic expansion", associated with Γ and ν. By this we mean an asymptotic expansion for functions, f, in $L^1_{\rm loc}((0,1] \times (0,1])$ of the form

$$f(\varepsilon,\omega) = \sum_{\substack{\alpha,\beta\in\Gamma_N \\ k\le\nu(\alpha),\, l\le\nu(\beta)}} f^{kl}_{\alpha\beta}\, \varepsilon^\alpha \log^k \varepsilon\, \omega^\beta \log^l \omega + R_N(f;\varepsilon,\omega), \tag{4.10}$$

with certain $f^{kl}_{\alpha\beta} \in \mathbb{C}$ and remainder estimate

$$|R_N(f;\varepsilon,\omega)| \le C_{N,\delta}(\varepsilon+\omega)^{N-\delta}, \tag{4.11}$$

uniformly in $0 < \varepsilon, \omega \le 1$, for every $\delta > 0$. We will mainly use the fact that (4.10) determines the coefficients $f^{kl}_{\alpha\beta}$ uniquely. We will have to determine the asymptotic expansion of expressions of the form

$$I(\sigma;z) := \int_0^\infty \sigma(x,xz)dx, \quad 1 \le z \to \infty, \tag{4.12}$$

where σ is in $C((0,\infty)\times(0,\infty),\mathbb{C})$. To ensure the existence of the integral and its asymptotic expansion, we introduce two assumptions.

($\mathbf{A}_0$) For any $C_0 > 0$ and $0 < x \le C_0$, $\xi \in (0,\infty)$, $N \in \mathbb{N}$, we have the expansion

$$\sigma(x,\xi) = \sum_{\substack{\alpha\in\Gamma_N \\ k\le\nu(\alpha)}} \sigma^0_{\alpha k}(\xi)\, x^\alpha \log^k x + R^0_N(x,\xi),$$

where $\sigma^0_{\alpha k} \in \mathcal{F}_{\Gamma,\nu}$ and

$$|R^0_N(x,\xi)| \le C(N,\delta,C_0)\, x^{N-\delta}\, \xi^{\alpha^0_0}(1+\xi)^{\alpha^0_\infty}$$

for $0 < \delta \le 1$ and certain numbers $\alpha^0_0 \le 0$, $\alpha^0_\infty \ge 0$.

($\mathbf{A}_\infty$) For any $C_0 > 0$ and $\xi \ge 1/C_0$, $x \in (0,\infty)$, $N \in \mathbb{N}$ we have the expansion

$$\sigma(x,\xi) = \sum_{\substack{\beta\in\Gamma_N \\ l\le\nu(\beta)}} \sigma^\infty_{\beta l}(x)\, \xi^{-\beta} \log^l \xi + R^\infty_N(x,\xi),$$

where $\sigma^\infty_{\beta l} \in \mathcal{F}_{\Gamma,\nu}$ and

$$|R^\infty_N(x,\xi)| \le C(N,\delta,C_0)\, \xi^{\delta-N}\, x^{\alpha^\infty_0}(1+x)^{\alpha^\infty_\infty}$$

for $0 < \delta \le 1$ and certain numbers $\alpha^\infty_0 \le 0$, $\alpha^\infty_\infty \ge 0$,

We point out the following consequence of the axioms (A_0) and (A_∞): as $x \to 0$, $\xi \to \infty$ we obtain the expansion, from (A_0),

$$\sigma(x,\xi) \sim \sum_{\substack{\alpha,\beta\in\Gamma \\ k\le\nu(\alpha),\, l\le\nu(\beta)}} [\sigma^0_{\alpha k}]^\infty_{\beta l}\, x^\alpha \log^k x\, \xi^{-\beta} \log^l \xi,$$

and from (A_∞),

$$\sigma(x,\xi) \sim \sum_{\substack{\alpha,\beta\in\Gamma \\ k\le\nu(\alpha),\, l\le\nu(\beta)}} [\sigma^\infty_{\beta l}]^0_{\alpha k}\, x^\alpha \log^k x\, \xi^{-\beta} \log^l \xi.$$

By uniqueness of the expansion coefficients, we conclude

$$[\sigma^0_{\alpha k}]^\infty_{\beta l} = [\sigma^\infty_{\beta l}]^0_{\alpha k} =: \sigma_{\alpha k,\beta l}. \tag{4.13}$$

Now we can formulate the main result of this section; the proof is given in [B2].

Theorem 4.1. *Let $\sigma \in C((0,\infty)\times(0,\infty),\mathbb{C})$. Under the assumptions* (A_0) *and* (A_∞) *we have the following asymptotic expansion, as $1 \le z \to \infty$:*

$$\begin{aligned} I(\sigma;z) \sim & \sum_{\substack{\alpha\in\Gamma \\ k\le\nu(\alpha)}} \int_0^\infty (x/z)^\alpha \log^k(x/z)\,\sigma^0_{\alpha k}(x)dx/z \\ & + \sum_{\substack{\beta\in\Gamma \\ l\le\nu(\beta)}} \int_0^\infty (xz)^{-\beta}\log^l(xz)\,\sigma^\infty_{\beta l}(x)dx \\ & + \sum_{\substack{\alpha\in\Gamma \\ k\le\nu(\alpha) \\ 0\le k''\le k,\ 0\le k'\le\nu(\alpha+1)}} z^{-\alpha-1}\log^{k+k'+1} z \binom{k}{k''}(-1)^{k-k'+1}(k+k'+1)^{-1}[\sigma^0_{\alpha k}]^0_{-\alpha-1,k'} \\ & + \sum_{\substack{\beta\in\Gamma \\ k\le\nu(\beta+1),\ l\le\nu(\beta) \\ 0\le k'\le k}} z^{-\beta}\log^{k+l+1} z \binom{k}{k'}(-1)^{k-k'}(k'+l+1)^{-1}\sigma_{\beta l;\beta-1,k}. \end{aligned}$$

In particular, $I(\sigma;z)\in\mathcal{F}^\infty_{\Gamma,\tilde\nu}$ with $\tilde\nu(\alpha)=\nu(\alpha)+\nu(\alpha+1)+1$.

5. Isolated singularities

We now use the analysis described in the previous sections to develop an approach to the resolvent expansion of Laplace type operators on certain isolated singularities. Our discussion will proceed axiomatically; it remains to be seen which specific examples, beyond metric horns, are actually covered by our method.

¿From now on we study a Riemannian manifold, M, which decomposes as

$$M =: M_1 \cup U.$$

Here, M_1 is a compact manifold with boundary $N := \partial M_1 = \partial U$, and

$$U = (0,\varepsilon_0)\times N.$$

It will be seen that our method also allows us to treat more general compact metric spaces than manifolds, provided we know enough about their spectral theory.

The "singularity" U comes with a distinguished coordinate $x\in(0,\varepsilon_0)$. We build our model operator according to the simplest possible separation of variables expressed through the form of the Laplacian: we assume that Δ, considered as an operator in $L^2(E|U)$ with domain $C_0^\infty(E|U)$, is unitarily equivalent to

$$-\partial^2/\partial x^2 + A(x), \tag{5.1}$$

acting in $L^2((0,\varepsilon_0),H)$ with domain $C_0^\infty((0,\varepsilon_0),H_1)$, where H_1 is the common domain of the family $A(x)_{x\in(0,\varepsilon_0)}$. We write $A_0 := A(\varepsilon_0/2)$. We remark that (5.1) is, at least for the geometric operators, a consequence of x being a normal coordinate in the sense that the metric, g, on U is given by

$$g = dx^2 + g_N(x), \tag{5.2}$$

with a smooth family $g_N(x)$ of metrics on N.

Our assumptions will prescribe functional analytic properties of the operators $A(x)$. The first assumption concerns smoothness and self-adjointness.

(O1) *The function*

$$(0,\varepsilon_0)\ni x\mapsto A(x)\in\mathcal{L}(H_1,H)$$

is smooth and all $A(x)$ are self-adjoint with domain H_1. Moreover, the operators $A^{(j)}(x)$ are of order 1 with respect to A_0, for all $j\in\mathbb{Z}_+$.

Next we assume that the operator coefficient, A, is extended to all of $(0,\infty)$ in such a way that (O1) remains valid and in addition

$$A(x) = A_0 \quad \text{for } x\ge 2\varepsilon_0. \tag{5.3}$$

Then the Friedrichs extension of the model operator exists in $\mathcal{H} := L^2(\mathbb{R}_+,H)$; it will be denoted by T.

To handle the necessary trace class estimates, we introduce a second axiom.

(O2) *For some positive constants C and α we have*

$$A(x) \geq C\,x^{-2-2\alpha} A_0.$$

Moreover, $A_0 \geq C$, and there is $q_0 > 0$ such that $A_0^{-1} \in \mathcal{C}_{q_0/2}(H)$.

Now we can derive the trace class properties of T.

Lemma 5.1. *$T \geq 0$, and if $2l > 1 + q_0$ then for any $\phi \in C_0^\infty(\mathbb{R})$ and $z > 0$ we have*

$$\phi(T + z^2)^{-2l} \in \mathcal{C}_1(\mathcal{H}).$$

Proof. The first assertion is clear from (O2). The second assertion is proved along the lines used in Lemma 3.1, Part 2. Note first the estimate

$$\|\partial_x u\| \leq (Tu, u), \tag{5.4}$$

valid for all $u \in C_0^2((0,\infty), H_1)$. Next we pick $R > 0$ such that $\operatorname{supp}\phi \subset [-R, R]$ and denote by T_R the Friedrichs extension of the operator (4.1) in $L^2((0,R),H)$. Then we compute

$$(T_R + z^2)\phi(T + z^2)^{-1} = \phi - (\phi'' + 2\phi'\partial_x)(T + z^2)^{-1}$$

Now (O2) allows to estimate the eigenvalues of T_R from below, by the max–min principle, and we conclude that $\phi(T+z^2)^{-1} \in \mathcal{C}_{(1+q_0)/2}$. This allows us to apply [B3, Lemma 4.1] and the assertion follows by induction on l. □

Now we observe that $(T + z^2)^{-l}$ has an operator kernel, by the Trace Lemma of [BS2], to be denoted by

$$\tilde{R}^l(x, y; z) \in \mathcal{L}(H), \quad x, y, z > 0.$$

Since $T \geq 0$, this kernel is nonnegative on the diagonal.

Next we bring in the unitary scaling map in $\mathcal{H}$, defined by

$$U_\delta f(x) := \delta^{1/2} f(\delta x), \;\; x, \delta > 0. \tag{5.5}$$

Then we obtain

$$\begin{aligned} T_\delta &:= \delta^2 U_\delta T U_\delta^* \\ &= -\partial_x^2 - \delta^2 A(\delta x) \\ &=: -\partial_x^2 - A_\delta(x). \end{aligned}$$

If we denote the kernel of $(T_\delta + z^2)^{-l}$ by $\tilde{R}_\delta^l$ then we deduce the identity

$$\tilde{R}^l(x, y; z) = \delta^{2l-1} \tilde{R}_\delta^l(x/\delta, y/\delta; \delta z). \tag{5.6}$$

With these preparations we can prove the trace formula which gives the basis for the application of the Singular Asymptotics Lemma stated in the previous section.

Lemma 5.2. *Choose $\phi \in C_0^\infty(-\varepsilon_0, \varepsilon_0)$ and $\psi \in C_0^\infty(1/2, 1)$ with*

$$\int_0^\infty \psi(x)dx = 1.$$

Then, with $\psi_x(y) := \phi(xy)\psi(y)y$ we have for $2l > 1 + q_0$ and all $N \in \mathbb{N}$

$$\operatorname{tr}_{L^2(E)}[\phi R^l(z)] = \int_0^\infty x^{2l-1} \operatorname{tr}_{\mathcal{H}}[\psi_x \tilde{R}_x^l(xz)]dx + C_N z^{-N}.$$

Proof. Using again [B3, Lemma 4.1], we see that

$$\operatorname{tr}_{L^2(E)}[\phi R^l(z)] = \operatorname{tr}_{L^2(E)}[\phi \tilde{R}^l(z)] + C_N z^{-N},$$

for all $N \in \mathbb{N}$. Then we calculate with the Trace Lemma and (5.6)

$$\begin{aligned}\operatorname{tr}_{L^2(E)}[\phi\tilde{R}^l(z)] &= \int_0^\infty\int_0^\infty \phi(x)\psi(y)\operatorname{tr}_H[\tilde{R}^l(x,x;z)]dydx\\ &= \int_0^\infty\int_0^\infty \phi(x)\psi(y)(x/y)^{2l-1}\operatorname{tr}_H[\tilde{R}^l_{x/y}(y,y;(x/y)z)]dxdy\\ &= \int_0^\infty\int_0^\infty \phi(xy)\psi(y)yx^{2l-1}\operatorname{tr}_H[\tilde{R}^l_x(y,y;xz)]dydx\\ &= \int_0^\infty x^{2l-1}\operatorname{tr}_{\mathcal{H}}[\psi_x\tilde{R}^l_x(xz)]dydx.\end{aligned}$$

This proves the lemma. □

With Lemma 5.2 we achieve the complete reduction of the expansion problem to the case treated in Section 2, but we have now also to control a small parameter; the structure of the trace formula also shows why it is necessary to apply something like the Singular Asymptotics Lemma. Hence Lemma 5.2 will be our basis for the resolvent expansion but in view of the very abstract setting, we have to introduce two more axioms. Before doing so, we pause to look at an instructive example, namely the case of metric horns mentioned before. By definition, in this case N is compact Riemannian and the metric is given by (5.2) with

$$g_N(x) = x^{2+2\alpha}g_N(\varepsilon_0/2),$$

with some $\alpha \geq 0$. For $\alpha = 0$ this is the conic case which has been studied intensively but, in view of (O2), *this case is excluded from the present discussion.* Indeed, it will become apparent that from the point of view of our analysis, the conic case is the most difficult.

For $\alpha > 0$, Axioms (O1) and (O2) are satisfied for the geometric operators with the possible exception of positivity; we will return to this problem in the next section. Typically (cf. [B1]), we will encounter the structure

$$A(x) = x^{-2-2\alpha}(A_1 + x^\alpha A_2 + x^{2\alpha}A_3), \tag{5.7}$$

which exhibits actually a complete asymptotic expansion of the operator function $A(x)$. This assumption was the basis of [B1] but it is considerably stronger than our present axioms; we have reason to believe that already for very simple algebraic singularities, no such expansion will exist.

Resuming our discussion of the asymptotic expansion of $\operatorname{tr}_{L^2(E)}[\phi R^l(z)]$, we state now the additional assumptions on A which we will need. Clearly, we need bounds on the derivatives, which take the following form.

(O3) *For $k \in \mathbb{N}$, $j_1,\dots,j_k \in \mathbb{Z}_+$, we have*

$$\|A^{(j_1)}(x)\cdots A^{(j_k)}(x)A(x)^{-k}\| \leq C_{j_1,\dots,j_k}x^{-k}.$$

Our final axiom concerns the existence of asymptotic expansions connected with the operator family $A(x)$, as dictated by Lemma 3.4. To formulate it, we denote by $J = (j_1,\dots,j_k) \in \mathbb{Z}_+^k$ and $L = (l_1,\dots,l_k) \in \mathbb{Z}_+^k$ multiindices and abbreviate

$$A_{(J,L)}(x,z) := A^{(j_1)}(x)(A(x)+z^2)^{-l_1}\cdots A^{(j_k)}(x)(A(x)+z^2)^{-l_k}.$$

(O4) *The operator $A_{JL}(x,z)$ is in $\mathcal{C}_1(H)$ for $\sum_{i=1}^N(2l_i-1) > q_0/2$, and its trace admits an asymptotic expansion as $z\to\infty$ of type Γ,ν, such that all expansion coefficients admit asymptotic expansions in x as $x \to 0$ of the same type.*

These assumptions are sufficient to obtain the following expansion result.

Theorem 5.3. *Assume that $2l > 1 + q_0$. Then $\operatorname{tr}_{L^2(E)}[\phi\tilde{R}^l(z)]$ admits a complete asymptotic expansion of type Γ,ν', for some weight $\nu' \geq \nu$.*

Proof. We want to apply Theorem 4.1 to the trace formula of Lemma 5.2. Since ψ_x admits a smooth Taylor expansion around $x = 0$, it is enough to check the assumptions of the theorem for $\operatorname{tr}_{\mathcal{H}}[\chi\tilde{R}^l_x(xz)]$, with any $\chi \in C_0^\infty(-\varepsilon_0, \varepsilon_0)$. To do so we rewrite

$$\begin{aligned} T_x + z^2 &= -\partial_y^2 + x^2 A(xy) + z^2 \\ &= -\partial_y^2 + x^2(A(xy) - C^2 x^{-2-2\alpha}) + x^{-2\alpha}(C^2 + (xz^{1/(1+\alpha)})^{2+2\alpha}) \\ &=: -\partial_y^2 + \tilde{A}_x(y) + x^{-2\alpha}(C^2 + (xw)^{2+2\alpha}). \end{aligned}$$

By (O2), we have $\tilde{A}_x(y) \geq 0$, and the resolvent parameter has been transformed to

$$\lambda = x^{-2\alpha}(C^2 + (xw)^{2+2\alpha}).$$

Thus, Theorem 4.1 can be applied with respect to w, and the λ–asymptotics of the above operator valued elliptic equation promise to produce remainder terms which are simultaneously small in x and w. Turning to Lemma 3.1 we see that this is, indeed, the case if we choose the free parameter as $\varepsilon = x^{-2\alpha}(C^2 + (xw)^{2+2\alpha})$, since the contributions of the derivatives are uniformly bounded, in view of (O3). This is also true, by the same token, for the remainder terms arising from taking Taylor expansions of all coefficients. Next, the remaining terms are expanded in λ using (O4), producing again terms which are small in x and w. Finally, again from (O4), we know that the resulting terms admit asymptotic expansions in x. All expansions are easily seen to be of type Γ, ν' with some weight $\nu' \geq \nu$.

By the uniformity of all remainder estimates, the assumptions of Theorem 4.1 are satisfied and the theorem follows. □

Let us emphasize that our axioms (O1) through (O4) simplify considerably if we assume a full asymptotic expansion of the operator function $A(x)$ as $x \to 0$, as assumed in [B1].

Having established the existence of the asymptotic expansion, we can investigate the structure of the expansion coefficients and attempt their explicit computation. As in the smooth case, we restrict our attention here to the most basic questions, in particular, we are interested in the separation of the coefficients into "regular" and "singular" terms. In view of Lemma 2.8 we have

$$\operatorname{tr}_{L(E)}[\phi R^l(z)] \sim \sum_{j \geq 0} \int_M \phi a_j, \tag{5.8}$$

for any $\phi \in C_0^\infty(M)$. Hence we may speculate that the expansion for $\phi = 1$ contains suitably regularized integrals of the a_j over M plus contributions of the singularities. The most natural regularization imitates the regularized integral introduced in Section 3: we let for $\varepsilon \leq \varepsilon_0$

$$M_\varepsilon := M_1 \cup (\varepsilon, \varepsilon_0) \times N,$$

and assume that, for all j,

$$\int_{M_\varepsilon} a_j \quad \text{admits an asymptotic expansion as } \varepsilon \to 0.$$

Then we can define, using "Res_0" again to denote the constant term in this asymptotic expansion,

$$⨍_M a_j := \mathrm{Res}_0 \int_{M_\varepsilon} a_j. \tag{5.9}$$

Next we introduce the space $C_c^\infty(M)$ of complex valued functions on M which are smooth on M and constant near the singularity; this space inherits a natural topology from $C^\infty(M)$. It is clear from what we have said that the expansion coefficients of $\operatorname{tr}_{L^2(E)}[\phi R^l(z)]$ define continuous linear functionals on $C_c^\infty(M)$ which, of course, would also be true of

$$F_j(\phi) := ⨍_M \phi a_j,$$

if we show that it is well defined. With this terminology we can formulate the following result.

Theorem 5.4.

1) *The distributions F_j are well defined.*
2) *The coefficient of z^{m-2l-j} in the expansion of $\operatorname{tr}_{L(E)}[\phi R^l(z)]$ as $z \to \infty$ can be written as a sum of F_j and a continuous linear functional on $C_c^\infty(M)$ which vanishes on $C_0^\infty(M)$. All other expansion coefficients vanish on $C_0^\infty(M)$.*

Proof. We choose variables $0 \leq \varepsilon \leq 1 \leq w$ and put $z := w/\varepsilon \geq 1$. We also choose $\phi \in C_0^\infty(-\varepsilon_0, \varepsilon_0)$ with $\phi = 1$ near 0, and put $\phi_\varepsilon(x) := \phi(x/\varepsilon)$. Then we write for $\psi \in C_c^\infty(M)$

$$\begin{aligned} \operatorname{tr}_{L^2(E)}[\psi R^l(z)] =: J(\psi, z) &= J(\psi, w/\varepsilon) \\ &= J(\phi_\varepsilon \psi, w/\varepsilon) + J((1-\phi_\varepsilon)\psi, w/\varepsilon) \\ &=: J'(\psi, w, \varepsilon) + J''(\psi, w, \varepsilon). \end{aligned}$$

It follows from the proof of Theorem 5.3 that $J'(\psi, w, \varepsilon)$ has an asymptotic expansion of type Γ, ν as $(\varepsilon + z^{-1}) \to 0$.

On the other hand, by Lemma 2.8 we can write for $N \in \mathbb{N}$:

$$J''(\psi, w, \varepsilon) =: \sum_{j=0}^{N} z^{m-2l-j} F_j((1-\phi_\varepsilon)\psi) + R_N(\psi, \varepsilon, z). \tag{5.10}$$

Let us assume for a moment that we have with some $\delta > 0$ the remainder estimate

$$|R_N(\psi, \varepsilon, z)| \leq C_{N,\psi}(\varepsilon + w^{-1})^{m-2l-j-\delta}.$$

Then, coupling the expansion variables by $w = \varepsilon^{-L}$ with L large, it is easily proved by induction on j that $F_j((1-\phi_\varepsilon)\psi)$ admits an asymptotic expansion as $\varepsilon \to 0$ proving 1). The second assertion is then obvious.

Hence it remains to prove (5.10). It is clearly enough to treat the case where $\psi \in C_0^\infty(-\varepsilon_0, \varepsilon_0)$. Then we can rewrite $J''(\psi, w, \varepsilon)$ using Lemma 5.2 to obtain an expansion in z, in terms of the local expansion coefficients of $\tilde{R}^l_x$, to be integrated over $[\varepsilon, \varepsilon_0]$. By the arguments in the proof of Theorem 5.3, the remainder term of order N can be estimated by $O_N(\int_\varepsilon^{\varepsilon_0} x^{2l-1}(xz)^{-N-\delta} dx) = O_N(w^{-N-\delta})$. Since the expansion coefficients are unique, (5.10) is proved. □

6. The general case

We now outline the treatment of the general case. This section should be considered as a research program rather than a summary of results but this does not seem entirely inappropriate, given the intentions of the present collection of articles.

First of all, it must be emphasized that the discussion we have presented needs to be substantiated by *identifying classes of singularities* which satisfy the axioms of the previous section. For the time being, we are confident that this class is rather large among all algebraic singularities so we propose to name it the class of *hyperconical singularities.* It is envisageable that their analysis, together with the rather well known conic case, will lead to a much better understanding of the spectral theory of algebraic varieties.

We turn next to the question of *essential self-adjointness* already discussed in Section 1. It seems likely that our axioms do actually imply that Δ with domain $C_0^\infty(E)$ is essentially self-adjoint in $L^2(E)$. On the other hand, simple examples show that the assumption of positivity for $A(x)$ is not very realistic at least for the Laplacian on forms. Our approach, however, lends itself to a fairly straightforward generalization by splitting the (discrete) operator coefficient into a "low eigenvalue part" and a "large eigenvalue part", such that the latter satisfies our assumption and the former is amenable to essentially known asymptotic analysis of ordinary differential equations. A similar approach has been taken in [BS2],[BS3]; we will deal with this question in a forthcoming publication.

Very little has been said here about the precise structure of the coefficients. In the smooth case, a lot of effort has been given to analyze it, building the basis for what is now called *spectral geometry* and a cornerstone of *inverse spectral theory.* In spite of all this work, no satisfying structure has evolved yet, except for the special case of the Local Index Theorem. A recent preprint of Polterovich ([P]) as well as unpublished work of Weingart and Frey indicate, however, that much more can be said about the expansion coefficients in general.

In the singular case, we should expect that the coefficients supported in the singular set contain very interesting *information on the nature of the singularities.* For example, we may hope to extract the most important numerical invariants. That this is a realistic hypothesis has been fully confirmed in the case of algebraic curves by the author and M. Lesch ([BL1]); in this case it is possible to distinguish smooths curves from those with singularities and, quite often, to determine all multiplicities.

A final question to be asked in the case of isolated singularities is whether or not *the model operator* (5.1) is really the most natural one. Most examples beyond the simple case of metric cones or horns lead to metrics with off-diagonal terms and hence to differential operators involving first order derivatives. Their analysis is considerably more complicated so it is important to clarify this point.

Let us now turn to the case of *non-isolated singularities* i.e. the case of strata with positive dimension. For a single stratum with conical fibers, the analysis described above has been developed and applied in [BS3]. This will serve us as a model for the general case which can be developed, under suitable positivity assumptions, quite analogously to the present discussion, relying heavily on the operator valued approach we have designed. In particular, we can admit in our axioms also operator coefficients which are defined as self-adjoint extensions of geometric operators on singular spaces i.e. the links of the given stratification; the necessary properties are then guaranteed by induction. The new feature emerging here is the need for more elaborate apriori estimates as derived in loc. cit. for the conic case; but in the required generality, they are not yet available.

The question of general self-adjoint extensions, however, becomes much more complicated in this case since the deficiency indices are infinite. One cannot hope for a complete analysis as in the previous case; one should perhaps concentrate on finding a description of the "regular" extensions in analogy to [BL2, Theorem 5.6] and extend the expansion result to this class.

Finally, we have to address the case of a general stratified space with a suitable metric. Knowing the case of a single singular stratum opens the way to an obvious inductive argument. Thus, we can produce an existence proof for the resolvent expansion along the lines of [BS3]. A more detailed knowledge of the structure of the coefficients, extending Theorem 5.4, is necessary, however, to derive the geometric index theorems. This requires some new combinatorial and analytic tools which seem, however, available.

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