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ON THE SPECTRAL THEORY OF MANIFOLDS WITH CUSPS

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ABSTRACT. – We are interested in the spectral properties of Dirac operators on Riemannian manifolds with cuspidal ends. We derive estimates for the essential spectrum and get formulas for the index in the Fredholm case. © 2001 Éditions scientifiques et médicales Elsevier SAS

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Introduction

Let M be a complete oriented Riemannian manifold. If M is closed and D is a formally self-adjoint elliptic operator which acts on the smooth sections of a Hermitian bundle E over M, then the essential spectrum of D is empty. In particular, D is a Fredholm operator. If M is noncompact, then the essential spectrum of D depends heavily on the coefficients of D and the end structure of M. In this work we are interested in the spectral properties of generalized Dirac operators in the sense of Gromov and Lawson, see [24]. We study the case when the ends of M are *cuspidal*, see Section 3 for the precise definition. In particular, we assume that each end U of M is of finite volume and diffeomorphic to a product $(0,\infty)\times N$, where $N = N_U$ is a closed manifold, and that the metric on U is of the form:

$$g = dt^2 + g_t, \tag{1}$$

where g_t is a family of metrics on N. The most important examples are complete Riemannian manifolds of finite volume and pinched negative curvature.

Let E be a graded Dirac bundle over M; that is, E is a bundle of left modules over the Clifford bundle $\mathbb{C}\mathrm{l}\,M$ of M with compatible Hermitian metric and connection together with a parallel unitary involution α which anticommutes with Clifford multiplication by vector fields. These data determine a Dirac operator D and a decomposition $E = E^+ \oplus E^-$ into the eigenspaces of α with eigenvalue ± 1. The main examples are the Clifford bundle and spinor bundles.

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In what follows, M is a noncompact manifold with finitely many cuspidal ends U, such that along each end U, the sectional curvature K of M is pinched between two negative constants,

$$-b^2 \leqslant K \leqslant -a^2, \tag{2}$$

where $0 < a < b$. It will be convenient to set $\dim M = n+1$.

THEOREM A. – *Let F be a flat Hermitian vector bundle over M and $D = d + d^*$ the Dirac operator on $\Lambda^* M \otimes F \cong \mathbb{C}\mathrm{l}\, M \otimes F$. Suppose that $n+1$ is even and that $(n+1)a - (n-1)b > 0$. Then D is a Fredholm operator.*

Note that D is a Fredholm operator if and only if the *twisted Hodge Laplacian* $\Delta = D^2$ is a Fredholm operator. Since F is flat, Δ preserves the degree of forms, and hence induces an operator Δ_k on k-forms. One may wonder whether Theorem A can be improved for Δ_k.

THEOREM B. – *Let F be a flat Hermitian vector bundle over M and Δ the twisted Hodge Laplacian on $\Lambda^* M \otimes F \cong \mathbb{C}\mathrm{l}\, M \otimes F$. Let $k < n/2$ and suppose that $(n-k)a - kb > 0$. Then Δ_k and Δ_{n+1-k} are Fredholm operators.*

The cases $k = (n+1)/2$ if $n+1$ is even and $k = n/2$ or $k = n/2 + 1$ if $n+1$ is odd are not covered. In fact, in the latter cases the essential spectrum of Δ on $\Lambda^k M$ is equal to $[0, \infty)$ if M is a noncompact quotient of finite volume of the real hyperbolic space H^{n+1}, see Example 1.15.

In Examples 4.2 and 5.5 we show that Theorems A and B are optimal with respect to the pinching conditions.

In the special case $k = 0$, Theorem B gives the result of McKean in [28]. Theorems A and B improve the corresponding results of Donnelly and Xavier in [15].

Except for the case of constant sectional curvature, the pinching condition of Theorem A is violated for locally symmetric spaces of finite volume and negative curvature. In this case, however, Borel and Casselman [9] have shown that D is a Fredholm operator if the dimension of M is even.

Suppose now that $E = E^+ \oplus E^-$ is a grading defined by an involution α as above. Then D maps sections of E^+ to sections of E^-. This restricted operator will be denoted by D^+. Note that D^+ is Fredholm if and only if D is. Our methods also allow to compute the index of D^+.

THEOREM C. – *Suppose that $n+1 = 2l$ is even and that $(n+1)a - (n-1)b > 0$. Let $\Lambda^{\pm} M \otimes F$ be the even–odd decomposition of $\Lambda^* M \otimes F$. Then*

$$\operatorname{ind} D^+ = \int_M \omega_{GB} + \sum_{U, i<l} (-1)^i b_i(N_U; F),$$

where ω_{GB} is the Gauss–Bonnet integrand and where, for each end U, $b_i(N_U; F)$ is the i-th Betti number of N_U with respect to the coefficient bundle F.

Suppose next that $\dim M = n+1$ is divisible by 4 and consider the splitting of $\mathbb{C}\mathrm{l}\, M \otimes F \cong \Lambda^* M \otimes F$ defined by multiplication with the complex volume element τ_M. Recall that when M is closed and $\Lambda^* M$ is untwisted, the index of D^+ is equal to the signature of M. For an end $U = (0, \infty) \times N_U$ we write $N_{U,t} := \{t\} \times N_U$.

THEOREM D. – *Suppose that $n+1$ is divisible by 4 and that $(n+1)a - (n-1)b > 0$. Let $\Lambda^{\pm} M \otimes F$ be the splitting of $\Lambda^* M \otimes F$ defined by multiplication with τ_M. Then*

$$\operatorname{ind} D^+ = \int_M \omega_L + \frac{1}{2} \sum_U \lim_{t \to \infty} \eta(A_t^+),$$

where ω_L is the integrand corresponding to the Hirzebruch L-form and where, for each end U, $A_t^+ = \tau_{N_{U,t}}(d_t + d_t^*)$ is the odd signature operator on $\Lambda^*(N_{U,t}) \otimes F$.

John Lott pointed out to us that it follows from the work of Cheeger and Gromov that $\lim_{t\to\infty} \eta(A_t^+)$ is a topological invariant of N_U, see [14, Theorem 4.1], and that this invariant has been studied – among others – by Meyerhoff, Ouyang and Rong, see [29,30].

We can also treat the Dirac operator on spinor bundles.

THEOREM E. – *Suppose that M is spin and that the spin structure of M is non-trivial along all ends of M. Let E be a spinor bundle over M. Then* $\operatorname{spec}_e D = \emptyset$ *and, in particular, D is a Fredholm operator.*

Suppose, furthermore, that $n+1$ is even and let $E^{\pm}$ be the splitting of E defined by multiplication with the complex volume form τ_M. Then

$$\operatorname{ind} D^+ = \int_M \omega_{AS} + \frac{1}{2}\sum_U \lim_{t\to\infty} \eta(A_t^+),$$

where ω_{AS} is the $\hat{A}$-form and where, for each end U, A_t^+ is the Dirac operator on the induced spinor bundle over $N_{U,t}$.

Our results fit into an obvious general scheme to describe the fundamental spectral properties of geometric differential operators on complete Riemannian manifolds: describe the essential spectrum as precisely as possible and compute the index, as the most accessible invariant of the situation, in the Fredholm case. In this work, we develop an axiomatic approach to fulfil this task which seems very promising in the case of manifolds with cuspidal ends. The axioms are inspired by [6] but contain some new ingredients, notably the construction of a flat connection $\bar{\nabla}$ of finite holonomy on the cuspidal ends which allows us to control the kernel of a crucial operator arising from separation of variables. The first two axioms together allow us to construct a first order ordinary differential operator on the half line with the same essential spectrum as D; this is related to the work of Lott ([26,27]). Our third axiom gives a sufficient condition for D to be Fredholm and allows us to compute the index of D^+. We check the axioms only in the case of twisted Clifford bundles, where the result is sharp in terms of the pinching constants, and for spinor bundles. Note that it follows from the work of Borel and Casselman in [9] that our axioms also hold in the case of twisted Clifford bundles over locally symmetric spaces of finite volume and negative curvature, see Remark 4.9 in the text.

For other relevant results related to the questions discussed in this paper see for example [4,5, 10,15,31], and [13].

The proof of Theorem B is contained in Section 5. It is quite elementary and follows the line of argument for the corresponding result in [15]. This section is more or less independent of the rest of the paper.

In Section 1 we discuss the general setup. In Section 2 we generalize the approach from [6]. This section contains the proofs of Theorems C, D, and E. In Section 3 we discuss the geometric structure of cusps and define and discuss the flat connection $\bar{\nabla}$. We note that $\bar{\nabla}$ is different from the connection constructed by Kanai in [21], although some of the problems we face in the discussion are similar. In Section 4 we derive estimates in the case of the Clifford bundle and prove Theorem A.

1. The setup

In this section we set the stage for our discussion. Let M be a complete Riemannian manifold of dimension m. We assume that M decomposes as:

$$M = M_0 \cup \{\text{ends}\}, \tag{1.1}$$

where M_0 is a compact manifold with boundary and each end, U, is diffeomorphic to a product, $U = (0, \infty) \times N$, where N is compact and connected and of dimension $n = m - 1$. Moreover, we assume that the metric on U has the form:

$$g = dt^2 + g_t, \tag{1.2}$$

where g_t and $\partial_t g_t$ are C^1 on U. In our discussion of cuspidal ends in Section 3 we will be more specific concerning the regularity of g.

Let $U = (0, \infty) \times N$ be an end of M. We let $T = \partial/\partial_t$ be the unit vector field in the t-direction. We use the prime $'$ to denote covariant differentiation in the direction of T.

The projection $U \to (0, \infty)$ onto the first coordinate is a Riemannian submersion. The fiber over t is the *cross section* $N_t := \{t\} \times N$. For each point $p = (\tau, x) \in U$, the curve $\gamma_p(t) = (t, x)$, $t > 0$, is a geodesic ray and $\gamma_p(\tau) = p$. The family of such rays is perpendicular to the family of cross sections N_t.

We denote by $S = S^t$ the second fundamental form and by $W = W^t$ the *Weingarten map* of N_t with respect to the normal vector field T,

$$WX = -\nabla_X T, \qquad S(X, Y) = \langle \nabla_X Y, T\rangle = \langle WX, Y\rangle, \tag{1.3}$$

where X and Y are vector fields tangent to N_t. The eigenvalues $\kappa_1, \ldots, \kappa_n$ of W^t are the *principal curvatures* of N_t. We suppose that there is a uniform bound

$$|\kappa_i| \leqslant b, \tag{1.4}$$

where $b \geqslant 0$ is a constant independent of t and U. This is equivalent to the pointwise bounds $\|S\| \leqslant b$ or $\|W\| \leqslant b$ of the operator norms of S or W. We let:

$$\kappa = \kappa_1 + \cdots + \kappa_n = \operatorname{tr} W. \tag{1.5}$$

Frequently, we will consider the shift map:

$$f_{t,\tau} : N_\tau \to N_t; \quad (\tau, x) \mapsto (t, x). \tag{1.6}$$

The Jacobian of $f_{t,\tau}$ is denoted by $j_{t,\tau}$. It satisfies the ordinary differential equation:

$$j'_{t,\tau} = -\kappa\, j_{t,\tau}. \tag{1.7}$$

This differential equation is the reason why we use κ instead of the mean curvature κ/n. This concludes our setup as far as the structure of the ends of M is concerned.

We now turn to the Dirac bundle. We say that a Dirac bundle E over a Riemannian manifold V is *geometric* if the pull back of E to the universal covering space $\tilde{V}$ of V is (isomorphic to) a Dirac bundle associated to the principal bundle $\mathrm{SO}(\tilde{V})$ of oriented orthonormal frames via a unitary representation of SO(m) or to a spin structure Spin$(\tilde{V})$ via a unitary representation of Spin(m), $m = \dim V$. Then the local formulas for induced connections and their curvature are available.

Let U be an end of M and $E \to U$ be a geometric Dirac bundle. Note that the restriction $E_t := E|N_t$ of E to a cross section N_t is a geometric Dirac bundle. Hence the Levi-Civita connection ∇^t of N_t induces a Hermitian connection on E_t which we also denote ∇^t. It will be convenient to denote the difference between the induced connections ∇ and ∇^t by S^t,

$$S^t := \nabla - \nabla^t, \tag{1.8}$$

since it is equal to the operator on E_t induced by the second fundamental form of N_t. The actual formula for S^t depends very much on the representation defining E. In any case, S^t is tensorial and, by (1.4), pointwise uniformly bounded independent of t.

Corresponding to the decomposition of ∇ in (1.8), we obtain a decomposition of the Dirac operator D,

$$\begin{aligned} D\sigma &= T\Big(\nabla_T\sigma - \sum TX_i\nabla^t_{X_i}\sigma - \sum TX_iS^t_{X_i}\sigma\Big) \\ &= T\Big(\Big(\nabla_T - \frac{\kappa}{2}\Big) + A_t + B_t\Big)\sigma \end{aligned} \tag{1.9}$$

with

$$A_t\sigma = -\sum TX_i\nabla^t_{X_i}\sigma \tag{1.10}$$

and

$$B_t\sigma = \frac{\kappa}{2}\sigma - \sum TX_iS^t_{X_i}\sigma. \tag{1.11}$$

The index t indicates that A_t and B_t act on sections of E_t. We recall that A_t is a Dirac operator on E_t and that B_t is symmetric, tensorial and, by (1.4), pointwise uniformly bounded independent of t.

We will work in the Hilbert space $L^2(E)$ which we view as the direct integral over $(0,\infty)$ of the Hilbert spaces $L^2(E_t)$. We use the notation $\langle\cdot,\cdot\rangle$ for the Hermitian product on E or E_t, respectively, and we write

$$(\sigma,\eta)_t := \int_{N_t} \langle\sigma,\eta\rangle \tag{1.12}$$

for L^2-sections σ,η of E_t. Then the scalar product on $L^2(E)$ is

$$(\sigma,\eta) := \int_0^\infty (\sigma_t,\eta_t)_t\,\mathrm{d}t, \tag{1.13}$$

where $\sigma_t = \sigma|E_t$ and $\eta_t = \eta|E_t$. Occasionally, we will consider only the direct integral over some interval I; this we will indicate by a subscript "I" for the relevant quantities. The following lemma is now immediate from (1.7).

LEMMA 1.14. – *Let $I = [t_0,t_1] \subset (0,\infty)$ be an interval and σ,η be C^1 sections of E. Then*

$$(\sigma',\eta)_I = (\sigma, -\eta' + \kappa\eta)_I + \big\{(\sigma,\eta)_{t_1} - (\sigma,\eta)_{t_0}\big\}.$$

In particular, $T(\nabla_T - \kappa/2)$ is symmetric on $C^1_0(E)$.

Example 1.15. – Let M_0 be a compact $(n+1)$-dimensional manifold with boundary $\partial M \cong T^n$, the n-dimensional torus. Attach an end $U = (0,\infty) \times T^n$ to M_0 and consider a Riemannian metric on $M = M_0 \cup U$ whose restriction to U is given by the hyperbolic metric:

$$g = dt^2 + \mathrm{e}^{-2at} g_0,$$

where g_0 is a flat metric on T^n, of volume 1, say. Then $S(X,Y) = a\langle X, Y\rangle T$. In particular, $\kappa = na$.

Let $X_1, \ldots, X_n$ be a global parallel orthonormal frame on T^n. In $\mathbb{C}\mathrm{l}\, U$, consider the multivector

$$X = \mathrm{e}^{akt} X_1 \wedge \cdots \wedge X_k.$$

Note that X has norm one and is parallel in the T-direction. By a straightforward computation, we obtain:

$$A_t X = 0, \qquad A_t(TX) = 0, \qquad B_t X = \left(\frac{n}{2} - k\right) aX, \qquad B_t(TX) = \left(k - \frac{n}{2}\right) aTX.$$

The density of the volume element of the warped metric is $\exp(-nat)$. Hence along the end U, the linear map:

$$\Psi : L^2(0,\infty) \otimes \mathbb{C}^2 \to L^2(E), \quad (u,v) \mapsto \mathrm{e}^{nat/2}(uX + vTX),$$

preserves L^2-norms. We obtain a Dirac system,

$$\Psi^{-1} D \Psi = \gamma(\partial_t - \tau\delta/2),$$

where

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\delta = (k - n/2)a$. In particular, if n is even (and $k = n/2$), then $\mathrm{spec}_e\, D = \mathbb{R}$.

2. An axiomatic approach

In this section we develop an axiomatic approach to the spectral theory of Dirac operators on Dirac bundles which are geometric over the ends of M. Our approach is inspired by our previous work on surfaces with cusps [6]. *We keep the notation and the assumptions on the ends introduced in Section* 1.

We fix an end U of M and a geometric Dirac bundle $E \to U$. We formulate the properties we need in our discussion as additional axioms on the structure of U and E. In Sections 3 and 4 we discuss an important example where we verify these axioms, namely the Clifford bundle.

AXIOM 1: There is a continuous Riemannian connection $\bar\nabla$ on U such that $\bar\nabla$ is flat in the sense that there is an open cover $\mathcal{V}$ of N such that for each $V \in \mathcal{V}$ we have a C^1 local orthonormal frame $(X_0, \ldots, X_n)$ of TU defined in $V \times (0,\infty)$ which is $\bar\nabla$–parallel and such that the transition functions between any two such frames are locally constant. If $X_0 = T$, then such a frame (and any associated frame for E) will be called *special*. Moreover, we assume that:

$$\bar{\nabla}_T = \nabla_T, \qquad \bar{\nabla} T = 0,$$
$$\text{and} \quad \bar{S} := \nabla - \bar{\nabla} \text{ is uniformly bounded.}$$

Note that the meaning of the operators S and $\bar{S}$ is quite different. However, the analogy in notation associated to the two different splittings of ∇ will be kept for mnemonic reasons. In accordance with this, the uniform bound on the pointwise operator norm of $\bar{S}$ on TU will be denoted $\bar{b}$,

$$\|\bar{S}\| \leqslant \bar{b}. \tag{2.1}$$

For any $t > 0$, the restriction of $\bar{\nabla}$ and $\bar{S}$ to N_t and E_t will be denoted $\bar{\nabla}^t$ and $\bar{S}^t$. We also consider $\bar{\nabla}^t$ as a first-order differential operator on $C^1(E_t)$ with values in $L^2(T^*N_t \otimes E_t)$. The formal adjoint of $\bar{\nabla}^t$ is denoted $(\bar{\nabla}^t)^*$.

Let $(T, X_1, \ldots, X_n)$ be a special frame of TU and (σ_I) be an associated special frame of E. Locally we write sections σ of E in the form

$$\sigma = \sum \varphi_I \sigma_I.$$

In terms of such linear combinations, we have:

$$\begin{aligned} &\bar{\nabla}^t \sigma = \sum d\varphi_I \otimes \sigma_I \quad \text{and} \\ &(\bar{\nabla}^t)^* \bar{\nabla}^t \sigma = -\sum \bigl\{ \bar{\nabla}^t_{X_i} \bar{\nabla}^t_{X_i} \sigma - \bar{\nabla}^t_{\bar{\nabla}^t_{X_i} X_i} \sigma \bigr\}. \end{aligned} \tag{2.2}$$

The kernel K_t of $\bar{\nabla}^t$ consists of sections σ of E_t which are parallel with respect to $\bar{\nabla}^t$. Since a parallel section is determined by its values at one point, we have $\dim K_t =: k \leqslant \operatorname{rk} E$. Now $\bar{\nabla}$ is flat, by Axiom 1, hence the family $K = (K_t)$ is parallel in the direction of T. Moreover, K is invariant under Clifford multiplication by T since T is parallel with respect to $\bar{\nabla}$.

Once and for all we choose a basis $\sigma_1, \ldots, \sigma_k$ of pointwise orthonormal sections spanning K. Note that these sections are defined on all of U. In terms of this basis, a section $\sigma \in L^2(E)$ is in K if and only if σ is of the form $\sigma = \sum_{i \leqslant k} \varphi_i \sigma_i$, where the coefficients φ_i depend on t only. Throughout, we choose our special frames (σ_I) such that $\sigma_1, \ldots, \sigma_k$ are its first k members.

Next we introduce the orthogonal projection $\bar{P}_0(t) : L^2(E_t) \to K_t$. Using the orthonormal frame $\sigma_1, \ldots, \sigma_k$, we have:

$$\bar{P}_0(t)\sigma = (\operatorname{vol} N_t)^{-1} \sum_{i \leqslant k} (\sigma, \sigma_i)_t \sigma_i.$$

We see that the family $(\bar{P}_0(t))_{t>0}$ integrates to an orthogonal projection $\bar{P}_0$ in $L^2(E)$. The following lemma is the analogue of [6, Lemmas 4.2 and 4.3].

LEMMA 2.3. – *On $C^1(E)$ we have*

$$\bar{P}_0 \nabla_T \bar{P}_0 = \nabla_T \bar{P}_0 \quad \textit{or equivalently} \quad (1 - \bar{P}_0)\nabla_T \bar{P}_0 = 0.$$

Furthermore, $T\bar{P}_0 = \bar{P}_0 T$ and

$$\|\bar{P}_0 D(1 - \bar{P}_0)\| = \|(1 - \bar{P}_0) D \bar{P}_0\| \leqslant n \|(1 - \bar{P}_0)\bar{S}\bar{P}_0\| \leqslant n \|\bar{S}\|,$$

where $\|\bar{S}\|$ denotes the pointwise uniform norm of $\bar{S}$ as a field of operators on E.

Proof. – The first assertions are clear since K consists precisely of sections of the form $\sum_{i\leqslant k}\varphi_i\sigma_i$, where the coefficients φ_i depend on t only. The next assertions follow since Clifford multiplication by T is orthogonal and leaves K invariant. As for the last assertions, we note that $\bar P_0 D(1-\bar P_0)=((1-\bar P_0)D\bar P_0)^*$, hence it suffices to estimate $\|(1-\bar P_0)D\bar P_0\|$. Now $\bar\nabla^t(\bar P_0\sigma)=0$ by the definition of $\bar P_0$, hence:

$$\begin{aligned}(1-\bar P_0)D\bar P_0\sigma &= (1-\bar P_0)\Big\{T\nabla_T(\bar P_0\sigma)+\sum X_i\bar\nabla_{X_i}(\bar P_0\sigma)+\sum X_i\bar S_{X_i}(\bar P_0\sigma)\Big\}\\ &= \sum\{(1-\bar P_0)X_i\bar S_{X_i}\bar P_0\}\sigma.\end{aligned}$$

Now $\|X_i\|=1$, hence the lemma follows. □

We now turn to the discussion of the Dirac operator. It is reasonable to write D as follows:

$$\begin{aligned}D\sigma &= T\bigg(\Big(\nabla_T-\frac{\kappa}{2}\Big)\sigma-\sum TX_i\bar\nabla^t_{X_i}\sigma+\Big(\frac{\kappa}{2}-\sum TX_i\bar S^t_{X_i}\Big)\sigma\bigg)\\ &=: T\bigg(\Big(\nabla_T-\frac{\kappa}{2}\Big)+\bar A_t+\bar B_t\bigg)\sigma,\end{aligned}\tag{2.4}$$

where we recall that $\bar\nabla_T=\nabla_T$. Using $\bar\nabla\sigma_I=0$ we obtain:

$$\bar A_t\sigma=-\sum\{X_i(\varphi_I)\}TX_i\sigma_I\tag{2.5}$$

and

$$\bar B_t\sigma=\frac{\kappa}{2}\sigma-\sum\varphi_I TX_i\{\nabla_{X_i}\sigma_I\}.\tag{2.6}$$

The following result is obvious from the assumptions in Axiom 1 and the definitions.

LEMMA 2.7. – *$\bar A_t$ is a first-order symmetric elliptic differential operator on E_t, and $\bar B_t$ is a zero-order symmetric differential operator on E_t with uniformly bounded norm. Furthermore, for any C^1 section σ of E_t we have the pointwise estimate*

$$\|\bar A_t\sigma\|\leqslant\sqrt{n}\,\|\bar\nabla^t\sigma\|.$$

The next axiom will enable us to study the essential spectrum of D. The axiom involves the smallest nonzero eigenvalue of $(\bar\nabla^t)^*\bar\nabla^t$,

$$2\lambda_t^2:=\inf\big\{\operatorname{spec}\big((\bar\nabla^t)^*\bar\nabla^t\big)\setminus\{0\}\big\}.\tag{2.8}$$

The normalization of λ_t will become clear from Lemma 2.9 below.

AXIOM 2: $\lim_{t\to\infty}\lambda_t=\infty$.

In the following lemma we show that Axiom 2 allows to reverse, in $L^2(E_t)$, the estimate in Lemma 2.7.

LEMMA 2.9. – *For all $t>0$, we have* $\ker\bar\nabla^t\subset\ker\bar A_t$. *For all sufficiently large t, we actually have* $\ker\bar\nabla^t=\ker\bar A_t$ *and*

$$\|\bar\nabla^t\sigma\|_t^2\leqslant 2\|\bar A_t\sigma\|_t^2$$

for any C^1 section σ of E_t. In particular, λ_t is a lower bound for the modulus of the non-zero eigenvalues of $\bar A_t$.

It will be convenient to shift the parameter t such that the assertions of Lemma 2.9 hold for all $t > 0$.

Proof of Lemma 2.9. – The first assertion is clear. To prove the other assertions, we need to compute $\bar{A}_t^2$. For this we let $\sigma = \sum \varphi_I \sigma_I$, where the coefficients φ_I are C^2. Then we have:

$$\begin{aligned}\bar{A}_t^2(\sigma) &= -\bar{A}_t\Big(\sum_{j,I} X_j(\varphi_I) T X_j \sigma_I\Big) = \sum_{i,j,I} X_i X_j(\varphi_I) T X_i T X_j \sigma_I \\ &= -\sum_{i,I} X_i X_i(\varphi_I)\sigma_I + \sum_{i<j,I} [X_i, X_j](\varphi_I) X_i X_j \sigma_I \\ &= (\bar{\nabla}^t)^* \bar{\nabla}^t \sigma + \sum_{i,j,I} \{(\nabla^t_{X_i} X_j)(\varphi_I)\} X_i X_j \sigma_I = (\bar{\nabla}^t)^* \bar{\nabla}^t \sigma + C_t \sigma.\end{aligned}$$

Since the fields X_i are parallel with respect to $\bar{\nabla}$, we have $\|\nabla^t_{X_i} X_j\| \leqslant b + \bar{b}$. Therefore,

$$2\|C_t\sigma\| \leqslant n(n+1)(b+\bar{b})\|\bar{\nabla}^t\sigma\|.$$

Now for any $\delta > 0$, we have

$$n(n+1)(b+\bar{b}) \leqslant \delta\lambda_t$$

for all $t > 0$ sufficiently large. For a section σ of E_t perpendicular to the kernel of $\bar{\nabla}^t$ this gives:

$$\|\bar{\nabla}^t\sigma\|_t^2 \leqslant 2\,\mathrm{Re}\big((\bar{\nabla}^t)^*\bar{\nabla}^t\sigma + C_t\sigma, \sigma\big)_t = 2\,\|\bar{A}_t\sigma\|_t^2.$$

In particular, σ is not in the kernel of $\bar{A}_t$. The lemma follows. □

It is now easy to deduce the following analogue of [6, Lemma 4.4]. Since $\bar{A}_t$ is essentially self-adjoint on $C^1(E_t)$ with discrete closure, we can form the spectral projection onto the positive and negative eigenspaces, to be denoted by $\bar{P}_{>0}(t)$ and $\bar{P}_{<0}(t)$, respectively.

LEMMA 2.10. – *Let $I = [t_0, t_1] \subset (0, \infty)$ be a compact interval and suppose that $\sigma \in C^1(E)$ satisfies $\bar{P}_{>0}(t_0)\sigma_{t_0} = 0$ and $\bar{P}_{<0}(t_1)\sigma_{t_1} = 0$. Then*

$$\|\bar{A}\sigma\|_I \leqslant \|D\sigma\|_I + C_1\,\|\sigma\|_I,$$

where $C_1 = 2\sqrt{n}b + \|\bar{B}\|$.

Proof. – We proceed exactly as in loc.cit. In terms of a special frame we write $\sigma = \sum \varphi_I \sigma_I$. By approximation we may assume that the coefficients φ_I are C^2. We have

$$[T, X_i] = \nabla_T X_i - \nabla_{X_i} T = W X_i,$$

where W is the Weingarten map as in (1.3). Furthermore,

$$\begin{aligned}-[\bar{A}, \nabla_T]\sigma &= \sum (X_i T - T X_i)(\varphi_I) T X_i \sigma_I \\ &= \sum [X_i, T](\varphi_I) T X_i \sigma_I = -\sum (W X_i)(\varphi_I) T X_i \sigma_I,\end{aligned}$$

and therefore, by the bound (1.4) on W,

$$\|[\bar{A}, \nabla_T]\sigma\|_I \leqslant \sqrt{n}b\,\|\bar{\nabla}^t\sigma\|_I.$$

By Lemma 1.14 we have

$$(\sigma', \bar{A}\sigma)_I = (\sigma, -\bar{A}\sigma' + \kappa\bar{A}\sigma)_I + \big(\sigma, [\bar{A}, \nabla_T]\sigma\big)_I + \big\{(\sigma, \bar{A}\sigma)_{t_1} - (\sigma, \bar{A}\sigma)_{t_0}\big\}.$$

Now $\bar{A}$ is symmetric, hence we conclude

$$2\,\mathrm{Re}(\sigma', \bar{A}\sigma)_I \geqslant -2\sqrt{n}b\,\|\sigma\|_I\,\|\bar{A}\sigma\|_I + \mathrm{Re}(\sigma, \kappa\bar{A}\sigma)_I,$$

where we use our assumption on the boundary values of σ and Lemma 2.9. Therefore,

$$\begin{aligned}\|D\sigma\|_I\|\bar{A}\sigma\|_I &\geqslant \mathrm{Re}(D\sigma, T\bar{A}\sigma)_I \\ &= \mathrm{Re}\left(\sigma' - \frac{\kappa}{2}\sigma, \bar{A}\sigma\right)_I + \|\bar{A}\sigma\|_I^2 + \mathrm{Re}(\bar{B}\sigma, \bar{A}\sigma)_I \\ &\geqslant \|\bar{A}\sigma\|_I^2 - \big(2\sqrt{n}b + \|\bar{B}\|\big)\|\sigma\|_I\|\bar{A}\sigma\|_I.\end{aligned}$$

This concludes the proof of the lemma. □

COROLLARY 2.11. – *Let $I = [t_0, t_1]$ and $\lambda_I := \inf\{\lambda_t \mid t \in I\}$. Then for σ as in Lemma* 2.10, *we have*:

$$(\lambda_I - C_1)\big\|(1 - \bar{P}_0)\sigma\big\|_I \leqslant \|D\sigma\|_I + C_1\|\bar{P}_0\sigma\|_I.$$

We denote by $\mathrm{spec}_e(D, U)$ the part of the essential spectrum $\mathrm{spec}_e\, D$ of D related to our end U. Now we show that our two axioms reduce the study of $\mathrm{spec}_e(D, U)$ to a Dirac system in one variable. We follow the line of argument given in [6, Theorem 5.7]. In particular, we recall the notions of *special Weyl sequence* and of *spectrum at infinity*, spec_∞, given in loc.cit. (5.2) and (5.5). The Dirac system in question is formed by means of the projection $\bar{P}_0$ as

$$D_0 := \bar{P}_0 D \bar{P}_0. \tag{2.12}$$

The following theorem is an analogue of Theorem 2 in [26] (and Theorem 5 in [27]), the proof is similar to the proof of the correponding Theorem 5.7 in [6].

THEOREM 2.13. – *We have* $\mathrm{spec}_\infty D_0 = \mathrm{spec}_e(D, U)$.

Proof. – We need to introduce the complete decomposition of D determined by $\bar{P}_0$. Thus we put $\bar{P}_1 := I - \bar{P}_0$ and write:

$$D_1 := \bar{P}_1 D \bar{P}_1, \qquad D_{10} := \bar{P}_1 D \bar{P}_0, \qquad D_{01} := \bar{P}_0 D \bar{P}_1.$$

We recall from Lemma 2.3 that D_{10} and $D_{01} = (D_{10})^*$ are uniformly bounded.

Now assume that $\lambda \in \mathrm{spec}_e(D, U)$. Then we can find a special Weyl sequence (σ_n) with support in U such that:

$$\lim_{n\to\infty} \|\sigma_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} (D - \lambda)\sigma_n = 0.$$

W.l.o.g. we assume that $\mathrm{supp}\,\sigma_n \subset [n, \infty) \times N$. We decompose

$$\sigma_n = \bar{P}_0\sigma_n + \bar{P}_1\sigma_n =: \sigma_{n0} + \sigma_{n1}.$$

Then we deduce from Lemma 2.10, for n sufficiently large, the estimate

$$\|\sigma_{n1}\| \leqslant \frac{2}{\lambda_n}\big(|\lambda| + C_1\big).$$

This implies that (σ_{n0}) is a special Weyl sequence. From the identity

$$(D_0 - \lambda)\sigma_{n0} = \bar{P}_0(D - \lambda)\sigma_n - D_{01}\sigma_{n1}$$

we infer that $\lim_{n\to\infty}(D_0 - \lambda)\sigma_{n0} = 0$. Hence we conclude that:

$$\operatorname{spec}_\infty D_0 \subset \operatorname{spec}_e(D, U).$$

Next assume that $\lambda \in \operatorname{spec}_\infty D_0$. Then we have a special Weyl sequence (σ_{n0}) for $\bar{P}_0(E)$ with

$$\lim_{n\to\infty} \|\sigma_{n0}\| = 1 \quad \text{and} \quad \lim_{n\to\infty} (D_0 - \lambda)\sigma_{n0} = 0.$$

W.l.o.g. we assume that $\operatorname{supp}\sigma_{n0} \subset [n+1, \infty) \times N$. We want to construct a sequence $(\sigma_{n1}) \subset C_0^1(\bar{P}_1(E))$ such that $\sigma_n := \sigma_{n0} + \sigma_{n1}$ is a special Weyl sequence for E with $\lim_{n\to\infty}(D - \lambda)\sigma_n = 0$. In view of the decomposition:

$$(D - \lambda)\sigma_n = (D_0 - \lambda)\sigma_{n0} + (D_1 - \lambda)\sigma_{n1} + D_{10}\sigma_{n0} + D_{01}\sigma_{n1},$$

we define

$$\sigma_{n1} := \phi_n (D_{1n} - \lambda)^{-1}(-D_{10}\sigma_{n0}).$$

Here we choose cut off functions $\phi_n \in C^\infty(0, \infty)$ with

$$\operatorname{supp}\phi_n \subset (n, \infty) \quad \text{and} \quad \sup_{n, t \geqslant n} \big(|\phi_n(t)| + |\phi_n'(t)|\big) \leqslant 3.$$

Moreover, D_{1n} denotes the self-adjoint extension of D in $L^2(\bar{P}_1(E))$ defined by Atiyah–Patodi–Singer boundary conditions at N_n. In view of Axiom 2 and Lemma 2.10, we have

$$\big\|(D_{1n} - \lambda)^{-1}\big\| = \mathrm{O}\big(\lambda_n^{-1}\big).$$

This implies $\lim_{n\to\infty} \|\sigma_{n1}\| = 0$; using this we deduce that $\lim_{n\to\infty}(D - \lambda)\sigma_n = 0$, proving

$$\operatorname{spec}_e(D, U) \subset \operatorname{spec}_\infty D_0,$$

as desired. □

It is now easy to get estimates on $\operatorname{spec}_e(D, U)$. We first note the following trivial consequence of Theorem 2.13.

COROLLARY 2.14. – *If* $\dim K = k = 0$, *then* $\operatorname{spec}_e(D, U) = \emptyset$.

Recall the estimate from Corollary 2.11. The corresponding estimate for $\bar{P}_0\sigma$ ensures the Fredholm property of D. We use the notation $U_t := (t, \infty) \times N$.

LEMMA 2.15. – *Suppose that there are positive constants δ and C_2 such that:*

$$\|\bar{P}_0\sigma\| \leqslant \delta^{-1}\big(\|D\sigma\| + C_2\big\|(1-\bar{P}_0)\sigma\big\|\big)$$

for all $\sigma \in C^1(E)$ with compact support in U. Then for any $\varepsilon > 0$ there is $t > 0$ such that

$$\|D\sigma\| \geqslant (\delta - \varepsilon)\|\sigma\|$$

for any $\sigma \in C^1(E)$ with compact support in U_t.

In particular, D is Fredholm if and only if the above apriori estimate holds for all ends U of M.

Proof. – Recall that D is Fredholm if and only if the estimate

$$\|\sigma\| \leqslant C\|D\sigma\|$$

holds for all $\sigma \in C_0^1(E)$ with support outside a sufficiently large compact subset of M; and the analogous estimate for D_0 is equivalent to $0 \notin \operatorname{spec}_\infty D_0$. Now the lemma follows from an easy calculation, using Axiom 2 and Theorem 2.13. □

In the Fredholm case, it is natural to study index problems. We will model our approach to such problems after the method we used in the surface case [6], which will lead to very explicit index formulas in the case of the Clifford bundle.

We assume now in addition that there is a *natural* involution α on E, that is, α is induced by an involution of the representation space which defines the given bundle E, and that α anticommutes with Clifford multiplication by tangent vectors. By naturality, α is parallel with repect to all connections involved. The induced parallel splitting of E we write as $E = E^+ \oplus E^-$. For any operator R acting on sections of E, we obtain the decomposition $R = R^+ + R^-$, where $R^\pm$ denotes the restriction of R to sections of $E^\pm$. Since E^+ and E^- are parallel with respect to ∇ and α anticommutes with Clifford multiplication by vector fields, $D^\pm$ maps sections of $E^\pm$ to sections of $E^\mp$. We will be concerned with the index of D^+.

By $P_{\geqslant 0}(t)$ we denote the spectral projection of A_t onto the eigenspaces with nonnegative eigenvalues. We use similar notation for other spectral projections of A_t and recall that the corresponding objects for $\bar{A}_t$ are decorated by a bar.

Our next axiom is formulated in such a way that we do not only get that D is a Fredholm operator but also an explicit index formula for D^+; it is inspired by the apriori estimate in Lemma 2.15.

AXIOM 3: For each end U and all sufficiently large t, there is an orthogonal projection $Q = Q(t)$ in $L^2(E_t)$ such that the following properties hold:

(1) $\alpha Q = Q\alpha$;
(2) $TQ = (1-Q)T$;
(3) $Q \geqslant \bar{P}_{<0}$;
(4) there are positive constants δ and C_2 such that:

$$\|\bar{P}_0\sigma\| \leqslant \delta^{-1}\big(\|D\sigma\| + C_2\big\|(1-\bar{P}_0)\sigma\big\|\big)$$

for all

$$\sigma \in \mathcal{D}_{U,t} := \big\{\sigma \in C_0^1(E|\bar{U}_t) \mid \big(1 - Q(t)\big)\sigma_t = 0\big\}.$$

Remark 2.16. – If $\ker \bar{A} = \{0\}$, then Axiom 3 holds with $Q := \bar{P}_{<0}$.

Throughout the rest of this section we assume that Axioms 1–3 are satisfied.

LEMMA 2.17. – *For all sufficiently large t, there is a subspace $K_Q = K_Q(t)$ of $K = K(t) = \ker \bar{A}_t$, such that $TK_Q = K_Q^\perp$ and*

$$\operatorname{im} Q = \operatorname{im} \bar{P}_{<0} + K_Q \quad \textit{and} \quad \operatorname{im}(1-Q) = \operatorname{im} \bar{P}_{>0} + K_Q^\perp.$$

Proof. – Note that Clifford multiplication by T anticommutes with $\bar{A}$, hence T preserves K and maps $\operatorname{im} \bar{P}_{<0}$ to $\operatorname{im} \bar{P}_{>0}$. By Axiom 3 we have $TQ = (1-Q)T$, hence the lemma. □

We say that a pair (P, Q) of orthogonal projections of a Hilbert space H is a *Fredholm pair* if $Q : \operatorname{im} P \to \operatorname{im} Q$ is a Fredholm operator. If (P, Q) is a Fredholm pair, then we set:

$$\operatorname{ind}(P, Q) := \operatorname{ind}(Q : \operatorname{im} P \to \operatorname{im} Q).$$

Recall that K has finite dimension. Hence by Lemma 2.17, $(\bar{P}_{<0}, Q)$ is a Fredholm pair with $\operatorname{ind}(\bar{P}_{<0}, Q) = -\dim K_Q$.

By assumption, Q commutes with the involution α, hence $Q(t)$ restricts to orthogonal projections $Q^+(t)$ in $L^2(E_t^+)$ and $Q^-(t)$ in $L^2(E_t^-)$, respectively. It follows that $(\bar{P}^+_{<0}(t), Q^+(t))$ is a Fredholm pair, too.

LEMMA 2.18. – *For all sufficiently large t, the pairs $(P_{\leqslant 0}(t), Q(t))$ and $(P^+_{\leqslant 0}(t), Q^+(t))$ are Fredholm pairs. Moreover, if μ satisfies $\|\bar{B} - B\| < -\mu < \lambda_t - \|\bar{B} - B\|$, then*

$$\operatorname{ind}\big(P^+_{\leqslant 0}(t), Q^+(t)\big) = \dim\ker A_t^+ + \dim\operatorname{im} P^+_{(\mu,0)}(t) - \dim K_Q^+(t).$$

Proof. – Recall that $A + B = \bar{A} + \bar{B}$. Hence $\operatorname{spec} A_t \cup (-\operatorname{spec} A_t)$ does not intersect the interval $(\|\bar{B} - B\|, \lambda_t - \|\bar{B} - B\|)$. By Corollary A.7 we have $\|\bar{P}_{<0} - P_{<\mu}\| < 1$, for t sufficiently large. It follows that:

$$(1 - \bar{P}_{<0} P_{<\mu}) : \operatorname{im} \bar{P}_{<0} \to \operatorname{im} \bar{P}_{<0} \quad \text{and} \quad (1 - P_{<\mu}\bar{P}_{<0}) : \operatorname{im} P_{<\mu} \to \operatorname{im} P_{<\mu}$$

have norm < 1, hence $\bar{P}_{<0} : \operatorname{im} P_{<\mu} \to \operatorname{im} \bar{P}_{<0}$ is an isomorphism. By assumption we have $\bar{P}_{<0} \leqslant Q$ with finite codimension, hence $(P_{<\mu}, Q)$ is a Fredholm pair and $\operatorname{ind}(P^+_{<\mu}, Q^+) = \operatorname{ind}(\bar{P}^+_{<0}, Q^+)$. We conclude that $(P_{\leqslant 0}, Q)$ is a Fredholm pair and that

$$\begin{aligned}\operatorname{ind}\big(P^+_{\leqslant 0}, Q^+\big) &= \dim\operatorname{im} P^+_{(\mu,0]} + \operatorname{ind}\big(P^+_{<\mu}, Q^+\big) \\ &= \dim\operatorname{im} P^+_{(\mu,0)} + \dim\ker A_t^+ + \operatorname{ind}\big(\bar{P}^+_{<0}, Q^+\big).\end{aligned}$$

The lemma follows. □

We obtain elliptic boundary value problems on $\mathcal{D}_{U,t}$ (from Axiom 3) for D on the noncompact manifolds U_t introduced above and on

$$\mathcal{D}_{\mathrm{int},t} := \big\{\sigma \in C^1(E|M_t) \mid Q(t)\sigma_t = 0 \text{ for each end } U\big\}$$

on the interior part $M_t = M \setminus \bigcup U_t$. The resulting self-adjoint operators we denote by $D_{U,t}$ and $D_{\mathrm{int},t}$, respectively. By the *splitting formula* for the index given by Brüning and Lesch [11] we have the following result:

LEMMA 2.19. – *For all sufficiently large t, we have* $\operatorname{ind} D^+ = \operatorname{ind} D^+_{\mathrm{int},t} + \sum_U \operatorname{ind} D^+_{U,t}$.

We remark that the quoted result from [11] is applicable in our situation since we can view $\nabla_T - \kappa/2$ as a connection on the Hilbert bundle $L^2(E)$ over $(0,\infty)$; the corresponding parallel transport gives the needed trivialization.

We proceed to compute the index of D^+. The first use to be made of Axiom 3 is to establish the vanishing of $\operatorname{ind} D^+_{U,t}$. We actually prove more.

LEMMA 2.20. – *For each end U and all sufficiently large t, we have* $\ker D_{U,t} = 0$.

Proof. – We note that $D_{U,t}$ is essentially self-adjoint in $L^2(E|U_t)$ with domain $\mathcal{D}_{U,t}$, cf. [11]. Thus, for $\sigma \in \ker D_{U,t}$ we have the *a priori* estimate of Axiom 3; and since $(1-Q) \geqslant \bar{P}_{>0}$ we also have the estimate from Corollary 2.11. Hence

$$\|\bar{P}_0\sigma\| \leqslant \frac{C_2}{\delta}\|(1-\bar{P}_0)\sigma\| \leqslant \frac{C_1C_2}{\delta(\lambda_{(t,\infty)} - C_1)}\|\bar{P}_0\sigma\|,$$

where $\lambda_{(t,\infty)} = \inf\{\lambda_\tau \mid t \leqslant \tau \leqslant \infty\}$. This implies the asserted vanishing for t large enough, in view of Axiom 2. □

Next we have to deal with $D_{\mathrm{int},t}$. For each sufficiently large t, we choose a Riemannian metric $g_0 = g_0(t)$ of M which coincides with the given metric g on M_t outside a neighborhood of ∂M_t and along ∂M_t and such that for each end U, $U_{t-\varepsilon}$ is a metric cylinder. We will use the following version of the Atiyah–Patodi–Singer index formula.

LEMMA 2.21. – *Let* ω_D^0 *denote the index form of* D^+ *with respect to* g_0 *given by the Local Index Theorem. Then*

$$\operatorname{ind} D^+_{\mathrm{int},t} = \int_{M_t} \omega_D^0 + \frac{1}{2}\big(\eta(A_t^+) - \dim\ker A_t^+\big) + \operatorname{ind}\big(P^+_{\leqslant 0}(t), Q^+(t)\big).$$

Proof. – The deformation $(1-s)g_0 + sg$, $0 \leqslant s \leqslant 1$, from g_0 to g induces a smooth deformation of first-order elliptic differential operators on the space $\mathcal{D}_{\mathrm{int},t}$. By our assumptions on Q and since $g = g_0$ along ∂M_t, all these operators are Fredholm, hence the index remains constant. Thus the result follows from the index formula of Atiyah, Patodi and Singer [3] and the relative index formula of Agranovic and Dynin [1], cf. Theorem 23.1 in [8]. □

We now discuss the terms in the above index formula. One aim is to eliminate the dependence on g_0 and t. To that end, *we assume from now on until the end of this section that M has finite volume and that, for each end U of M, the volume of the cross sections* N_t *tends to* 0 *as t tends to infinity.*

Then the first term on the right-hand side of the index formula can be easily dealt with using Chern–Weil theory.

LEMMA 2.22. – *Let* ω_D *denote the index form of* D^+ *with respect to g given by the Local Index Theorem. Then*

$$\lim_{t\to\infty} \int_{M_t} \omega_D^0 = \int_M \omega_D.$$

Proof. – We have $\omega_D^0 - \omega_D = d\Phi$, where Φ is the transgression form; by Lemma 5 in [23, p. 297], Φ is a universal polynomial in the connection forms and curvature tensors of g and g_0. By assumption g_0 agrees with g along ∂M_t, hence the connection forms of g_0 and g differ by the second fundamental form S^t of ∂M_t with respect to g. Furthermore, the Gauss and Codazzi equations express the difference of the curvature tensor of g_0 and g along ∂M_t in terms of S^t. It follows that Φ is a universal polynomial in the curvature tensor R of g and S^t, see loc. cit. Hence the pointwise norm of Φ is uniformly bounded. By assumption, the volume of ∂M_t tends to 0 as $t \to \infty$, hence $\int_{\partial M_t} \Phi \to 0$. Now

$$\int_{M_t} \omega_D^0 = \int_{M_t} \omega_D + \int_{\partial M_t} \Phi,$$

hence the lemma. □

We proceed to formulate an additional axiom which will allow us to compute $\operatorname{ind}(\bar{P}^+_{<0}(t), Q(t)^+)$ explicitly.

AXIOM 3′: For each end U, there is a symmetric involution $\bar{\beta}$ of E over U and a number $\delta > 0$ such that:

(1) $\alpha\bar{\beta} = \bar{\beta}\alpha$;
(2) $\bar{P}_0\bar{\beta} = \bar{\beta}\bar{P}_0$;
(3) $T\bar{\beta} = -\bar{\beta}T$;
(4) $\nabla_T\bar{\beta} = 0$;
(5) $(\bar{B}\sigma, \bar{\beta}\sigma) \geqslant \delta\|\sigma\|^2$ for all sections σ of E in the kernel of $\bar{P}_0$.

We denote by $\bar{P}_0(t) =: (1 - Q_0)(t) + Q_0(t)$ the splitting of $\operatorname{im}\bar{P}_0(t) = \ker\bar{A}_t$ induced by the positive and negative eigenspaces of $\bar{\beta}(t)$ acting on $K(t)$, and we define:

$$Q(t) := \bar{P}_{<0}(t) + Q_0(t). \tag{2.23}$$

In other words, the subspace $K_Q(t)$ of $K(t)$ in Lemma 2.17 is equal to $\operatorname{im} Q_0(t)$, the negative eigenspace of $\bar{\beta}(t)$ on $K(t)$.

LEMMA 2.24. – *With this choice of Q, Axiom 3′ implies Axiom 3 (with the same δ as there). Moreover,* $\operatorname{ind}(\bar{P}^+_{\leqslant 0}(t), Q(t)^+) = \dim\ker\bar{A}^+_t - \dim\operatorname{im} Q^+_0(t)$ *is independent of t.*

Proof. – The proof of the first claim is the same as the proof of Lemma 4.5 in [6]. The formula for $\operatorname{ind}(\bar{P}^+_{\leqslant 0}(t), Q(t)^+)$ is clear from the definition of Q. By assumption $\bar{\beta}$ is parallel in the T-direction, hence the last claim. □

Since $\|A - \bar{A}\| = \|\bar{B} - B\|$ and the non-zero eigenvalues of $\bar{A}_t$ have modulus at least λ_t, we have:

$$\dim\ker A^+_t + \dim\operatorname{im} P^+_{(-\mu,0)}(t) \leqslant \dim\operatorname{im} P^+_{(-\mu,\mu)}(t) = \dim\ker\bar{A}^+_t$$

for all $\mu \in (\|\bar{B} - B\|, \lambda_t - \|\bar{B} - B\|)$. In particular, if $\dim\ker A^+_t = \dim\ker\bar{A}^+_t$, then $\dim\operatorname{im} P^+_{(-\mu,0)} = 0$. From Lemmas 2.18–2.22 and 2.24 we get the index formula as we need it in our applications.

LEMMA 2.25. – *Suppose that* $\dim\ker A^+_t = \dim\ker\bar{A}^+_t$ *for all ends U and sufficiently large t. Then we have:*

$$\operatorname{ind} D^+ = \int_M \omega_D + \frac{1}{2}\Big\{\lim_{t\to\infty}\eta\big(A^+_t\big) + \dim\ker A^+_t\Big\} - \dim\operatorname{im} Q^+_0(t).$$

Proof of Theorem C. – Let $E = \mathbb{C}l\, M \otimes F \cong \Lambda^*(TM) \otimes F$, where F is a flat Hermitian bundle with a compatible connection. Let $D = d + d^*$ and let α be the involution corresponding to the even–odd decomposition of E.

Let U be an end of M. By extending the map $X \mapsto TX$ to the Clifford algebra, we identify $\Lambda^*(TN_t) \otimes F$ with E_t^+. Under this identification, A_t^+ corresponds to $-(d_t + d_t^*)$ on $L^2(\Lambda^*(TN_t) \otimes F)$. In particular, the kernel of A_t^+ corresponds to the space of harmonic forms on N_t with values in F.

Since all ends of M are cuspidal, Axiom 1 is satisfied, by Corollary 3.16. By assumption, along each end U of M, the curvature bounds satisfy $(n+1)a > (n-1)b$. Then Axiom 2 holds, by Lemma 3.23. Furthermore, we let $\bar{\beta}$ be as in (4.7). Then Axiom 3′ holds, by Lemma 4.8. We conclude:

(1) Since $d_t + d_t^*$ anticommutes with the involution defining the even–odd decomposition of $\Lambda^*(N_t) \otimes F$, we have $\eta(A_t^+) = 0$.
(2) By Hodge theory, $\dim\ker A_t^+$ is equal to the sum of the Betti numbers of N_t with coefficients in F. Moreover, $\dim\ker A_t^+ = \dim\ker \bar{A}_t^+$, by Lemma 4.4.
(3) To determine $\dim\operatorname{im} Q_0^+(t)$ we have to use the definition of $\bar{\beta}$ as given in (4.7). For $k < l$ odd or $k \geqslant l$ even, respectively, a section of $\Lambda^*(TN_t) \otimes F$ of degree k corresponds, under the above identification, to an eigenvector of $\bar{\beta}$ with eigenvalue -1, and for $k \geqslant l$ odd or $k < l$ even, respectively, to an eigenvector of $\bar{\beta}$ with eigenvalue $+1$. Using Poincaré duality, we get:

$$\frac{1}{2}\dim\ker A_t^+ - \dim\operatorname{im} Q_0^+(t) = \sum_{i<l} (-1)^i b_i(N_t; F).$$

Collecting terms we get the asserted formula for the index of D^+ in Theorem 0.2 C. □

Proof of Theorem D. – For E and D as before, we now let α be the involution of E corresponding to multiplication by the complex volume form τ_M.

Let U be an end of M. We use the map $\sigma \mapsto \frac{1}{\sqrt{2}}(\sigma + \tau_M \sigma)$ to identify $\Lambda^*(TN_t) \otimes F$ with E_t^+. Under this identification, A_t^+ corresponds to the odd signature operator $\tau_{N_t}(d_t + d_t^*)$ on $L^2(\Lambda^*(TN_t) \otimes F)$. In particular, the kernel of A_t^+ corresponds to the space of harmonic forms on N_t with values in F.

Since all ends of M are cuspidal, Axiom 1 is satisfied. Again we assume that along each end U of M, the curvature satisfies $(n+1)a > (n-1)b$, and then Axiom 2 holds, by Lemma 3.23. With $\bar{\beta}$ as in (4.7), Axiom 3′ holds and we have:

(1) $\dim\ker A_t^+ = \dim\ker \bar{A}_t^+$ by Lemma 4.4.
(2) Exactly half of the kernel of $\bar{A}_t^+$ consists of eigenvectors of $\bar{\beta}$ for the eigenvalues 1 and -1, respectively. Hence

$$\frac{1}{2}\dim\ker A_t^+ - \dim\operatorname{im} Q_0^+(t) = 0.$$

Collecting terms we get the formula for the index of D claimed in Theorem 0.2 D. □

Proof of Theorem E. – In this case, E is the spinor bundle of a spin structure of M. Since the ends of M are cuspidal, Axiom 1 holds. The holonomy of E is at most twice the holonomy of $\mathbb{C}l\, U$, hence it is finite, by Lemma 3.17. Therefore Axiom 2 holds, by Lemma 3.23.

Now if the spin structure is non-trivial along each end U of M, then $\ker \bar{A}_t = \{0\}$ for each end U and all $t > 0$. Then Axiom 3 holds, by Remark 2.16, and we get the claimed index formula. □

3. Cuspidal ends

In this section we study a special situation where the axiomatic approach from Section 2 applies. Let $U = (0, \infty) \times N$, where N is a closed manifold, and let $g = dt^2 + g_t$ be a Riemannian metric on U, where g_t is a family of Riemannian metrics on N. Then the projection $U \to (0, \infty)$ onto the first coordinate is a Riemannian submersion. The fiber over t is the cross section $N_t = \{t\} \times N$.

We let $T = \partial_t$ be the unit vector field in the t-direction. We use the prime $'$ to denote covariant differentiation in the direction of T.

For each point $p = (\tau, x) \in U$, the curve $\gamma_p(t) = (t, x)$, $t > 0$, is a geodesic ray perpendicular to the family of cross sections N_t. For $u \in T_pU$ we denote by $J = J_u(t)$ the Jacobi field along γ_p corresponding to variations $\gamma(t, s) = \gamma_{c(s)}(t + \tau(s) - \tau)$ of γ_p, where $c = c(s) = (\tau(s), x(s))$, $-\varepsilon < s < \varepsilon$, is a C^1 curve with $c(0) = p$ and $\partial_s c(0) = u$. Note that $J_u(\tau) = u$ and that $T = J_T$ along γ_p. The Jacobi fields $J = J_u$ satisfy the ordinary differential equation

$$J' = -WJ, \tag{3.1}$$

where W is the Weingarten map as in (1.3), and W satisfies the *Riccati equation*:

$$W' = W^2 + R(\cdot, T)T. \tag{3.2}$$

We assume that U has the geometry of an end of a complete Riemannian manifold of finite volume and pinched negative curvature. That is, we assume that there are constants $0 < a \leqslant b$ such that

$$-b^2 \|X\|^2 \leqslant \langle R(X, T)T, X\rangle \leqslant -a^2 \|X\|^2 \tag{3.3}$$

for all vector fields X perpendicular to T and that, for $t > \tau$,

$$e^{-b(t-\tau)} \|J(\tau)\| \leqslant \|J(t)\| \leqslant e^{-a(t-\tau)} \|J(\tau)\| \tag{3.4}$$

for Jacobi fields $J = J_u$ as above which are perpendicular to the corresponding ray. That is, we assume that such Jacobi fields are *stable* in the sense of dynamical systems.

There is the question of regularity. For the application in the case where U is an end of a complete Riemannian manifold of finite volume and pinched negative curvature, the map which establishes the diffeomorphism of U with $(0, \infty) \times N$ is only C^2, see [19], such that g may only be C^1. However, it turns out that $\partial_t g$ is also C^1, and hence (3.2) and (3.3) are defined and meaningful.

We will need a bit more, though. For example, we will need to estimate parallel translation along small loops. To that end we assume that there is a C^3-atlas on U such that g is C^2 with respect to this atlas. Then the curvature tensor R of g is defined. We assume that the pointwise norm of R is uniformly bounded on U. We denote this uniform bound by $\|R\|$, thus

$$\|R(X, Y)Z\| \leqslant \|R\| \, \|X\| \, \|Y\| \, \|Z\| \tag{3.5}$$

uniformly on U. We say that $U = (0, \infty) \times N$ is a *cusp* or a *cuspidal end* if N is closed and connected and the metric $g = dt^2 + g_t$ on U satisfies the above smoothness assumption and the estimates (3.3), (3.4), and (3.5).

The assumptions (3.3) and (3.4) imply bounds for the second fundamental forms S of the cross sections,

$$a \leqslant S \leqslant b, \tag{3.6}$$

and, equivalently, for the principal curvatures,

$$a \leqslant \kappa_i \leqslant b. \tag{3.7}$$

These estimates will be crucial in our discussion below.

Let $p = (\tau, x) \in U$, $u \in T_pU$ and $c = c(s) = (\tau(s), x(s))$ be a curve in U with $c(0) = p$ and $\partial_s c(0) = u$. Consider the 1-parameter family $\gamma(t,s) = \gamma_{c(s)}(t + \tau(s) - \tau)$ of rays and denote by $J = J(t,s)$ the corresponding family of stable Jacobi fields. Then $[T, J] = 0$. Let $v, w \in T_pU$ and $X = X(t,s)$, $Y = Y(t,s)$ be vector fields along γ with $X(\tau, 0) = v$, $Y(\tau, 0) = w$ and $\nabla_T X = \nabla_T Y = 0$. Then

$$T\langle \nabla_J X, Y\rangle = \langle \nabla_T \nabla_J X, Y\rangle = \big\langle R(T,J)X, Y\big\rangle.$$

Define a tensor field $\bar{S}$ with values in TU by:

$$\big\langle \bar{S}(u,v), w\big\rangle = -\int_\tau^\infty \big\langle R(T,J)X, Y\big\rangle(t,0)\,\mathrm{d}t. \tag{3.8}$$

By (3.4) and (3.5) the integral converges uniformly, hence $\bar{S}$ is well defined.

LEMMA 3.9. – *The tensor field $\bar{S}$ is continuous and uniformly bounded,*

$$\big\|\bar{S}(u,v)\big\| \leqslant \frac{1}{a}\|R\|\|u\|\|v\|.$$

Proof. – Fix a point $p = (\tau, x) \in U$. Let J be the stable Jacobi field along γ_p with $J(\tau) = u$ and X and Y be the parallel vector fields along γ_p such that $X(\tau) = v$ and such that Y is of length one and points in the direction of $\bar{S}(u,v)$ at p. Then by (3.4) and (3.5),

$$\big|\big\langle R(T,J)X, Y\big\rangle\big|\big(\gamma_p(t)\big) \leqslant \mathrm{e}^{-a(t-\tau)}\|R\|\|u\|\|v\|.$$

This uniform bound on the integrand in the definition (3.8) of $\bar{S}$ implies that $\bar{S}$ is continuous and satisfies the asserted bound. □

Define a new connection $\bar{\nabla}$ on U by

$$\bar{\nabla} = \nabla - \bar{S}. \tag{3.10}$$

LEMMA 3.11. – *The connection $\bar{\nabla}$ is continuous and Riemannian.*

Proof. – Continuity of $\bar{\nabla}$ follows from the continuity of $\bar{S}$. Now the integrand in the definition (3.8) of $\bar{S}$ is skewsymmetric in X and Y. Hence $\bar{\nabla}$ is Riemannian. □

In general, the connection $\bar{\nabla}$ may not be smooth. However, parallel translation with respect to $\bar{\nabla}$ is well defined.

If $\gamma = \gamma(t,s)$ is a 1-parameter family of rays, $J = J(t,s)$ the corresponding family of stable Jacobi fields along γ and X and Y are vector fields along γ such that $\nabla_T X = \nabla_T Y = 0$, then

$$T\langle \bar{\nabla}_J X, Y\rangle = 0 \quad \text{and} \quad \bar{\nabla}_T X = 0. \tag{3.12}$$

Using the approximation of cross sections by large geodesic spheres, we also get that

$$\bar{\nabla} T = 0. \tag{3.13}$$

We denote by $\overline{\mathrm{Tor}}$ the torsion tensor field of $\bar{\nabla}$.

COROLLARY 3.14. – *The torsion tensor field is uniformly bounded,*

$$\|\overline{\mathrm{Tor}}(X, Y)\| \leqslant \frac{2}{a} \|R\| \|X\| \|Y\|.$$

Proof. – We have $\overline{\mathrm{Tor}}(X, Y) = -\bar{S}(X, Y) + \bar{S}(Y, X)$. Now Lemma 3.9 applies. □

The curvature tensor $\bar{R}$ of $\bar{\nabla}$ may not be defined since $\bar{\nabla}$ may only be continuous. Nevertheless, the statement $\bar{R} = 0$ is still meaningful.

LEMMA 3.15. – *In the sense of parallel translation, $\bar{R} = 0$. That is, parallel translation with respect to $\bar{\nabla}$ depends only on homotopy classes of curves.*

Proof. – It suffices to prove this locally along a cross section N_τ. Let $p = (\tau, x) \in N_\tau$ and $c = c(s)$, $0 \leqslant s \leqslant 1$, be a C^1 loop in N at x which is homotopically trivial. Let X be a $\bar{\nabla}$-parallel vector field along the curve $c_\tau(s) = (\tau, c(s))$ in N_τ. Since c is C^1, X solves an ordinary differential equation with continuous coefficients; hence X is also C^1.

Let $\gamma(t, s) = (t, c(s))$ and extend X along γ by parallel translation in the t-direction (which coincides with respect to ∇ and $\bar{\nabla}$). Then $\bar{\nabla}_{\partial_s} X = 0$ by (3.12), hence $\bar{\nabla} X = 0$ along γ. Therefore, by (3.4) and Lemma 3.9,

$$\|\nabla_{\partial_s} X\|(t, s) = \|\bar{\nabla}_{\partial_s} X - \nabla_{\partial_s} X\|(t, s) \leqslant \mathrm{const} \cdot \mathrm{e}^{-a(t-\tau)}.$$

Therefore, if h_t denotes parallel translation along the curve $c_t(s) = (t, c(s))$ with respect to ∇, then

$$\|h_t(X(t, 0)) - X(t, 1)\| \leqslant \mathrm{const} \cdot \mathrm{e}^{-a(t-\tau)}.$$

Now choose a proper C^1 contraction C of c_τ in N_τ to the point curve and denote by A_τ the area of this contraction. Then $C_t = f_{t,\tau} \circ C \circ f_{\tau,t}$ is a contraction of c_t, and by (3.4), the area A_t of C_t is bounded by

$$A_t \leqslant \mathrm{e}^{-2a(t-\tau)} A_\tau.$$

Now

$$\|X(t, 0) - h_t(X(t, 0))\| \leqslant A_t \|R\| \|X\|,$$

see Inequality 6.2.1 in [12]. On the other hand, $X(t, 1) - X(t, 0)$ is parallel along γ_p, hence $\|X(t, 1) - X(t, 0)\|$ is independent of t, and hence $X(t, 1) = X(t, 0)$. □

COROLLARY 3.16. – *Suppose $V \subset N$ is open and simply connected. Then there is an orthonormal frame $(T, X_1, \ldots, X_n)$ of TU over $(0, \infty) \times V$ which is parallel with respect to $\bar{\nabla}$.*

This verifies Axiom 1 for TU, hence also for any geometric bundle $E \to U$.

In the proof of the following lemma, we use one of the crucial ideas and results in Gromov's proof of his celebrated theorem on almost flat manifolds [17].

LEMMA 3.17. – *The holonomy of $\bar{\nabla}$ is finite.*

Proof. – The second fundamental form S of the cross sections is uniformly bounded, see (3.6), hence the curvature tensor of the cross sections is defined in the sense of the Gauss formula and is uniformly bounded. Hence for each t, the metric on N_t can be C^1-approximated by a C^2-metric with uniformly bounded curvature, where the bound does not depend on t. Hence the standard arguments from comparison geometry apply to the cross sections.

The result of Gromov we use is as follows: Let $\theta > 0$. Then, by Proposition 3.4.1 and Corollary 3.4.2 in [12], there is $\rho > 0$ such that for all t sufficiently large, the ∇-holonomy along a loop in N_t of length $< \rho$ rotates vectors by at most θ if it rotates vectors by at most $1/3$.

Now let $c = c(s)$ be a loop in N. Then the $\bar{\nabla}$-holonomy $\bar{h}_t$ along the loop $c_t(s) = (t, c(s))$ in N_t is parallel in the t-direction. We claim that $\bar{h}_t = \mathrm{id}$ if $\bar{h}_t$ rotates vectors by at most $1/3$. For this we note that the length of c_t is as small as we please if only t is sufficiently large, and that the difference between the holonomy along c_t with respect to ∇ and $\bar{\nabla}$ respectively can be estimated in terms of the length of c_t and the norm of $\bar{S} = \nabla - \bar{\nabla}$. Hence the above result of Gromov applies and shows that $\bar{h}_t$ is as close to the identity as we please if only t is sufficiently large. On the other hand, $\bar{h}_t$ is parallel in the t-direction, hence $\bar{h}_t = \mathrm{id}$.

Let $p \in U$ and denote by $\bar{H}_p$ the image of the holonomy representation of $\pi_1(U, p) \cong \pi_1(N)$ in the orthogonal group $O(T_pU)$. By the above, if B is the ball of radius $1/3$ about the identity in $O(T_pU)$, then $\bar{H}_p \cap B = \{\mathrm{id}\}$. Now $\bar{H}_p$ is a subgroup of $O(T_pU)$, hence $\bar{H}_p$ is finite. □

COROLLARY 3.18. – *There is a finite cover $\tilde{N}$ of N such that the holonomy of $\bar{\nabla}$ on $\tilde{U} = (0, \infty) \times \tilde{N}$ is trivial. In other words, $\tilde{U}$ admits a global orthonormal frame which is parallel with respect to $\bar{\nabla}$.*

Example 3.19 (Doubly warped products). – The warped metric $dt^2 + \mathrm{e}^{-2at} g_n$ on $\mathbb{R} \times \mathbb{R}^n$, where g_n is the Euclidean metric on $\mathbb{R}^n$, is a model for hyperbolic space of curvature $-a^2$. We consider the metric:

$$dt^2 + \mathrm{e}^{-2at} g_m + \mathrm{e}^{-2bt} g_n$$

on $U = (0, \infty) \times \mathbb{R}^m \times \mathbb{R}^n$ with $a, b > 0$. Let $\alpha_0 T + X_0 + Y_0$ and $\alpha_1 T + X_1 + Y_1$ be vector fields on U, where X_0 and X_1 are tangent to the factor $\mathbb{R}^m$ and Y_0 and Y_1 are tangent to the factor $\mathbb{R}^n$. Then using the formulas in Lemma 7.4 in [7], a straightforward computation shows that:

$$\begin{aligned}
&\big\langle R(\alpha_0 T + X_0 + Y_0, \alpha_1 T + X_1 + Y_1)(\alpha_1 T + X_1 + Y_1), \alpha_0 T + X_0 + Y_0\big\rangle \\
&\quad = -a^2\big\{\|\alpha_0 T + X_0\|^2 \|\alpha_1 T + X_1\|^2 - \langle \alpha_0 T + X_0, \alpha_1 T + X_1\rangle^2\big\} \\
&\qquad - b^2\big\{\|\alpha_0 T + Y_0\|^2 \|\alpha_1 T + Y_1\|^2 - \langle \alpha_0 T + Y_0, \alpha_1 T + Y_1\rangle^2\big\} \\
&\qquad - ab\big\{\|X_0\|^2 \|Y_1\|^2 + \|X_1\|^2 \|Y_0\|^2 - 2\langle X_0, X_1\rangle\langle Y_0, Y_1\rangle\big\}.
\end{aligned}$$

The first term on the right-hand side reflects the fact that the submanifolds

$$H_y := (0, \infty) \times \mathbb{R}^m \times \{y\}, \quad y \in \mathbb{R}^n,$$

are totally geodesic hyperbolic spaces of curvature $-a^2$, the second term that the submanifolds

$$H_x := (0, \infty) \times \{x\} \times \mathbb{R}^n, \quad x \in \mathbb{R}^m,$$

are totally geodesic hyperbolic spaces of curvature $-b^2$. The third term mediates between the two first terms. It is linear in a and b and gives constant sectional curvature $-a^2 = -b^2$ for

$a = b$. It follows that the sectional curvature satisfies

$$-b^2 \leqslant K \leqslant -a^2 < 0$$

if $a \leqslant b$. Now let $p = (\tau, x, y) \in U$ and let $u, v \in T_pU$ be perpendicular to $T(p)$. Then since the submanifolds H_y and H_x are totally geodesic hyperbolic spaces of curvature $-a^2$ and $-b^2$, respectively, we have:

$$\bar{S}(u,v) = \begin{cases} 0 & \text{if } u \text{ is tangent to } H_a \text{ and } v \text{ to } H_b, \\ a\langle u,v\rangle T & \text{if } u \text{ and } v \text{ are tangent to } H_a, \\ b\langle u,v\rangle T & \text{if } u \text{ and } v \text{ are tangent to } H_b. \end{cases}$$

Hence $\bar{S} = S$ and, therefore, $\bar{\nabla} = \nabla^t$. Now for constant vector fields v on $\mathbb{R}^m$ and w on $\mathbb{R}^n$, the restriction of the vector fields

$$X(t,x,y) = e^{at}(0, v, 0) \quad \text{and} \quad Y(t,x,y) = e^{bt}(0,0,w)$$

to the hypersurfaces $\{t = \text{constant}\}$ are parallel with respect to ∇^t. It follows that they are globally parallel on U with respect to $\bar{\nabla}$.

This example will be of importance in our discussion of the pinching constants below.

Example 3.20 (The complex hyperbolic plane). – Let H be the Heisenberg group of dimension 3. The Lie algebra $\mathfrak{h}$ of H admits a basis of left invariant vector fields X, Y, and Z with

$$[X, Y] = 2Z$$

and such that the other Lie brackets between X, Y, and Z vanish. In particular, H is 2-step nilpotent. Let $f_t : H \to H$ be the automorphism such that

$$f_{t*}X = e^{-t}X, \qquad f_{t*}Y = e^{-t}Y, \quad \text{and} \quad f_{t*}Z = e^{-2t}Z,$$

and let $S = \mathbb{R} \times H$ be the corresponding semidirect product. Then S is solvable.

Define an S-invariant metric $g = dt^2 + g_t$ on S by requiring X, Y, and Z to be orthonormal. Then S is a model of the complex hyperbolic plane with sectional curvature in $[-4, -1]$. In fact, the product $S = \mathbb{R}H$ corresponds to the factor AN in the Iwasawa decomposition $G = KAN$ of the component of the identity G of the group of isometries of the complex hyperbolic plane. If M is a noncompact quotient of finite volume of S, then the ends of M are of the form $U = (0, \infty) \times N$, where N is a compact quotient of H.

A straightforward computation shows that the vector fields X, Y, and Z are parallel along the rays $\gamma = \gamma_p$ and that they are parallel with respect to $\bar{\nabla}$. Hence $\bar{\nabla}$ corresponds to the canonical flat connection on S.

The situation for the other symmetric spaces of negative curvature and their finite volume quotients is similar.

We now consider the restriction $\bar{\nabla}^t$ of $\bar{\nabla}$ to $E_t = E|N_t$. The kernel $K(t) = \ker \bar{\nabla}^t$ of $\bar{\nabla}^t$ consists of sections of E_t which are parallel with respect to $\bar{\nabla}^t$. It is a finite-dimensional vector space and the family of these spaces is parallel in the T-direction. For a C^1 section σ of E_t, the pointwise norm of $\bar{\nabla}^t\sigma$ is given by:

$$\|\bar{\nabla}^t\sigma\|^2 = \sum \|\bar{\nabla}^t_{X_i}\sigma\|^2, \tag{3.21}$$

where $X_1, \ldots, X_n$ is a local orthonormal frame of N_t.

Denote by $\Pi_{t,\tau}$ parallel translation from E_τ to E_t along the rays γ_p, $p \in N_\tau$. The next lemma implies that the non-zero eigenvalues of $(\bar\nabla^t)^*\bar\nabla^t$ grow exponentially if the sectional curvature of U is sufficiently pinched.

LEMMA 3.22. – *Assume that* $(n+1)a - (n-1)b =: \delta > 0$. *Let* σ_τ *be a* C^1*-section of* E_τ *and assume that*

$$\left\|\bar\nabla^\tau \sigma_\tau\right\|_\tau \geqslant \lambda \|\sigma_\tau\|_\tau$$

for some $\lambda > 0$. *Let* $t > \tau$ *and set* $\sigma_t = \Pi_{t,\tau} \circ \sigma_\tau \circ f_{\tau,t}$. *Then*

$$\left\|\bar\nabla^t \sigma_t\right\|_t \geqslant e^{\delta(t-\tau)/2} \lambda \|\sigma_t\|_t.$$

Proof. – Let $p = (\tau, x) \in N_\tau$ and $v_1, \ldots, v_n$ be an orthonormal basis of $T_p N_\tau$ such that their images $w_1, \ldots, w_n$ under the differential of $f_{t,\tau}$ are pairwise orthogonal at $(t,x) \in N_t$. Let $\alpha_i = \ln\|w_i\|$ and assume $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$. By (3.4),

$$e^{-b(t-\tau)} \leqslant e^{\alpha_i} \leqslant e^{-a(t-\tau)}.$$

Choose a special frame (σ_I) of E. Then $\sigma_\tau = \sum \varphi_I \sigma_I(\tau,\cdot)$ with appropriate functions $\varphi_I = \varphi_I(x)$. Since $\bar\nabla_T \sigma_I = 0$, we have $\sigma_t = \sum \varphi_I \sigma_I(t,\cdot)$ and hence

$$\begin{aligned}
\left\|\bar\nabla^t \sigma_t(t,x)\right\|^2 j_{t,\tau} &= \sum \left\|\left\{e^{-\alpha_i} w_i(\varphi_I)\right\}\sigma_I(t,x)\right\|^2 e^{\alpha_1+\cdots+\alpha_n} \\
&\geqslant \sum \left\|\left\{w_i(\varphi_I)\right\}\sigma_I(t,x)\right\|^2 e^{-\alpha_1+\alpha_2+\cdots+\alpha_n} \\
&= \sum \left\|\left\{v_i(\varphi_I)\right\}\sigma_I(\tau,x)\right\|^2 e^{-\alpha_1+\alpha_2+\cdots+\alpha_n} \\
&= \left\|\bar\nabla^\tau \sigma_\tau(\tau,x)\right\|^2 e^{-\alpha_1+\alpha_2+\cdots+\alpha_n} \\
&\geqslant \left\|\bar\nabla^\tau \sigma_\tau(\tau,x)\right\|^2 e^{(a-(n-1)b)(t-\tau)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\bar\nabla^t \sigma_t\right\|_t^2 &= \int_{N_\tau} \left(\left\|\bar\nabla^t \sigma_t\right\|^2 \circ f_{t,\tau}\right) j_{t,\tau} \\
&\geqslant e^{(a-(n-1)b)(t-\tau)} \int_{N_\tau} \left\|\bar\nabla^\tau \sigma_\tau\right\|^2 \\
&\geqslant e^{(a-(n-1)b)(t-\tau)} \lambda^2 \int_{N_\tau} \|\sigma_\tau\|^2 \\
&\geqslant e^{((n+1)a-(n-1)b)(t-\tau)} \lambda^2 \int_{N_\tau} \left(\|\sigma_t\|^2 \circ f_{t,\tau}\right) j_{t,\tau} \\
&= e^{\delta(t-\tau)} \lambda^2 \|\sigma_t\|_t^2.
\end{aligned}$$

This is the asserted inequality. □

LEMMA 3.23. – *Let* U *be a cuspidal end and* $E \to U$ *be a geometric bundle. Let* $(\bar\nabla^t)^*\bar\nabla^t$ *be the connection Laplacian associated to* $\bar\nabla^t$. *Let* $\lambda_t^2 > 0$ *be the smallest nonzero eigenvalue of* $(\bar\nabla^t)^*\bar\nabla^t$. *Suppose that at least one of the following three conditions is satisfied:*

(1) *The holonomy of* $\bar{\nabla}$ *is finite.*
(2) $(n+1)a-(n-1)b>0$.
(3) $\|\nabla R\|$ *is uniformly bounded on* U.
Then there exists a positive constant c such that $\lambda_t \geqslant c\exp(ct)$.

Proof. – In the first case we can assume, by passing to a finite covering space $\tilde{U}=(0,\infty)\times\tilde{N}$ of U if necessary, that E admits a global special frame (σ_I). Then $(\bar{\nabla}^t)^*\bar{\nabla}^t$ is equal to $\operatorname{rk}E$ copies of the standard Laplace operator of N_t, acting on the different components φ_I of sections with respect to the frame. The asserted estimate then is immediate from the corresponding and well known eigenvalue estimate of Li and Yau, see Theorem 7 in [25] (and [32] for an improved estimate).

In the second case we recall that λ_t can be characterized as the maximum of all possible

$$\lambda_V=\inf\{\|\bar{\nabla}^t\sigma\|^2 \mid \sigma\in V,\ \|\sigma\|_t=1\},$$

where the maximum is taken over all complementary closed subspaces V of the kernel of $\bar{\nabla}^t$ in $L^2(E_t)$. Hence Lemma 3.22 applies.

In the third case we recall that a simplification of the arguments in [20] shows that the ratios $j_{t,\tau}(x)/j_{t,\tau}(y)$ of the Jacobians $j_{t,\tau}$ of the shift maps $f_{t,\tau}$ are uniformly bounded in terms of a, b and a bound on $\|\nabla R\|$. □

Lemma 3.23 verifies Axiom 2 for all geometric bundles which satisfy its assumptions. In particular, for these bundles we obtain the characterization of the essential spectrum given in Theorem 2.13.

4. The Clifford bundle

Throughout this section we assume that $E=\mathbb{C}l\,U\otimes F\cong\Lambda^*U\otimes F$, where F is a flat Hermitian bundle with a compatible connection, and that $D=d+d^*$. Our main result is Lemma 4.8 which verifies Axiom 3′ if the dimension $n+1$ of M is even and along all the ends, the sectional curvature satisfies the pinching condition $(n+1)a>(n-1)b$. The following estimate is a consequence of Lemma 2.15 and Lemma 4.8 and shows that D is Fredholm under these assumptions.

THEOREM 4.1. – *Suppose that* $\dim U=n+1=2l$ *is even and that the sectional curvature on* U *satisfies the pinching condition*

$$la-(l-1)b=\delta>0.$$

Then for any $\varepsilon>0$ *there is* $t>0$ *such that*

$$\|D\sigma\|\geqslant\frac{\delta-\varepsilon}{2}\|\sigma\|$$

for any C^1 *section* σ *of* E *with compact support in* $U_t=(t,\infty)\times N$.

Example 4.2. – Suppose that $\dim U=n+1=2l$ and consider the case $F=\mathbb{C}$. Suppose furthermore that $(n+1)a\geqslant(n-1)b$, where $0<a<b$. Let M_0 be a compact $2l$-dimensional manifold with boundary $\partial M\cong T^n$, the n-dimensional torus. Attach an end $U=(0,\infty)\times T^n$ to M_0 and consider a Riemannian metric on $M=M_0\cup U$ whose restriction to U is the metric

considered in Example 3.19, where the m there equals l here and n there is equal to $l-1$. Let $v_1, \ldots, v_l$ be an orthonormal basis of the factor $\mathbb{R}^l$. Consider the multivector

$$X = \mathrm{e}^{alt} v_1 \wedge \cdots \wedge v_l$$

which has norm one and is parallel in the T-direction. Then the *tangential parts* $A_t X$ and $A_t(TX)$ respectively are zero.

Recall that the density of the volume element of the warped metric is $\exp(-\kappa t)$ with $\kappa = la + (l-1)b$. Hence the linear map

$$\Psi : L^2(0,\infty) \otimes \mathbb{C}^2 \to L^2(E), \quad (u,v) \mapsto \mathrm{e}^{\kappa t/2}(uX + vTX),$$

preserves L^2-norms. We obtain a Dirac system,

$$\Psi^{-1} \mathring{D} \Psi = \gamma(\partial_t - \tau\delta/2),$$

where γ and τ are as in Example 1.15 and $\delta = la - (l-1)b$. In particular, $\pm\delta/2$ belong to $\operatorname{spec}_e D$. Hence Theorem 4.1 is optimal with respect to the pinching constant and D is not Fredholm if $\delta = 0$.

For a special frame $T = X_0, X_1, \ldots, X_n$ of TU, a naturally associated special frame of E consists of sections of the form $X_I \otimes \Phi_J = (X_{i_1} \cdots X_{i_k}) \otimes \Phi_J$ and $TX_I \otimes \Phi_J = (TX_{i_1} \cdots X_{i_k}) \otimes \Phi_J$, respectively, where I is a multiindex with $i_j \geqslant 1$ and (Φ_J) is a local parallel orthonormal frame of F.

As above, we denote by ∇^t the Levi-Civita connection of N_t. Then for the induced connection on $TU|N_t$ (viewed as a geometric bundle over N_t) and the Clifford bundle $\mathbb{C}\mathrm{l}\, U|N_t$ we have $\nabla^t T = 0$.

Recall the definition of the operators A_t, B_t in (1.10), (1.11) and $\bar{A}_t$, $\bar{B}_t$ in (2.5), (2.6). Now $\bar{A}_t + \bar{B}_t = A_t + B_t$ and $\bar{P}_0 \bar{A}_t \bar{P}_0 = 0$, hence:

$$\bar{P}_0 \bar{B}_t \bar{P}_0 = \bar{P}_0 A_t \bar{P}_0 + \bar{P}_0 B_t \bar{P}_0. \tag{4.1}$$

For a section σ we call $\bar{P}_0 A_t \bar{P}_0 \sigma$ the *tangential part* and $\bar{P}_0 B_t \bar{P}_0 \sigma$ the *normal part*. The tangential part is determined by the interior geometry of the cross sections, the normal part by their exterior geometry.

LEMMA 4.4. – *If $n \leqslant 2$, or if $n \geqslant 3$ and $2a > b$, then* $\dim\ker A_t = \dim\ker \bar{A}_t$ *and*

$$\|\bar{P}_0 A_t \bar{P}_0\|_t \to 0 \quad \textit{as } t \to \infty.$$

Remark 4.5. – The example of the complex hyperbolic plane shows that the pinching assumption $2a > b$ is optimal for $n \geqslant 3$, see Example 3.20.

Proof of Lemma 4.4. – If $n = 1$, then N_t is a circle. If $n = 2$, then N_t is a torus (since M is orientable). If $n \geqslant 3$, then $a^2/b^2 > 1/4$. Then Gromov's theorem on almost flat manifolds applies and shows that N_t is a finite quotient of a torus, see Corollary 1.5.2 in [12]. Hence in each of the cases we may assume, by passing to a finite cover if necessary, that $U = (0,\infty) \times N$, with $N = T^n$.

Consider the splitting $E = E^+ + E^-$ into even and odd part, which is parallel with respect to ∇ and $\bar{\nabla}$. Furthermore, the restrictions E_t^+ and E_t^- are each isomorphic to $\mathbb{C}\mathrm{l}\, N_t \otimes F$ such that

A_t corresponds to $d_t + d_t^*$, where d_t is exterior differentation on N_t. Hence by Hodge theory,

$$\dim\ker A_t = 2\sum b_i(N;F).$$

Now $N = T^n$ is a torus of dimension n, hence $\sum b_i(N;F)$ is equal to 2^n times the dimension of the trivial representation in the representation of $\pi_1(N)$ defining F.

On the other hand, for t sufficiently large, we have

$$\dim\ker \bar{A}_t = \dim\ker \bar{\nabla}^t,$$

and the latter consists of globally $\bar{\nabla}$-parallel sections of E_t. Now by Lemma 3.17 the latter is equal to $\operatorname{rk}\mathbb{C}\mathrm{l}\, U$, again times the dimension of the trivial representation in the representation of $\pi_1(N)$ defining F. Hence $\dim\ker A_t = \dim\ker \bar{A}_t$.

By Lemma 3.23, there is a positive function $\lambda = \lambda_t$ with $\lim_{t\to\infty}\lambda_t = \infty$ such that

$$\|\bar{A}_t\sigma\| \geqslant \lambda_t\|\sigma\|$$

for all σ in the domain of $\bar{A}_t$ and perpendicular to the kernel of $\bar{A}_t$. Moreover, $\|A_t - \bar{A}_t\|$ is uniformly bounded, hence Corollary A.7 applies and gives

$$\|\bar{P}_0 - P_0\|_t \to 0 \quad \text{as } t \to \infty.$$

Since

$$\bar{P}_0 A_t \bar{P}_0 = (\bar{P}_0 - P_0)(A_t - \bar{A}_t)\bar{P}_0,$$

the lemma is proved. □

LEMMA 4.6. – *Let $p \in U$. Choose a special frame $T = X_0, X_1, \dots, X_n$ of TU such that $S_{X_i}X_j = \delta_{ij}\kappa_j T$ at p. Consider an associated special frame $X_I \otimes \Phi_J$, $TX_I \otimes \Phi_J$ of E, where $I = i_1 \cdots i_k$ is a multiindex with $i_j \geqslant 1$ and (Φ_J) is a parallel local orthonormal frame of E. Then at p,*

$$B_t(X_I \otimes \Phi_J) = \frac{1}{2}\Big\{\sum_{i\notin I}\kappa_i - \sum_{i\in I}\kappa_i\Big\} X_I \otimes \Phi_J,$$
$$B_t(TX_I \otimes \Phi_J) = \frac{1}{2}\Big\{\sum_{i\in I}\kappa_i - \sum_{i\notin I}\kappa_i\Big\} TX_I \otimes \Phi_J.$$

Proof. – At p we have

$$\begin{aligned}\sum TX_i S_{X_i}(X_I \otimes \Phi_J) &= \sum_{i,j}(TX_iX_{i_1}\cdots S_{X_i}X_{i_j}\cdots X_{i_k}) \otimes \Phi_J \\ &= \sum_j \kappa_{i_j}(TX_{i_j}X_{i_1}\cdots T\cdots X_{i_k}) \otimes \Phi_J \\ &= \Big(\sum_j \kappa_{i_j}\Big) X_I \otimes \Phi_J = \Big(\sum_{i\in I}\kappa_i\Big) X_I \otimes \Phi_J.\end{aligned}$$

Therefore,

$$B_t(X_I \otimes \Phi_J) = \frac{\kappa}{2} X_I \otimes \Phi_J - \Big(\sum_{i\in I}\kappa_i\Big) X_I \otimes \Phi_J = \frac{1}{2}\Big\{\sum_{i\notin I}\kappa_i - \sum_{i\in I}\kappa_i\Big\} X_I \otimes \Phi_J.$$

This proves the first equation. The second follows by a similar computation, observing that $\nabla_{X_i} T = -\kappa_i T$ at p. □

We assume now that $\dim U = n+1 = 2l$ is even. We let $\bar{\beta}$ be the field of endomorphism of the Clifford bunlde $E \to U$ such that in terms of a special frame $X_I \otimes \Phi_J$, $(TX_I) \otimes \Phi_J$ of E as above

$$\begin{aligned} \bar{\beta}(X_I \otimes \Phi_J) &= \begin{cases} X_I \otimes \Phi_J & \text{if } |I| < l, \\ -X_I \otimes \Phi_J & \text{if } |I| \geqslant l, \end{cases} \\ \bar{\beta}(TX_I \otimes \Phi_J) &= \begin{cases} TX_I \otimes \Phi_J & \text{if } |I| \geqslant l, \\ -TX_I \otimes \Phi_J & \text{if } |I| < l. \end{cases} \end{aligned} \tag{4.7}$$

In other words, $\bar{\beta} = \operatorname{sign} B$.

Note that $\bar{\beta}$ only depends on the degree $k = |I|$ and on the absence or non-absence of the factor T. Now T is parallel with respect to $\overset{\circ}{\nabla}{}^t$ and $\bar{\nabla}$. It follows that $\bar{\beta}$ is well defined independently of the choice of frame and that it is parallel with respect to ∇^t and $\bar{\nabla}$. The following lemma establishes the last requirement of Axiom 3′.

LEMMA 4.8. – *Suppose that* $\dim U = n+1 = 2l$ *and that* $\delta = la - (l-1)b > 0$. *Then for any* $\varepsilon > 0$ *there is* $t > 0$ *such that*:

$$(\bar{P}_0 \bar{B}_t \bar{P}_0 \sigma, \bar{\beta} \bar{P}_0 \sigma)_t \geqslant \frac{\delta - \varepsilon}{2} \|\bar{P}_0 \sigma\|_t^2$$

for any C^1 *section* σ *of* E *with compact support in* U_t.

Proof. – We first note that $\bar{\beta}\bar{P}_0 = \bar{P}_0\bar{\beta}$. Hence it suffices to estimate $(\bar{B}_t \bar{P}_0 \sigma, \bar{\beta}\bar{P}_0\sigma)$. Now $a \leqslant \kappa_i \leqslant b$ and $n = 2l-1$ is odd, hence

$$\left| \sum_{i \in I} \kappa_i - \sum_{i \notin I} \kappa_i \right| \geqslant la - (l-1)b$$

for each multiindex I. Moreover, by Lemma 4.4, the error term $(\bar{P}_0 A_t \bar{P}_0 \sigma, \bar{\beta}\bar{P}_0\sigma)_t$ from (4.1) is as small as we please if only t is sufficiently large. □

Remark 4.9. – By the work of Borel and Casselman [9], the Dirac operator on the Clifford bundle is a Fredholm operator if M is an even-dimensional locally symmetric space of finite volume and negative curvature. For each cuspidal end U of such a space M, $\bar{B}$ is parallel in the direction of T. It follows that $\bar{\beta} := \operatorname{sign} \bar{B}$ is an operator as required in our Axiom 3, hence the method developed in Section 2 applies to M.

5. The Hodge Laplacian

In this section we improve results of Donnelly and Xavier in [15]. Let U be an open subset of a Riemannian manifold M of dimension $n+1$ and $E = \Lambda^*(TU) \otimes F$, where F is a Hermitian bundle with a compatible connection.

For the following, compare page 123 in [24]. The Riemannian metric induces a contraction "$\llcorner$" on E,

$$\langle X \llcorner \sigma, \eta \rangle := \langle \sigma, X \wedge \eta \rangle, \tag{5.1}$$

where σ and η are sections of E. For a vector field X and a multivector $X_I = X_{i_1} \wedge \cdots \wedge X_{i_k}$, we have

$$X \llcorner (X_I \otimes \Phi) = \sum (-1)^{j+1} \langle X_{i_j}, X \rangle (X_{i_1} \wedge \cdots \wedge \hat{X}_{i_j} \wedge \cdots \wedge X_{i_k}) \otimes \Phi.$$

Furthermore, if σ is a section of $\Lambda^k(TU)$ and η a section of E, then

$$X \llcorner (\sigma \wedge \eta) = (X \llcorner \sigma) \wedge \eta + (-1)^k \sigma \wedge (X \llcorner \eta).$$

We identify $\mathbb{C}l(U) = \Lambda^*(TU)$. With respect to this identification, Clifford multiplication by a vector field X from the left and right, respectively, is given by

$$X \cdot \sigma = X \wedge \sigma - X \llcorner \sigma,$$
$$\sigma \cdot X = (-1)^k (X \wedge \sigma + X \llcorner \sigma) \quad \text{on } \Lambda^k(TU) \otimes F.$$

Exterior differentiation d and its adjoint d^* are given by

$$d\sigma = \sum X_i \wedge \nabla_{X_i} \sigma \quad \text{and} \quad d^*\sigma = -\sum X_i \llcorner \nabla_{X_i} \sigma,$$

where $X_0, \ldots, X_n$ is a local orthonormal frame. Clifford multiplication from the right is well defined and leads to the *right handed* Dirac operator:

$$\hat{D}\sigma = \sum \nabla_{X_i} \sigma \cdot X_i.$$

We have

$$D = d + d^* \quad \text{and} \quad \hat{D}\sigma = (-1)^k (d - d^*) \quad \text{on } \Lambda^k(TU) \otimes F.$$

If F is flat, then $d^2 = (d^*)^2 = 0$ and

$$D^2 = \hat{D}^2 = \Delta,$$

where Δ is the *twisted Hodge Laplacian.*

Let T be a prefered vector field on U and ∇T be the covariant derivative of T. Recall that ∇T is skewsymmetric if T is a Killing field and that ∇T is symmetric if T is the gradient of a function on U locally. We are interested in the symmetric part of ∇T.

Let p be a point in U. The quadratic form $q_T(v) = \langle \nabla_v T, v \rangle$ on T_pU only depends on the symmetric part of $\nabla T(p)$. Let $\alpha_0 \geqslant \alpha_1 \geqslant \cdots \geqslant \alpha_n$ be the characteristic values of q_T and set:

$$\delta_k = \delta_k(p) = \sum_{i > n-k} \alpha_i - \sum_{i \leqslant n-k} \alpha_i, \quad 0 \leqslant k \leqslant n. \tag{5.2}$$

Then the divergence $\operatorname{div} T = -\delta_0$. If T is the gradient of a function f, then $\Delta f = \delta_0$.

In this section we use an index U to denote the L^2-norm on U. Our main result is as follows:

THEOREM 5.3. – *Suppose $\|T\| \leqslant 1$. Then*

$$\|D\sigma\|_U \|\sigma\|_U, \|\hat{D}\sigma\|_U \|\sigma\|_U \geqslant \frac{1}{2} \int_U \delta_k \|\sigma\|^2$$

for any differential form σ of degree k or $n + 1 - k$ with compact support in U.

This improves the main Theorem 2.2 in [15]. The applications of that theorem in [15] can be improved accordingly. In this work we concentrate on cuspidal ends as before and let $T = \partial_t$. Then $\|T\| = 1$, $\alpha_0 = 0$ and $-a \geqslant \alpha_i = -\kappa_i \geqslant -b$ for all $i \geqslant 1$. Hence

$$\delta_k \geqslant (n-k)a - kb.$$

In conclusion, we have the following application to cuspidal ends.

COROLLARY 5.4. – *Suppose F is flat and $(n-k)a - kb = \delta > 0$ (where $k < n/2$). Then we have*:

$$\|\Delta\sigma\|_U \geqslant \frac{\delta^2}{4}\|\sigma\|_U$$

for any differential form σ of degree k or $n+1-k$ with compact support in U.

In the special case $k = 0$, Corollary 5.4 gives the estimate of McKean in [28]. It improves Theorem 5.3 in [15] and is optimal with respect to the pinching constants, see Example 5.5 below. The cases $k = (n+1)/2$ if $n+1$ is even, respectively $k = n/2, n/2+1$ if $n+1$ is odd, are not covered. In fact, in the latter case the essential spectrum of Δ on $L^2(\Lambda^k(TM))$ is equal to $[0,\infty)$ for noncompact quotients of finite volume of the real hyperbolic space H^{n+1}, see Example 1.15.

Example 5.5. – Let $k < n/2$ and consider doubly warped products as in Example 3.19, where the dimensions m and n there correspond to $n-k$ and k here, respectively. Let $w_1, \dots, w_k$ be an orthonormal basis of the factor $\mathbb{R}^k$. Consider the multivector

$$Y = \mathrm{e}^{kbt}\, w_1 \wedge \cdots \wedge w_k,$$

which has norm one and is parallel in the T-direction. Recall that the density of the volume element of the warped metric is $\exp(-\kappa t)$ with $\kappa = (n-k)a + kb$. Hence the linear map

$$\Psi : L^2(0,\infty) \otimes \mathbb{C}^2 \to L^2(E), \quad (u,v) \mapsto \mathrm{e}^{\kappa t/2}(uY + vTY),$$

preserves L^2-norms. We obtain a Dirac system,

$$\Psi^{-1} D \Psi = \gamma(\partial_t + \tau\delta/2),$$

where γ and τ are as in Example 1.15 and $\delta = (n-k)a - kb$. In particular,

$$\Psi^{-1} D^2 \Psi = -\partial_t^2 + \delta^2/4,$$

hence the estimate in Corollary 5.4 is optimal with respect to the pinching constant.

Proof of Theorem 5.3. – Let σ be a C^1 section of $\Lambda^k(TU) \otimes F$ with compact support. Let V be the vector field on U such that:

$$\langle V, W\rangle = \langle T \llcorner \sigma, W \llcorner \sigma\rangle \tag{5.6}$$

for any vector field W. In a first step, we compute the divergence of V. To that end, fix $p \in U$. Choose a local orthonomal frame $X_0, X_1, \dots, X_n$ in a neighborhood of p such that $\nabla X_i(p) = 0$ for all i. Then the divergence of V in p is given by:

$$\begin{aligned}\operatorname{div} V &= \sum X_i\langle T\llcorner\sigma, X_i\llcorner\sigma\rangle\\ &= \sum\big\langle(\nabla_{X_i}T)\llcorner\sigma, X_i\llcorner\sigma\big\rangle + \sum\big\langle T\llcorner(\nabla_{X_i}\sigma), X_i\llcorner\sigma\big\rangle - \langle T\llcorner\sigma, d^*\sigma\rangle\\ &=: \mathrm{I} + \mathrm{II} - \langle T\llcorner\sigma, d^*\sigma\rangle.\end{aligned}$$

To compute the first term I on the right-hand side, decompose $\nabla T = A + B$, where A is the symmetric part of ∇T and B the skewsymmetric part. Then

$$\mathrm{I} = \sum\big\langle(AX_i)\llcorner\sigma, X_i\llcorner\sigma\big\rangle + \sum\big\langle(BX_i)\llcorner\sigma, X_i\llcorner\sigma\big\rangle.$$

Using the normal form of skewsymmetric endomorphisms, it is easy to see that the second term on the right-hand side is purely imaginary. Now choose the frame (X_i) such that $AX_i = \alpha_i X_i$ in p. Then in p,

$$\operatorname{Re}\mathrm{I} = \sum\alpha_i\langle X_i\llcorner\sigma, X_i\llcorner\sigma\rangle = \sum\alpha_i\|X_i\llcorner\sigma\|^2.$$

To compute the second term II, we choose the frame (X_i) such that T is a multiple of X_0 in p. Then in p,

$$\begin{aligned}\mathrm{II} &= \sum\big\langle X_i\wedge(T\llcorner\nabla_{X_i}\sigma), \sigma\big\rangle\\ &= -\sum\big\langle T\llcorner(X_i\wedge\nabla_{X_i}\sigma), \sigma\big\rangle + \big\langle X_0\wedge(T\llcorner\nabla_{X_0}\sigma), \sigma\big\rangle + \big\langle T\llcorner(X_0\wedge\nabla_{X_0}\sigma), \sigma\big\rangle\\ &= -\langle T\llcorner d\sigma, \sigma\rangle + \langle\nabla_T\sigma, \sigma\rangle = -\langle d\sigma, T\wedge\sigma\rangle + \langle\nabla_T\sigma, \sigma\rangle.\end{aligned}$$

Now σ has compact support, hence the integral of the divergence $\operatorname{div} V$ of V vanishes. Therefore,

$$\begin{aligned}&\operatorname{Re}\big\{(d\sigma, T\wedge\sigma)_U + (T\llcorner\sigma, d^*\sigma)_U\big\}\\ &\quad= \operatorname{Re}\bigg\{\int\sum\alpha_i\|X_i\llcorner\sigma\|^2 + \bigg(\Big(\nabla_T + \frac{1}{2}\operatorname{div} T\Big)\sigma, \sigma\bigg)_U - \frac{1}{2}\int\operatorname{div} T\,\|\sigma\|^2\bigg\}.\end{aligned}$$

Here the integrand in the first term on the right-hand side has to be understood pointwise as explained above. The second term is imaginary since $\nabla_T + \frac{1}{2}\operatorname{div} T$ is skew Hermitian. Hence we obtain the following version of the integral formula (2.7) of Donnelly and Xavier in [15],

$$\operatorname{Re}\big\{(d\sigma, T\wedge\sigma)_U + (T\llcorner\sigma, d^*\sigma)_U\big\} = \int_U\bigg\{\sum_i\alpha_i\|X_i\llcorner\sigma\|^2 - \frac{1}{2}\operatorname{div} T\,\|\sigma\|^2\bigg\}. \tag{5.7}$$

Pointwise we have $\langle d\sigma, T\llcorner\sigma\rangle = \langle d^*\sigma, T\wedge\sigma\rangle = 0$ since the degrees of the factors are different in each case. Hence the left-hand side of (5.7) is equal to:

$$\operatorname{Re}(d\sigma + d^*\sigma, T\wedge\sigma + T\llcorner\sigma)_U = (-1)^k\operatorname{Re}(D\sigma, \sigma\cdot T)_U$$

and also to

$$\operatorname{Re}(d\sigma - d^*\sigma, T\wedge\sigma - T\llcorner\sigma)_U = (-1)^k\operatorname{Re}(\hat{D}\sigma, T\cdot\sigma)_U.$$

As for the right-hand side of (5.7), we express σ locally as a linear combination, $\sigma = \sum\sigma_{IJ}X_I\otimes\Phi_J$. Here I denotes a multiindex, $I = (i_1, \ldots, i_k)$ with $i_1 < i_2 < \cdots < i_k$, and $X_I = X_{i_1}\wedge\cdots\wedge X_{i_k}$ is the corresponding multivector. Furthermore, Φ_J is a local orthonormal frame of F. Now

$$\sum_i\alpha_i\big\|X_i\llcorner(X_I\otimes\Phi_J)\big\|^2 = \sum_{i\in I}\alpha_i$$

and $\operatorname{div} T = \sum \alpha_i$. Therefore,

$$\sum_i \alpha_i \big\| X_i \llcorner (X_I \otimes \Phi_J) \big\|^2 - \frac{1}{2} \operatorname{div} T = \frac{1}{2} \Big\{ \sum_{i \in I} \alpha_i - \sum_{i \notin I} \alpha_i \Big\}.$$

Hence the integral formula (5.7) can also be written in the following way,

$$\text{(5.8)} \qquad \operatorname{Re}(D\sigma, \sigma \cdot T)_U = \operatorname{Re}(\hat{D}\sigma, T \cdot \sigma)_U = \frac{(-1)^k}{2} \int_U \sum_{I,J} \Big\{ \sum_{i \in I} \alpha_i - \sum_{i \notin I} \alpha_i \Big\} |\sigma_{IJ}|^2.$$

Now the claimed estimate in Theorem 5.3 in the case where σ is a k-form is an immediate consequence of (5.8) since

$$\delta_k \leqslant \sum_{i \in I} \alpha_i - \sum_{i \notin I} \alpha_i$$

by the definition of δ_k. Now for a multiindex I of length $n+1-k$ we also have:

$$\delta_k \leqslant \sum_{i \notin I} \alpha_i - \sum_{i \in I} \alpha_i.$$

Hence the claimed estimate in Theorem 5.3 in the case where σ is a form of degree $n+1-k$ also follows from (5.8), where k is substituted by $n+1-k$. □

Remark 5.9. – Instead of considering the vector field V as in (5.6), it is also possible to consider the vector fields V and $\hat{V}$ defined by

$$\langle V, W \rangle = \langle \sigma \cdot T, W \cdot \sigma \rangle \quad \text{and} \quad \langle \hat{V}, W \rangle = \langle T \cdot \sigma, \sigma \cdot W \rangle.$$

Then integrating the divergence of V and $\hat{V}$ one gets the integral formula (5.8) for $\operatorname{Re}(D\sigma, \sigma \cdot T)_U$ and $\operatorname{Re}(\hat{D}\sigma, T \cdot \sigma)_U$, respectively. The integral of the divergence of the vector field V defined by:

$$\langle V, W \rangle = \langle T \cdot \sigma, W \cdot \sigma \rangle$$

does not give any information, it results in the equality $0 = 0$.

Appendix. A lemma from spectral theory

The following lemma (and its corollaries) are used several times in our considerations. It is basically well known and implicit, e.g., in Section 5 of [22, Chapter VI], but we have been unable to locate a reference for the following statement:

LEMMA A.1. – *Let A_j, $j = 1, 2$ be self-adjoint operators in some Hilbert space H, with common dense domain H_1. Consider $z_0 \in \mathbb{R} \setminus \operatorname{spec} A_1 \cup \operatorname{spec} A_2$ and put, for any self-adjoint operator A,*

$$\beta(A, z_0) := \operatorname{dist}\{z_0, \operatorname{spec} A\}.$$

Consider next a family, $B(z)$, of closed operators in H, defined for $\operatorname{Re} z = z_0$ *and satisfying for these z the relation*

$$\text{(A.2)} \qquad \mathcal{D}\big(B(z)\big) \supset H_1,$$

and the estimate

$$\big\| |A_1 - z_0|^{-\alpha_1} B(z) |A_2 - z_0|^{-\alpha_2} u \big\| \leqslant 2C_0 \|u\|, \tag{A.3}$$

for $u \in H_1$ and some numbers $\alpha_j \in [0, 1/2]$.

Then the integral

$$I := \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathrm{Re}\, z = z_0} (A_1 - z)^{-1} B(z) (A_2 - z)^{-1} \,\mathrm{d}z \tag{A.4}$$

is strongly convergent and defines a bounded operator in H with

$$\|I\| \leqslant C_0 \beta(A_1, z_0)^{\alpha_1 - 1/2} \beta(A_2, z_0)^{\alpha_2 - 1/2}. \tag{A.5}$$

If $\alpha_j < 1$ for some j, then I converges in the operator norm. If, in addition, A_j has compact resolvent, then I is compact.

Proof. – Since z_0 is real we may and will assume that $z_0 = 0$. Then, for $z \in \mathbb{R}$, we have the estimate

$$\begin{aligned} \big\| (A_j - \mathrm{i}z)^{-1} |A_j|^{\alpha_j} \big\| &= \sup\big\{ |\lambda|^{\alpha_j} |\lambda - \mathrm{i}z|^{-1};\ \lambda \in \operatorname{spec} A_j \big\} \\ &\leqslant \big(\beta(A_j, 0)^2 + z^2 \big)^{(\alpha_j - 1)/2}. \end{aligned} \tag{A.6}$$

Assume next that $\alpha_1 = 1/2 - 2\varepsilon$, for some $\varepsilon > 0$. Then we write

$$\begin{aligned} &(A_1 - \mathrm{i}z)^{-1} B(\mathrm{i}z) (A_2 - \mathrm{i}z)^{-1} \\ &\quad = |A_1|^{-\varepsilon} (A_1 - \mathrm{i}z)^{-1} |A_1|^{\alpha_1 + \varepsilon} |A_1|^{-(\alpha_1 + \varepsilon)} B(\mathrm{i}z) |A_2|^{-\alpha_2} (A_2 - \mathrm{i}z)^{-1} |A_2|^{\alpha_2} \\ &\quad =: |A_1|^{-\varepsilon} C(\varepsilon, z). \end{aligned}$$

We deduce from (A.3) and (A.6) that $C(\varepsilon, z)$ is bounded in H and has, for large $|z|$, the norm estimate

$$\big\| C(\varepsilon, z) \big\| \leqslant C_0 \big(\beta^2 + z^2 \big)^{-(1+\varepsilon)/2},$$

where $\beta := \min\{\beta(A_1, 0), \beta(A_2, 0)\}$. This implies the norm convergence in this case, and also the compactness of I if A_1^{-1} is compact.

Consider now the general case $\alpha_1, \alpha_2 \leqslant 1/2$. Then we form, with $(\cdot\,,\ \cdot)$ the scalar product in H, the bilinear form

$$c(u, v)(z) := \big((A_1 - \mathrm{i}z)^{-1} B(\mathrm{i}z) (A_2 - \mathrm{i}z)^{-1} u, v \big),$$

which is defined for $z \in \mathbb{R}$ and $u, v \in H$. In view of (A.3) and the Cauchy–Schwarz inequality it is enough to estimate, for $T \in \mathbb{R}$,

$$\begin{aligned} \int\limits_T^\infty \big\| |A_j|^{\alpha_j} (A_j - \mathrm{i}z)^{-1} u \big\|^2 \,\mathrm{d}z &= \int\limits_T^\infty \int\limits_{-\infty}^\infty |\lambda|^{2\alpha_j} \big(\lambda^2 + z^2 \big)^{-1} \,\mathrm{d}\big\| P_{\leqslant \lambda}(A_j) u \big\|^2 \,\mathrm{d}z \\ &= \int\limits_{-\infty}^\infty |\lambda|^{2\alpha_j - 1} \big(\pi/2 - \operatorname{arctg}\big(T/|\lambda|\big) \big) \,\mathrm{d}\big\| P_{\leqslant \lambda}(A_j) u \big\|^2 \\ &\leqslant \pi \|u\|^2. \end{aligned}$$

This implies the desired estimate and, by dominated convergence, also the strong convergence of the integral. □

COROLLARY A.7. – *Let A_j be as in* Lemma A.1 *and assume that $A_2 = A_1 + B$, where B satisfies* (A.3). *Then*

$$\|P_{>z_0}(A_1) - P_{>z_0}(A_2)\| = \|P_{<z_0}(A_1) - P_{<z_0}(A_2)\| \leqslant C_0\beta(A_1, z_0)^{\alpha_1 - 1/2}\beta(A_2, z_0)^{\alpha_2 - 1/2}.$$

Proof. – We observe the well known formula (see, e.g., Lemma 5.6 in [22, Chapter VI])

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{\mathrm{Re}\, z = z_0} (A_j - z)^{-1}\, \mathrm{d}z = \frac{1}{2}\big(P_{>z_0}(A_j) - P_{<z_0}(A_j)\big), \tag{A.8}$$

where the integral is strongly convergent. Taking the difference of these operators for $j = 1$ and $j = 2$ and estimating it with the lemma easily implies the assertion. □

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We would like to add: Gilles Carron pointed out to us that the definition of the η-invariant needs further explanation in the case where the metric on the cross sections is not sufficiently smooth. This explanation will be supplied in a forthcoming joint paper of the authors with Carron.