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A Global Semiclassical Description of the Spectrum of the Two-Dimensional Magnetic Schrödinger Operator with a Periodic Potential[1]

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Classical and quantum models describing the motion of particles in a constant magnetic and a periodic electric field have very curious properties even in two dimensions (see, e.g., [1–8]). If the magnetic field is strong, then small parameters appear in both classical and quantum problems. This makes it possible to use averaging [9] and semiclassical approximations [10]. This circle of ideas was studied in many papers (see, e.g., [2–8]). Nevertheless, based on the topological theory of Hamiltonian systems [13], we propose here a global point of view of the spectrum of the corresponding Schrödinger operator.

1. FORMULATION OF THE PROBLEM

We want to describe certain asymptotic spectral properties of the magnetic Schrödinger operator in $L^2(\mathbb{R}^2)$ (see [1, 5])

$$\hat{H}\Psi \equiv \left[\frac{1}{2}\left(-ih\frac{\partial}{\partial x_1}+x_2\right)^2 + \frac{1}{2}\left(-ih\frac{\partial}{\partial x_2}\right)^2 + \varepsilon v(x_1, x_2)\right]\Psi = E\Psi. \quad (1)$$

Here $h = \frac{(2\pi l_M)^2}{L_0^2} \equiv \frac{(2\pi)^2\hbar c}{(|eB|L_0^2)}$ and $\varepsilon = \frac{(2\pi)^2 V c^2 m}{(L_0 eB)^2}$, where $e, \hbar, c, m, B, V$, and L_0 are well-known physical constants and parameters of the model. We assume that the potential $v(x_1, x_2)$ is real and analytic in a certain $\mathbb{C}^2$-neighborhood of the plane $\mathbb{R}^2$ and is periodic with respect to the lattice Γ generated by the vectors $a_1 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \equiv \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}$ and $a_2 = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$; i.e., we have $v(x + a_1) = v(x + a_2) = v(x)$. Then, $\hat{H}$ is essentially self-adjoint on functions from $\mathbf{C}_0^\infty$.

It is well known that the spectral properties of $\hat{H}$ depend crucially on the parameter (flux) $\eta = \frac{a_{22}}{h}$. If $\eta = \frac{N}{M}$ is rational, then the spectrum of $\hat{H}$ has a band structure, since $\hat{H}$ has the Kadison property in that case (see [4, 5]). The spectrum is then the union of subbands, in each of which the dispersion relation defines the spectral parameter E as a function of quasi-momenta $q = (q_1, q_2) \in [0, 1/M] \times [0, 1]$. Moreover, each q is associated with a basis of M generalized eigenfunctions $\Psi^j(q, x)$ $(j = 1, 2, \ldots, M)$ with the properties (see [5])

$$\Psi^j(q, x+a_1) = \Psi^j(q, x)e^{-2\pi i(q_1 - \eta(j-1))}, \quad j = 1, 2, \ldots, M,$$

$$\Psi^j(q, x+a_2) = \Psi^{j+1}(q, x)e^{-i\eta(x_1 + a_{21}/2)}, \quad j = 1, 2, \ldots, M-1, \quad (2)$$

$$\Psi^M(q, x+a_2) = \Psi^1(q, x)e^{-i\eta(x_1 + a_{21}/2) - 2\pi i q_2}.$$

The structure of spec$\hat{H}$ becomes much more complicated if η is irrational. In particular, Cantor sets may arise in this case [7, 8]. We consider the situation when both parameters ε and h are small (then $\eta \gg 1$) and obtain asymptotic information about spec$\hat{H}$ by means

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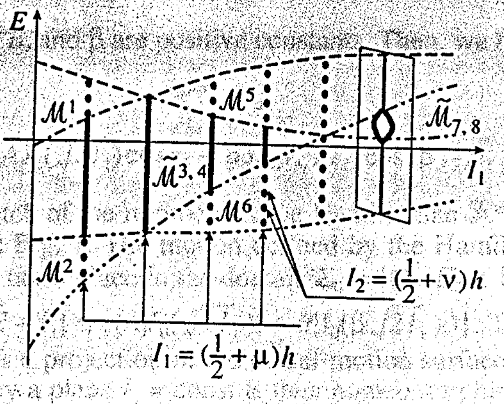

Figure.

of the semiclassical approximation[1] 1 as $h \to 0$.

2. AVERAGING, ALMOST INTEGRABILITY, AND CLASSIFICATION OF CLASSICAL MOTIONS

For the operator (1), the classical Hamiltonian is

$$H(p, x) = H_0 + \varepsilon v(x_1, x_2),$$

$$H_0 = \frac{1}{2}(p_1 + x_2)^2 + \frac{1}{2}p_2^2.$$

The trajectories of the Hamiltonian system generated by H_0 on the plane (x_1, x_2) are the cyclotron circles of radius $\sqrt{2I_1}$ and angle φ_1 centered at (y_1, y_2) (see, e.g., [12]). In the phase space, they induce new canonical variables: the generalized momenta I_1, y_1 (or P, y_1) and the positions φ_1, y_2 (or Q, y_2) given by the formulas

$$x_1 = Q + y_1, \quad p_1 = -y_2, \quad x_2 = P + y_2,$$

$$p_2 = -Q, \quad P = \sqrt{2I_1}\cos\varphi_1, \quad Q = \sqrt{2I_1}\sin\varphi_1.$$

In these variables,

$$H = I_1 + \varepsilon v(\sqrt{2I_1}\sin\varphi_1 + y_1, \sqrt{2I_1}\cos\varphi_1 + y_2).$$

Then, we introduce the averaged Hamiltonian

$$\mathcal{H}^{av}(y, I, \varepsilon) = \frac{1}{2\pi}\int_0^{2\pi} H\, d\varphi$$

$$\equiv I_1 + \varepsilon J_0(\sqrt{-2I_1\Delta_y})v(y_1, y_2),$$

where $J_0(z)$ is a Bessel function, $\Delta_y = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$, and $J_0(\sqrt{-2I_1\Delta_y})$ is a pseudo-differential operator [10].

Theorem 1. *Let the potential v satisfy the conditions specified above. Then, for any $K > 0$, there are $\varepsilon_1(K, v) > 0$ and a canonical change of variables in the domain $P^2 + Q^2 \le K$, given by $y_j = y'_j + \varepsilon U_j(y', P', Q', \varepsilon)$, $j = 1, 2$, $P = P' + \varepsilon \tilde{U}_3(y', P', Q', \varepsilon)$, $Q = Q' + \varepsilon \tilde{U}_4(y', P', Q')$, $\varepsilon \le \varepsilon_1$. With the new action variable $I' = \frac{1}{2}(P'^2 + Q'^2)$, we obtain the representation $H = \mathcal{H}(y', I', \varepsilon) + O\left(\exp\left(-\frac{K_1}{\varepsilon}\right)\right)$, where $\mathcal{H}(y', I', \varepsilon) = \mathcal{H}^{av}(y', I', \varepsilon) + O(\varepsilon^2)$, with some $K_1(K, v) > 0$. There is also $K_2(K, v) > 0$ such that, in the domain of the coordinate change, $|U_j| < K_2$, $j = 1, 2, 3, 4$. Moreover, $\mathcal{H}$ and U_j are real analytic Γ-periodic functions.*

The proof of Theorem 1 follows immediately from the general result in [9] (using the angle-action variables I_1, φ_1) for the case $I > K_3 > 0$. To extend this result to $I \to 0$ (where H is not analytic with respect to I_1), we modify the approach [9] by using an appropriately defined canonical transformation that is analytic in P and Q as $I_1 \to 0$.

Remark. Theorem 1 says that, modulo $O(\varepsilon^\infty)$, the Hamiltonian system with a sufficiently small ε becomes integrable. The first application of averaging in this context was apparently due to van Alven. Later, this approach was refined and extended by many authors [as a rule, in variables other than (I_1, φ_1, y)]. Theorem 1 seems to include the most complete result about the averaging of the system under consideration.

Since v is Γ-periodic, the Hamiltonian $\mathcal{H}$ for almost all I_1 may be viewed as a Morse function on the two-torus $\mathbb{T}_y^2 = \mathbb{R}^2/\Gamma$. Using the topological theory of Hamiltonian systems [13], for each fixed I_1, we can separate the motion defined by the averaged Hamiltonian into different topological regimes, which are conveniently described in terms of its Reeb graph as a Morse function $\mathcal{H}|_{I_1} = \text{const}$ (cf. [14]). By translating the action variable I_1 from 0 to $+\infty$, these regimes are obtained on the (action–energy) half-plane $\{(I_1, E) \in \mathbb{R}^2; I_1 \ge 0\}$ as the sets of points in the phase space that correspond to topologically similar edges of the Reeb graph. The union of all regimes gives the set Σ of actual motions with a structure similar to a Riemann surface. We restrict ourselves to the simplest nontrivial situation, when, for almost all I_1, the Morse function $\mathcal{H}$ has exactly one nondegenerate minimum and one maximum (see [13]). This situation is illustrated by the following example featuring the potential

$$v = A\cos x_1 + B\cos(\beta x_2), \tag{3}$$

[1] Apparently, Onsager and Azbel were the first to point out that the parameter $\frac{1}{B}$ plays the same role for H as the Planck constant for the ordinary Schrödinger equation. We emphasize that all assumptions on h, ε, and a_j are essential for our method. For example, if $|a_j| \sim h$, then $v = v(x_1/h, x_2/h)$ and, instead of standard semiclassical methods, the Born–Oppenheimer (adiabatic) approximation has to be used in this situation, which leads to quite different results (see [6]).

where A, B, and β are positive constants. Then, we find

$$\mathcal{H}^{av} = I_1 + \varepsilon(AJ_0(\sqrt{2I_1})\cos y_1 + BJ_0(\beta\sqrt{2I_1})\cos(\beta y_2)).$$

A sketch of the regimes for the Hamiltonian $\mathcal{H}^{av}$ is shown in Fig. 1. The motion defined by the Hamiltonian $\mathcal{H}^{av}$ takes place in the domain $\Sigma_0 = \{(I_1, E) \in \mathbb{R}^2: I_1 \geq 0, |E - I_1| \leq \varepsilon(A|J_0(\sqrt{2I_1})| + B|J_0(\beta\sqrt{2I_1})|)\}$. This domain is a projection of the actual motion surface Σ; any cut by a plane $I_1 = \text{const}$ is then homeomorphic to the Reeb graph of the Morse function $\mathcal{H}$ (see Fig. 1).

Moreover, Σ decomposes into regimes $\tilde{\mathcal{M}}^r$ (lying in the interior) and $\mathcal{M}^r$ (containing boundary curves) separated by the curves $E = I_1 \pm \varepsilon|A|J_0(\sqrt{2I_1})| \pm B|J_0(\beta\sqrt{2I_1})||$, which form the common boundaries of $\mathcal{M}^r$ and $\tilde{\mathcal{M}}^r$ (r is the index of a regime). It is natural to distinguish between regular and singular boundaries of regimes, according to whether they are external or internal. For each fixed I_1, the regular boundaries correspond to minima or maxima of $\mathcal{H}|_{I_1 = \text{const}}$, and the singular boundaries correspond to saddle points. The boundaries may have intersection points, which are boundary singularities. The function $\mathcal{H}|_{I_1 = \text{const}}$ has a degenerate saddle point at singularities formed by singular boundary components. At singularities formed by regular boundary components, this function depends on a single variable (y_1 or y_2) and is, strictly speaking, not a Morse function. In the case of example (3), there exists a one-to-one map from $\mathcal{M}^r$ to its image $\pi_{I_1,E}(\mathcal{M}^r)$, and each image $\pi_{I_1,E}(\tilde{\mathcal{M}}^r) \subset \Sigma_0$ of $\tilde{\mathcal{M}}^r$ has two preimages on the surface Σ.

Each internal point of a regime is associated with a family of closed trajectories on $\mathbb{T}^2$ and, hence, with a family of closed (for $\mathcal{M}^r$) or open (for $\tilde{\mathcal{M}}^r$) trajectories on the universal covering space $\mathbb{R}^2_y$. These are, in turn, associated with a family of Lagrangian (or Liouville) tori Λ^r_l (for $\mathcal{M}^r$) and with a family of Lagrangian (or Liouville) cylinders $\tilde{\Lambda}^r_k$ (for $\tilde{\mathcal{M}}^r$) in the original phase space $\mathbb{R}^4_{p,x}$. With the tori or cylinders (and, hence, with a regime), we may thus associate (a) the vector $\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ of a drift in the original configuration space $\mathbb{R}^2_x$, or, equivalently, the rotation number $\frac{d_1}{d_2}$ of the related closed trajectory on the torus; and (b) the Maslov indices of the related tori or cylinders. The rotation number of a boundary regime is formally equal to $\frac{0}{0}$ (there is no drift). The Maslov indices of natural cycles on each torus are equal to 2. On the other hand, the rotation number of an internal regime is nontrivial: there exists a preferred direction, but each cylinder has only one (natural) cycle and, hence, only one Maslov index, which again is equal to 2. In each regime, one can introduce the second action variable I_2 and can find (c) the form of the Hamiltonian in action variables, $\mathcal{H} = \mathcal{H}_r(I_1, I_2, \varepsilon)$, which depends on the regimes.

The drift vector and the function $\mathcal{H}_r(I_1, I_2, \varepsilon)$ may change discontinuously and nonsmoothly, respectively, in transition from one regime to another. The drift vectors related to a fixed I_1 coincide up to direction. For each (I_1, I_2), the tori $\Lambda^r_l(I_1, I_2)$ in the phase space $\mathbb{R}^4_{p,x}$ can be obtained from $\Lambda^r_0(I_1, I_2)$ by the shift $p_1 \mapsto p_1 + 2\pi l_1 + a_{21}, p_2 \mapsto p_2, x \mapsto x + a_1 l_1 + a_2 l_2$. Likewise, the cylinders $\tilde{\Lambda}^r_k(I_1, I_2)$ can be obtained from $\tilde{\Lambda}^r_0(I_1, I_2)$ by these shifts, with $l(k) = k(\mathbf{f}_2 a_1 - \mathbf{f}_1 a_2)$, $k \in \mathbb{Z}$, where $\mathbf{f}_1$ and $\mathbf{f}_2$ are fixed integers satisfying the condition $\mathbf{f}_1\mathbf{d}_1 + \mathbf{f}_2\mathbf{d}_2 = 1$. It is natural to enumerate the tori Λ^r_l by the multi-index $l = (l_1, l_2) \in \mathbb{Z}^2$ and the cylinders $\tilde{\Lambda}^r_k$ by the index $k \in \mathbb{Z}^1$.

The corrections $\mathcal{H} - \mathcal{H}^{av}$ do not change the above rough description of the classical motion or the general asymptotic description of the spectrum, even though a complete analysis of the effected changes may be important in certain problems. Here, we analyze neither the classical motion in the neighborhood of singular boundaries nor the behavior of the corresponding part of the spectrum. Thus, introducing a small number δ (independent of h or ε) and removing a δ-neighborhood of the singular boundary from all regimes $\mathcal{M}^r$ and $\tilde{\mathcal{M}}^r$, we obtain new sets $\mathcal{M}^{r,\delta}$ and $\tilde{\mathcal{M}}^{r,\delta}$, which are also referred to as regimes.

3. THE GLOBAL ASYMPTOTIC STRUCTURE OF THE SPECTRUM

Next, we quantize the regimes $\mathcal{M}^{r,\delta}$, $\tilde{\mathcal{M}}^{r,\delta}$ according to the Bohr–Sommerfeld rules to obtain quantized regimes on the surface Σ. After projecting onto the energy axis, we find the asymptotic approximation to the spectrum of the original operator. The quantization of I_1 gives the Landau levels: $I_1^\mu = \left(\frac{1}{2} + \mu\right)h$. The action I_2 is quantized only in boundary regimes, according to the rule $I_2^\nu = \left(\frac{1}{2}+\nu\right)h$. Here, μ and ν are integers chosen so that $(I_1^\mu, I_2^\nu) \in \mathcal{M}^{r,\delta}$. The energy values that we expect to approximate the spectrum of $\hat{H}$ are now given by the numbers $\mathcal{H}_r(I_1^\mu, I_2^\nu)$ for $(I_1^\mu, I_2^\nu) \in \mathcal{M}^{r,\delta}$ and by the functions $\mathcal{H}_r(I_1^\mu, I_2)$ for $(I_1^\mu, I_2) \in \tilde{\mathcal{M}}^{r,\delta}$. Thus, we obtain points and intervals [the situation for example (3) is sketched in Fig. 1].

Theorem 2. *For any integers L, K and* $(I_1^\mu, I_2^\nu) \in \mathcal{M}^{r,\delta}$ *or* $(I_1^\mu, I_2) \in \tilde{\mathcal{M}}^{r,\delta}$, *there exist corrections* $g_{r,L}^{\mu,\nu}(\varepsilon, h) = O(h^2)$ *and* $\tilde{g}_{r,L}^{\mu}(I_2, \varepsilon, h) = O(h^2)$ *such that* $\operatorname{dist}\{\mathcal{H}_r(I_1^\mu, I_2^\nu, \varepsilon) + g_{r,L}^{\mu,\nu}(\varepsilon, h), \operatorname{spec}\hat{H}\} = O(h^L + \varepsilon^K)$ *and* $\operatorname{dist}\{\mathcal{H}_r(I_1^\mu, I_2, \varepsilon) + \tilde{g}_{r,L}^{\mu}(I^2, \varepsilon, h), \operatorname{spec}\hat{H}\} = O(h^L + \varepsilon^K)$.

The proof of Theorem 2 is based on the construction of asymptotic generalized eigenfunctions (quasi-modes) $\psi_l^{\mu,\nu,r}(x, h, \varepsilon) = \psi_0^{\mu,\nu,r}(x - l\cdot a, h, \varepsilon)e^{-\frac{i}{h}x_1 l_2 a_{22}}$ $(l \in \mathbb{Z}^2)$ corresponding to the tori Λ_l^r and $\tilde{\psi}_k^{\mu,r}(x, I_2, \varepsilon, h) = \tilde{\psi}_0^{\mu,r}(x \pm k a_2, I_2, \varepsilon, h)e^{\frac{\pm ik}{h}(a_{22}x_1 + k a_{22}a_{21})}$ $(k \in \mathbb{Z})$ corresponding to the cylinders $\tilde{\Lambda}_k^r$ and the drift vectors $\mathbf{d} = \begin{pmatrix}\pm 1\\ 0\end{pmatrix}$. These functions are defined using the Maslov canonical operator [10] associated with Λ_l^r or $\tilde{\Lambda}_k^r$ and are supported [mod $O(h^\infty)$] in small neighborhoods of the projections of these tori or cylinders onto $\mathbb{R}_x^2$. The functions $\tilde{\psi}_k^{\mu,r}(x, I_2, \varepsilon, h)$ also satisfy the identity $\tilde{\psi}_k^{\nu_1,r}(x + a_1, I_2, \varepsilon, h) = \tilde{\psi}_k^{\mu,r}(x, I_2, \varepsilon, h)e^{\frac{\pm 2\pi i I_2}{h}}e^{\frac{\pm ik a_{11}a_{22}}{h}}$. Thus, we can apply Theorem 2.12 in [10] or its Γ-periodic modification to obtain the estimates in Theorem 2.

Theorem 2 does not depend on the choice of coordinates (x_1, x_2) or the gauge in the original problem (1), but the structure of the eigenfunctions simplifies if the drift vector of the corresponding Landau level is $\mathbf{d} = \begin{pmatrix}\pm 1\\ 0\end{pmatrix}$. For a rational η, this means that, for each Landau level, it is good to match conditions (2) with the drift. Hence, one has to rotate the coordinates x and to change the gauge when moving from one Landau level to another. We assume that the coordinates (x_1, x_2) are chosen in accordance with this rule.

4. THE RELATIONSHIP BETWEEN THE ACTUAL AND ASYMPTOTIC SPECTRA: THE CASE OF A RATIONAL FLUX

Each asymptotic spectral value is infinitely degenerate, which stems from the fact that we are dealing with a continuous spectrum. It should be emphasized that this construction does not depend on the rationality of η: the asymptotic approximation used does not feel this effect. Note also that the construction of quantized regimes on the surface Σ gives more information than the energy values alone. For example, quantized regimes allow us to estimate the width of Landau levels. In example (3), the width is estimated as $2\varepsilon\left(A|J_0\left(\sqrt{2I_1^\mu}\right)| + B|J_0\left(\beta\sqrt{2I_1^\mu}\right)| + O(h)\right)$.

Physical considerations indicate that the actual (continuous) spectrum of the operator $\hat{H}$ lies in an exponentially small neighborhood of the constructed Bohr–Sommerfeld set on the (I_1, E)-plane (cf. [2–8]). Moreover, all these neighborhoods are disconnected and have a finite number of components if the flux η is rational and, possibly, an infinite number of components if η is irrational. This conjecture has some evidence in the case of a rational flux, $\eta = \frac{N}{M}$. Then, the spectrum of $\hat{H}$ has a band structure. This means that, in our approximation, points may appear as the traces of exponentially small (in h) bands, and intervals may hide exponentially small gaps that cannot be detected by the methods we use here. At least heuristically, the situation becomes clearer if one looks at the quasi-modes $\psi_l^{\mu,\nu,r}(x, \varepsilon, h)$ and $\tilde{\psi}_k^{\mu,r}(x, I_2, \varepsilon, h)$, taking into account (2). They do not satisfy (2) as it stands, but this can be achieved by a purely algebraic construction. Specifically, one can use these quasi-modes as a basis for the construction of Bloch functions in the form of the Gelfand–Landau–Zak representation (see, e.g., [5]). To simplify the formulas, we assume that $a_{21} = 0$.

Theorem 3. (i) *For any* $q_1 \in \left[0, \frac{1}{M}\right]$, $q_2 \in [0, 1]$ *and any* $I_1^\mu, I_2^\nu \in \mathcal{M}^{r,\delta}$ *from M, there exist exactly M linearly independent M-dimensional collections of functions*

$$\Psi_{j,s}^{\mu,\nu,r}(x, h, \varepsilon, q) = \sum_{l\in\mathbb{Z}^2} C_l^{j,s}(q) \times \psi_0^{\mu,\nu,r}(x - l\cdot a, h, \varepsilon)e^{-\frac{i}{h}x_1 l_2 a_{22}} \tag{4}$$

satisfying (2). *The coefficients* $C_j^{j,s}(q)$ *of these functions may be chosen so that* $C_{l_1,l_2}^{j,s}(q) =$

$C_s e^{-2\pi i(q_1 l_1 + q_2 n) + 2\pi i \eta l_1 (s-1)}$ *if* $l_2 = s - j + nM$ *and* $C^{j,s}_{l_1,l_2}(q) = 0$ *otherwise. Here,* $l_1 = \pm 1, \ldots; j, s \in \{1, 2, \ldots, M\}$ *(s is the index of the corresponding collection); and* C_s *is a constant.*

(ii) *The functions*

$$\tilde{\Psi}^{\mu,r}_j = \sum_{k \in \mathbb{Z}^1} C^j_k(q_1, q_2, r)\tilde{\psi}^{\mu,r}_0(x \pm ka_2, I_2, \varepsilon, h)e^{\frac{\pm ik}{h}(a_{22}x_1)} \quad (5)$$

satisfy the Bloch conditions (2) *if and only if*

$$I_2 = I_2(q, n^{\pm}) = \mp a_{22}\left(\frac{q_1 M}{N} + \frac{n^{\pm}}{N}\right) \equiv \mp h\left(q_1 + \frac{n^{\pm}}{M}\right), \quad q_1 \in \left[0, \frac{1}{M}\right], \quad (6)$$

where $n^{\pm}$ *are integers such that* $(I^{\mu}_1, I_2(q_1, n^{\pm})) \in \mathcal{M}^{r,\delta}$. *Here,* $C^j_k = Ce^{-2\pi i \tilde{n} q_2}$ *if* $k \mp (N_1 n^{\pm} + j - 1 + \tilde{n}M) \equiv 0$ *and* $C^j_{\mp k} = 0$ *otherwise;* M_1 *and* N_1 *are a (nonunique) pair of integers chosen in such a way that* $M_1 M + N_1 N = 1$; $\tilde{n} = 0, \pm 1, \ldots; j = 1, 2, \ldots, M$; *and C is a constant. The signs are determined by the direction of the drift vector* $\mathbf{d} = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$.

Each of the functions (4), (5) is localized in strips separated by $M - 1$ strips not supporting [mod $(O(h^{\infty}))$] this generalized eigenfunction on the (x_1, x_2)-plane. (This is not so when the drift vector is not parallel to the x_1-axis: all strips are then filled.) Thus, if η approaches an irrational limit as $M \to \infty$, we obtain an infinite number of functions for any fixed spectral parameter, and each of the functions is localized in a neighborhood of a single horizontal strip.

The fact that there exist M collections of M Bloch functions of the type (4) probably indicates that the points in $\mathcal{M}^{r,\delta}$ obtained by the Bohr–Sommerfeld rule actually split into M exponentially small subsets separated by exponentially small gaps such that their projections onto the E-axis give exponentially small subbands and gaps. Note that, using analogues of the so-called Lifshits [1] and Herring [11] formulas, one can find the asymptotics of the dispersion relation in the form of a trigonometric polynomial in the quasi-momenta q with exponentially small coefficients (which can be determined only with the use of tunneling).

Let us describe certain points that should be interpreted as traces of gaps. Specifically, substituting the Landau levels I^{μ}_1 and the functions (6) into the corresponding Hamiltonian $\mathcal{H}_r$, we obtain the dispersion curves $E = \mathcal{H}\left(I^{\mu}_1, \mp h\left(q_1 + \frac{n^{\pm}}{M}\right)\right)$. By Theorem 3, each edge of the Reeb graph is connected with a set of M asymptotic generalized eigenfunctions. The dispersion curves intersect at the end points of the segment $q \in \left[0, \frac{1}{M}\right]$ and at its middle point $q = \frac{1}{2M}$. It is natural to assume that they describe the traces of gaps as mentioned before. As a function of q_1, each curve is defined by Hamiltonians corresponding to the same edge with the drift vector $\mathbf{d} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ before $q_1 = \frac{1}{2M}$ and with the drift vector $\mathbf{d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ beyond $q_1 = \frac{1}{2M}$. The points $I_2 = I_2(0, n^{\pm})$ and $I_2 = I_2\left(\frac{1}{2M}, n^{\pm}\right)$ are resonances: here, the family of eigenfunctions with one drift vector jumps to a family with the opposite orientation of the drift vector. Although a rigorous description of these effects must be based on tunneling, this explanation nevertheless allows us to heuristically find the number of subbands for each Landau level. Let us show this for example (3). Neglecting the boundary effects, we can count the number of Bohr–Sommerfeld points on the edges of the Reeb graph related to finite motion. It is equal to the difference between the actions I_2 at the end and the beginning of an edge divided by the parameter h. Multiplying these numbers by M, we obtain the number of exponentially small subbands. The same idea gives the number of subbands for regions with infinite motion. Again, we obtain the variation of the action, but we now have to divide it by $\frac{2h}{M}$, according to the above considerations. Using the Kirchhoff law for actions on the Reeb graph [13], we obtain the analogue of the Weyl formula (for eigenvalues): *The number of subbands in each Landau level is approximately equal to the numerator N of the rational flux* η.

Finally, a naive quantization of the averaged Hamiltonian $\mathcal{H}$ leads to a Harper-type difference equation and corresponding quasi-modes. This allows us to compare the generalized eigenfunctions constructed above with the eigenfunctions of the difference equation. This comparison shows that the functions featuring in Theorem 3 correspond to solutions to difference equations with a δ-like structure [cf. 7, 14, 15].

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