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Averaging for Hamiltonian Systems with One Fast Phase and Small Amplitudes

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Abstract—In this paper we consider an analytic Hamiltonian system differing from an integrable system by a small perturbation of order ε. The corresponding unperturbed integrable system is degenerate with proper and limit degeneracy: all variables, except two, are at rest and there is an elliptic singular point in the plane of these two variables. It is shown that by an analytic symplectic change of the variable, which is $O(\varepsilon)$-close to the identity substitution, the Hamiltonian can be reduced to a form differing only by exponentially small ($O(e^{-\operatorname{const}/\varepsilon})$) terms from the Hamiltonian possessing the following properties: all variables, except two, change slowly at a rate of order ε and for the two remaining variables the origin is the point of equilibrium; moreover, the Hamiltonian depends only on the "action" of the system linearized about this equilibrium.

Key words: *Hamiltonian system with fast phase and small amplitudes, averaging for Hamiltonian systems, Neishtadt method, small perturbation.*

1. INTRODUCTION

The problem of averaging for systems with one fast phase was studied on numerous occasions from different points of view in a multitude of papers. The studies in question were based on the well-known Krylov–Bogolyubov averaging method [1] and on methods of KAM theory. The most complete and definitive results were obtained by Neishtadt in [2]; in that paper, the coefficients of the equations were assumed to depend on the variables "action–angle" as real analytic functions. However, there exists a large number of problems, interesting from the point of view of applications, in which the analyticity assumption fails in the neighborhood of certain points.

For example, consider the motion of a charged particle in a small periodic electric field and a constant magnetic field. The Hamiltonian of such a system is of the form

$$H=\frac{1}{2}((p_1+x_2)^2+p_2^2)+\varepsilon V(x_1,x_2).$$

By the canonical change of variables

$$x_1=Q+y_1,\quad p_1=-y_2,\quad x_2=P+y_2,\quad p_2=-Q,\qquad Q=\sqrt{2I}\cos\varphi,\quad P=\sqrt{2I}\sin\varphi,$$

it can be rewritten as

$$H=I+\varepsilon V\big(\sqrt{2I}\cos\varphi+y_1,\sqrt{2I}\sin\varphi+y_2\big).$$

This system depends on one fast phase φ; it is not analytic in I ($\sim\sqrt{I}$) in the neighborhood of the point $I=0$. From a mathematical point of view, the measure of this region is small. Nevertheless, this region can play a crucial role for physical applications. For example, in the problem of semiclassical quantization, the neighborhood $I = 0$ is associated with the so-called lower Landau levels related to Hall conductance.

The Neishtadt procedure [2] is based on successive changes of variables, as in the KAM method. The goal of this paper is to show that we can choose a transformation such that the averaging procedure [2] will be applicable in the neighborhood of the point $I = 0$; moreover, the transition from $I > \varkappa > 0$ to $I = 0$ will turn out to be uniform.

2. STATEMENT OF THE PROBLEM AND MAIN RESULTS

Consider the Hamiltonian

$$H = \mathcal{H}_0(I) + \varepsilon g_0(q, p, y_1, y_2), \qquad I = \frac{p^2 + q^2}{2}, \tag{1}$$

where $0 < \varepsilon < \varepsilon_0$ is a small parameter and $\mathcal{H}_0$ and g_0 are real analytic functions in a complex δ-neighborhood of the domain

$$D := D_{2n}\{y_1, y_2\} \times D_2^{\varkappa}\{q, p\}, \qquad D_{2n} \subset \mathbb{R}^{2n}, \quad D_2^{\varkappa} = \{(q, p) \in \mathbb{R}^2 \mid I < \varkappa\}.$$

Suppose that the following conditions are satisfied in D:

$$|\mathcal{H}_0| \le C, \qquad |g_0| \le C, \qquad \left|\frac{\partial \mathcal{H}_0}{\partial I}\right| \ne 0.$$

Just as in [1, 2], we wish to show that for each $m > 0$, $m \in \mathbb{N}$, there exists a real analytic canonical change of variables $(q, p, y_1, y_2) \to (Q, P, z_1, z_2)$, which is close to the identity substitution and is defined by the formulas

$$\begin{aligned} q &= Q + \varepsilon Q^1(Q, P, z_1, z_2, \varepsilon), \qquad & y_1 &= z_1 + \varepsilon Z_1^1(Q, P, z_1, z_2, \varepsilon), \\ p &= P + \varepsilon P^1(Q, P, z_1, z_2, \varepsilon), \qquad & y_2 &= z_2 + \varepsilon Z_2^1(Q, P, z_1, z_2, \varepsilon), \end{aligned} \tag{2}$$

where $|Q^1| + |P^1| + |Z_1^1| + |Z_2^1| \le C$, such that the Hamiltonian (1) is transformed to the form

$$H = \mathcal{H}_m\left(\frac{Q^2 + P^2}{2}, z_1, z_2, \varepsilon\right) + \varepsilon g_m(Q, P, z_1, z_2, \varepsilon), \tag{3}$$

and the following theorem (announced in [3]) is valid.

Theorem 1. *Suppose that the conditions given above are satisfied in $(Q, P, z_1, z_2) \in D + \frac{1}{2}\delta$. Then there exist an interval $(0, \varepsilon_1]$, an integer r, and a real analytic canonical change of variables (2) that takes the Hamiltonian (1) to (3) with an exponentially small g:*

$$|g_r| + |\nabla g_r| < c_2 \exp\left(-\frac{1}{c_1 \varepsilon}\right), \qquad |Q^1| + |Z_2^1| + |P^1| + |Z_1^1| < c_3, \qquad |\mathcal{H}_r - \mathcal{H}_0| < c_4 \varepsilon. \tag{4}$$

Here $\varepsilon \in (0, \varepsilon_1]$ and ε_1, r, c_i, $i = 1, 2, 3, 4$, are constants depending on ε_0, δ, C, $\varkappa$.

3. AUXILIARY LEMMAS

Suppose that $w(I,\mu)$ and $g(q,p,\mu)$ are analytic functions of I and (q,p), respectively, and also of the parameter vector μ, $\partial w/\partial I \neq 0$, $I=(q^2+p^2)/2$. Denote $\partial w/\partial q = q\partial w/\partial I$, $\partial w/\partial p = p\partial w/\partial I$. Consider the equation

$$\frac{\partial w}{\partial p}\frac{\partial W}{\partial q}-\frac{\partial w}{\partial q}\frac{\partial W}{\partial p}+g(q,p,\mu)=\bar g\biggl(\frac{q^2+p^2}{2},\mu\biggr), \tag{5}$$

$$\bar g\biggl(\frac{q^2+p^2}{2},\mu\biggr)=\int_0^{2\pi} g(q(\varphi,I),p(\varphi,I),\mu)\,d\varphi\bigg|_{\left(\substack{\varphi=\varphi(q,p)\\ I=I(q,p)}\right)}, \tag{6}$$

$$g(q,p,\mu)=\bar g\biggl(\frac{q^2+p^2}{2},\mu\biggr)-\tilde g(q,p,\mu), \tag{7}$$

where $\bar g((q^2+p^2)/2,\mu)$ is the average value of $g(q,p,\mu)$ over the angle and $\tilde g(q,p,\mu)$ denotes the remaining part of g, which, for convenience, is taken with the minus sign.

Lemma 1. *Equation* (5) *has a solution which is analytic in* (q,p) *and in the parameter* μ. *The function* $W(q,p,\mu)$, *given by the formula*

$$\begin{aligned}W(q,p,\mu)=\frac{1}{\partial w/\partial I}\biggl(&\frac12\int_0^{\varphi}\tilde g(q(\psi,I),p(\psi,I),\mu)\,d\psi\\ &+\frac12\int_{\pi}^{\varphi}\tilde g(q(\psi,I),p(\psi,I),\mu)\,d\psi\biggr)\bigg|_{\varphi=\varphi(q,p),\,I=I(q,p)},\end{aligned} \tag{8}$$

is an analytic solution of Eq. (5).

Remark. The general solution of Eq. (5) is of the form

$$W(q,p,\mu)=\biggl(\frac{1}{\partial w/\partial I}\int_0^{\varphi}\tilde g(q(\psi,I),p(\psi,I),\mu)\,d\psi+W_0(I,\mu)\biggr)\bigg|_{\varphi=\varphi(q,p),\,I=I(q,p)},$$

where $W_0(I,\mu)$ is the constant of integration. It is obtained by applying the method described in [2] for the variables (q,p). It will be shown that, by choosing the constant of integration according to (8), we can integrate Eq. (5) in the neighborhood of the point $q=0$, $p=0$, preserving at the same time the analyticity of its solution.

Proof. Let us perform the canonical change of variables:

$$q=\frac{u+iv}{\sqrt2},\qquad p=\frac{v+iu}{\sqrt2}.$$

Equation (5) takes the form

$$\biggl(\frac{\partial w}{\partial v}\frac{\partial W}{\partial u}-\frac{\partial w}{\partial u}\frac{\partial W}{\partial v}\biggr)=\tilde g\biggl(\frac{u+iv}{\sqrt2},\frac{v+iu}{\sqrt2},\mu\biggr). \tag{9}$$

Now let us expand $\tilde g$ in a Taylor series. The expansion is of the form

$$\tilde g\biggl(\frac{u+iv}{\sqrt2},\frac{v+iu}{\sqrt2},\mu\biggr)=\sum_{k,l\in\mathbb N}\tilde g_{kl}(\mu)u^kv^l\frac{(k+l)!}{k!l!}.$$

Using the coordinates (φ, ρ): $u = \rho e^{i\varphi}$, $v = \rho e^{-i\varphi}$, we can easily integrate Eq. (9):

$$\begin{aligned} W &= \frac{\rho}{\partial w/\partial \rho}\biggl(\int_0^{\varphi} \tilde g\biggl(\frac{u+iv}{\sqrt2}, \frac{v+iu}{\sqrt2}, \mu\biggr)\,d\psi\biggr) + W_0(\rho,\mu) \\ &= \frac{\rho}{\partial w/\partial \rho}\biggl(\sum_{k,l\in\mathbb N} \tilde g_{kl}(\mu)\frac{u^k(\varphi,\rho)v^l(\varphi,\rho)}{i(k-l)}\frac{(k+l)!}{k!l!} \\ &\qquad\qquad - \sum_{k,l\in\mathbb N} \tilde g_{kl}(\mu)\frac{|uv|^{(k+l)}}{i(k-l)}\frac{(k+l)!}{k!l!}\biggr) + W_0(\rho,\mu), \end{aligned} \tag{10}$$

where $W_0(\rho,\mu)$ is the constant of integration.

Obviously, at the lower limit of integration a nonanalytic ($\sim (uv) = \sqrt{2I}$) dependence on I can occur at the point ($u = 0$, $v = 0$). But we can choose $W_0(\rho,\mu)$ so that the nonanalytic summand disappears. Define W_0 as follows:

$$\begin{aligned} W_0(\rho,\mu) &= \frac{\rho}{\partial w/\partial\rho}\biggl(\frac12\int_{\pi}^{\varphi} \tilde g(u(\psi,\rho), v(\psi,\rho),\mu)\,d\psi - \frac12\int_0^{\varphi} \tilde g(u(\psi,\rho), v(\psi,\rho),\mu)\,d\psi\biggr) \\ &= \frac{\rho}{\partial w/\partial\rho}\biggl(\sum_{k,l\in\mathbb N} \tilde g_{kl}(\mu)\frac{|uv|^{(k+l)}}{i(k-l)}\frac{(k+l)!}{k!l!}\biggr). \end{aligned} \tag{11}$$

Obviously, the expression (11) "destroys" the nonanalytic term in (10), and the function W becomes an analytic function of I at the point $I = 0$. It is also obvious that $\partial w/(\rho\partial\rho) = \partial w/\partial I$. The function $W(q,p,\mu)$ is analytic in μ by construction: at each step the dependence on the parameter μ is analytic. Therefore, W is an analytic function of (q,p) and the parameter μ. The lemma is proved. $\square$

In the proof of the main theorem, we use the generating function to construct a canonical transformation.

Suppose that $S(q,P,y_1,z_2,\varepsilon)$ is an analytic function of all of its variables (q,P,y_1,z_2) and of a small parameter $\varepsilon \in [0,\varepsilon_0]$,

$$(q,p,y_1,y_2) \in U, \qquad (Q,P,z_1,z_2) \in U-\delta, \quad \delta > 0, \quad U \subset \mathbb R^{2n+2};$$

$U-\delta$ is the set of points appearing in U together with their δ-neighborhoods [5]. Consider the system of equations

$$\begin{aligned} Q &= q + \varepsilon\frac{\partial S(q,P,y_1,z_2,\varepsilon)}{\partial P}, \qquad & z_1 &= y_1 + \varepsilon\frac{\partial S(q,P,y_1,z_2,\varepsilon)}{\partial z_2}, \\ p &= P + \varepsilon\frac{\partial S(q,P,y_1,z_2,\varepsilon)}{\partial q}, \qquad & y_2 &= z_2 + \varepsilon\frac{\partial S(q,P,y_1,z_2,\varepsilon)}{\partial y_1}. \end{aligned} \tag{12}$$

Suppose that $\max\{|\partial S/\partial q|, |\partial S/\partial P|, |\partial S/\partial y_1|, |\partial S/\partial z_2|\} < C_U$ in the domain U.

Using the following lemma, we can prove that the transformation introduced above is a change of variables and construct estimates required for corrections.

Lemma 2. *If $\varepsilon < \delta/(2C_U(n+1))$, where n is the dimension of y_1, then system (12) has a solution of the form*

$$\begin{aligned} q &= Q + \varepsilon Q^1(Q,P,z_1,z_2,\varepsilon), \qquad & y_1 &= z_1 + \varepsilon Z_1^1(Q,P,z_1,z_2,\varepsilon), \\ p &= P + \varepsilon P^1(Q,P,z_1,z_2,\varepsilon), \qquad & y_2 &= z_2 + \varepsilon Z_2^1(Q,P,z_1,z_2,\varepsilon), \end{aligned} \tag{13}$$

where Q^1, P^1, Z_1^1, Z_2^1 are analytic functions in the domain $U-\delta\times[0,\varepsilon_0]$ with

$$\max\{|Q^1|, |P^1|, |Z_1^1|, |Z_2^1|\} < C_U$$

in the domain $U-\delta$.

Proof. Let us express system (12) as

$$\begin{aligned} &f_1(q, p, y_1, y_2\,; Q, P, z_1, z_2, \varepsilon) = 0, \qquad && f_3(q, p, y_1, y_2\,; Q, P, z_1, z_2, \varepsilon) = 0, \\ &f_2(q, p, y_1, y_2\,; Q, P, z_1, z_2, \varepsilon) = 0, \qquad && f_4(q, p, y_1, y_2\,; Q, P, z_1, z_2, \varepsilon) = 0 \end{aligned} \tag{14}$$

and $F = (f_1, f_2, f_3, f_4)$. By the implicit function theorem, system (14) is solvable with respect to (Q, P, z_1, z_2) if $\det|DF/D(q, p, y_1, y_2)| \neq 0$, and its solution is analytic on $U - \delta \times [0, \varepsilon_0]$. In our case

$$\det\left|\frac{DF}{D(q, p, y_1, y_2)}\right| = \det\begin{vmatrix} 1 + \varepsilon\dfrac{\partial^2 S}{\partial P \partial q} & \varepsilon\dfrac{\partial^2 S}{\partial P \partial y_1} \\ \varepsilon\dfrac{\partial^2 S}{\partial z_2 \partial q} & 1 + \varepsilon\dfrac{\partial^2 S}{\partial z_2 \partial y_1} \end{vmatrix},$$

where $S(q, P, y_1, z_2, \varepsilon)$ is an analytic function of ε. Using Cauchy estimates for analytic functions [4], we obtain

$$\det\left|\frac{DF}{D(q, p, y_1, y_2)}\right| - 1 < \sum_{i=1}^{n+1}\left(\varepsilon\frac{C_U}{\delta}\right)^i \frac{(n+1)!}{(n+1-i)!} < \sum_{i=1}^{\infty}\left(\varepsilon\frac{C_U}{\delta}(n+1)\right)^i < 1. \tag{15}$$

Therefore, if $\varepsilon < \delta/(2C_U(n+1))$, then inequality (15) is always satisfied in U and the system is solvable in U. The uniqueness of the solution is proved in the same way as in [5]. The solution must satisfy Eqs. (12), including the case $\varepsilon = 0$, so that it can be expressed as (13). Let us substitute the solution into Eq. (12). From the equations for q and y_1, we obtain the following expressions for Q^1 and Z_1^1:

$$Q^1 = -\frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial P}, \qquad Z_1^1 = -\frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial z_2}.$$

Therefore, $|Q^1| < C$, $|Z_1^1| < C$. For p and y_2, we use the formulas

$$\begin{aligned} p &= P + \varepsilon\frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial q} = P + \varepsilon\frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial Q}, \\ y_2 &= z_2 + \varepsilon\frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial y_1} = z_2 + \varepsilon\frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial z_1}. \end{aligned}$$

Thus

$$P^1 = \frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial Q}, \qquad Z_2^1 = \frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial z_1}$$

and once again $|P^1| < C$, $|Z_2^1| < C$. The lemma is proved. □

4. PROOF OF THE MAIN THEOREM

We construct the change of variables as the composition of successively defined canonical transformations yielding a dependence of the Hamiltonian on q and p of the form $(q^2 + p^2)/2$, of progressively higher degree in ε.

4.1. Procedure of successively defined changes of variables

Assume that the Hamiltonian obtained after i changes of variables is of the form

$$H = \mathcal{H}_i\biggl(\frac{q^2+p^2}{2}, y_1, y_2, \varepsilon\biggr) + \varepsilon g_i(q, p, y_1, y_2, \varepsilon), \tag{16}$$

$$(q, y_2, p, y_1) \in D_i, \qquad D_i = D_1 - 2(i-1)K\varepsilon, \quad D + \frac{\delta}{2} \subset D_i \subset D + \delta, \quad D_1 = D + \frac{3}{4}\delta.$$

At the $(i+1)$st step, it is necessary to construct a canonical change of variables (see [2, 4, 5]) $(q, p, y_1, y_2) \to (Q, P, z_1, z_2)$,

$$\begin{aligned} q &= Q + \varepsilon Q^1(Q, P, z_1, z_2, \varepsilon), \qquad & y_1 &= z_1 + \varepsilon Z_1^1(Q, P, z_1, z_2, \varepsilon), \\ p &= P + \varepsilon P^1(Q, P, z_1, z_2, \varepsilon), \qquad & y_2 &= z_2 + \varepsilon Z_2^1(Q, P, z_1, z_2, \varepsilon) \end{aligned} \tag{17}$$

close to the equivalent one so that the Hamiltonian takes the form

$$\begin{gathered} H = \mathcal{H}_{i+1}\biggl(\frac{Q^2+P^2}{2}, z_1, z_2, \varepsilon\biggr) + \varepsilon g_{i+1}(Q, P, z_1, z_2, \varepsilon), \\ \mathcal{H}_{i+1} = \mathcal{H}_i + \varepsilon \bar{g}_i, \qquad g_{i+1} = O(\varepsilon^{i+1}), \end{gathered} \tag{18}$$

where $\mathcal{H}_{i+1}$ contains terms of order $i+1$ in ε and the operation "bar" is defined in Sec. 3 (cf. Eq. 6).

This can be performed using the generating function

$$S = S(q, P, y_1, z_2, \varepsilon) = qP + y_1 z_2 + \varepsilon S^1(q, P, y_1, z_2, \varepsilon).$$

(Note that the transformation thus obtained will immediately be canonical.) All the other variables can be expressed in terms of (q, P, y_1, z_2) and $S(q, P, y_1, z_2)$:

$$\begin{aligned} Q &= q + \varepsilon\frac{\partial S^1(q, P, y_1, z_2, \varepsilon)}{\partial P}, \qquad & z_1 &= y_1 + \varepsilon\frac{\partial S^1(q, P, y_1, z_2, \varepsilon)}{\partial z_2}, \\ p &= P + \varepsilon\frac{\partial S^1(q, P, y_1, z_2, \varepsilon)}{\partial q}, \qquad & y_2 &= z_2 + \varepsilon\frac{\partial S^1(q, P, y_1, z_2, \varepsilon)}{\partial y_1}. \end{aligned} \tag{19}$$

Let us substitute (19) into (16) and (18) and equate the Hamiltonians in these "mixed" old–new variables:

$$\begin{aligned} &\mathcal{H}_i\biggl(\biggl(\frac{q^2+(P+\varepsilon\,\partial S^1/\partial q)^2}{2}\biggr), y_1, z_2 + \varepsilon\frac{\partial S^1}{\partial y_1}, \varepsilon\biggr) + \varepsilon g_i\biggl(q, P + \varepsilon\frac{\partial S^1}{\partial q}, y_1, z_2 + \varepsilon\frac{\partial S^1}{\partial y_1}, \varepsilon\biggr) \\ &\quad = \mathcal{H}_i\biggl(\biggl(\frac{(q+\varepsilon\,\partial S^1/\partial P)^2+P^2}{2}\biggr), y_1 + \varepsilon\frac{\partial S^1}{\partial z_2}, z_2, \varepsilon\biggr) \\ &\qquad + \varepsilon\bar{g}_i\biggl(\frac{(q+\varepsilon\,\partial S^1/\partial P)^2+P^2}{2}, y_1 + \varepsilon\frac{\partial S^1}{\partial z_2}, z_2, \varepsilon\biggr) + \varepsilon g_{i+1}\biggl(q + \varepsilon\frac{\partial S^1}{\partial P}, P, y_1 + \varepsilon\frac{\partial S^1}{\partial z_2}, z_2, \varepsilon\biggr). \end{aligned}$$

Expanding in a Taylor series, let us write out terms of identical order in ε and take into account the relation $\partial\mathcal{H}_i/\partial y_1 = \partial\mathcal{H}_i/\partial z_2 = O(\varepsilon)$:

$$\mathcal{H}_i\biggl(\frac{q^2+P^2}{2}, y_1, z_2, \varepsilon\biggr) = \mathcal{H}_i\biggl(\frac{q^2+P^2}{2}, y_1, z_2, \varepsilon\biggr), \tag{20}$$

$$\frac{\partial\mathcal{H}_i}{\partial P}\frac{\partial S^1}{\partial q} - \frac{\partial\mathcal{H}_i}{\partial q}\frac{\partial S^1}{\partial P} + g_i(q, P, y_1, z_2, \varepsilon) = \bar{g}_i\biggl(\frac{q^2+P^2}{2}, y_1, z_2, \varepsilon\biggr). \tag{21}$$

Equation (21) can be integrated using Lemma 1:

$$S(q, P, y_1, z_2, \varepsilon) = W(q, P, \mu), \qquad \text{where} \quad \mu = (y_1, z_2, \varepsilon).$$

The function S is an analytic function of all of its variables and of a small parameter ε, so we can solve system (19) using Lemma 2. The solution is continuous and has continuous derivatives; therefore, it determines the change of variables. We found the canonical transformation (17). On substituting into (16), we obtain an explicit formula for $\mathcal{H}_{i+1}$ and εg_{i+1}.

4.2. Estimates

Suppose that r steps have been made. The domain D_i in which our Hamiltonian is considered after i steps is defined as $D_{i+1} = D_1 - 2(i-1)K\varepsilon$, where $D_1 = D + (3/4)\delta$ and K is a positive constant to be determined later.

At the first step, it is readily shown that if $(q, p, y_1, y_2) \in D_1$ and ε is sufficiently small, then (17) is defined and the following conditions are satisfied:

$$|g_1| + |\nabla g_1| < k_1\varepsilon, \qquad |Q^1 + P^1 + Z_1^1 + Z_2^1| < k_2, \qquad |\mathcal{H}_1 - \mathcal{H}_0| < k_3\varepsilon,$$

where the $\{k_i\}$ are positive constants. Indeed, from (18) we readily see that $g_1 = O(\varepsilon^2)$. All the other inequalities follow from the estimation procedure defined below.

Take the inductive conjecture that for i: $1 \le i \le r$ the following estimates are satisfied:

$$|\mathcal{H}_i| < 2C, \qquad c < \left|\frac{\partial \mathcal{H}_i}{\partial I}\right| < 2C, \qquad |\nabla g_i| + |g_i| < M_i, \qquad M_i = \frac{k_1\varepsilon}{2^{i-1}}, \tag{22}$$

where $I = (q^2 + p^2)/2$. Now we must find ε_1 and K so that for $0 < \varepsilon < \varepsilon_1$ we have

$$(Q, P, z_1, z_2) \in D_{r+1} = D_r - 2K\varepsilon,$$

system (17) is defined with $i = r$, and the estimates (22) are satisfied with $i = r + 1$.

For $\partial S^1/\partial\varphi$, we have the following estimates from the definition of $\tilde{g}_i$ (7), Eq. (21) and the form of the solution (8):

$$\left|\frac{\partial S^1}{\partial \varphi}\right| < |\tilde{g}_i| \le |g_i| < M_i.$$

Estimates for $\partial S^1/\partial J$, $J = (q^2 + P^2)/2$, are obtained as follows:

$$\begin{aligned} \frac{\partial S^1}{\partial J} &= \frac{\partial}{\partial J}\left(\frac{1}{\partial \mathcal{H}_i/\partial J}\left(\frac{1}{2}\int_0^{\varphi} \tilde{g}(\psi, J)\,d\psi + \frac{1}{2}\int_{\pi}^{\varphi} \tilde{g}(\psi, J)\,d\psi\right)\right) \\ &= -\frac{\partial^2 \mathcal{H}_i}{\partial J^2}\frac{S^1}{\partial \mathcal{H}_i/\partial J} + \frac{1}{\partial \mathcal{H}_i/\partial J}\frac{\partial}{\partial J}\left(\frac{1}{2}\int_0^{\varphi} \tilde{g}(\psi, J)\,d\psi + \frac{1}{2}\int_{\pi}^{\varphi} \tilde{g}(\psi, J)\,d\psi\right). \end{aligned} \tag{23}$$

We can estimate $\partial^2\mathcal{H}_i/\partial J^2$:

$$\left|\frac{\partial^2 \mathcal{H}_i}{\partial J^2}\right| < \left|\frac{\partial^2 \mathcal{H}_0}{\partial J^2}\right| + \varepsilon\left|\frac{\partial^2 \bar{g}_0}{\partial J^2}\right| + \varepsilon\sum_{j=1}^{i}\left|\frac{\partial^2 \bar{g}_j}{\partial J^2}\right|.$$

From the original conditions we can obtain $|\partial^2\mathcal{H}_0/\partial J^2| < m_1$ and $|\partial^2\bar{g}_0/\partial J^2| < k_4$, where the $\{m_i\}$ are positive constants and are independent of the step i. For $|\partial^2\bar{g}_j/\partial J^2|$, we use Cauchy estimates for the analytic functions [4]:

$$\left|\frac{\partial^2 \bar{g}_i}{\partial J^2}\right| < \frac{M_i}{K\varepsilon}, \qquad \sum_{j=1}^{i}\left|\frac{\partial^2 \bar{g}_j}{\partial J^2}\right| < \sum_{j=1}^{i}\frac{k_1\varepsilon}{2^{j-1}}\frac{1}{K\varepsilon} < \frac{2k_1}{K}.$$

As a result, we obtain

$$\left|\frac{\partial^2\mathcal{H}_i}{\partial J^2}\right| < m_1 + k_4\varepsilon + \frac{2k_1}{K}\varepsilon < k_5. \tag{24}$$

For $\partial g_i/\partial J$, from (22) we obtain the estimates

$$\left|\frac{\partial g_i}{\partial J}\right| = \left|\frac{\partial \rho}{\partial J}\right|\left|\frac{\partial g_i}{\partial \rho}\right| < \frac{1}{\rho}\left(\left|\frac{\partial g_i}{\partial P}\right| + \left|\frac{\partial g_i}{\partial q}\right|\right) < \frac{1}{\rho}2M_i,$$

where $\rho^2 = J$.

Then the integral in (23) can be estimated as follows:

$$\left|\frac{1}{\partial\mathcal{H}_i/\partial J}\left(\frac{1}{2}\int_0^{\varphi}\frac{\partial\tilde g(\psi,J)}{\partial J}\,d\psi + \frac{1}{2}\int_{\pi}^{\varphi}\frac{\partial\tilde g(\psi,J)}{\partial J}\,d\psi\right)\right| < m_2\left(\frac{4\pi M_i}{\rho}\right) < \frac{m_3M_i}{\rho}.$$

Further, the expression (23) can be estimated as

$$\left|\frac{\partial S^1}{\partial J}\right| < \left(k_5 + \frac{m_3}{\rho}\right)M_i.$$

We have obtained estimates for $\partial S^1/\partial\varphi$ and $\partial S^1/\partial J$ and can now obtain estimates for $\partial S^1/\partial q$ and $\partial S^1/\partial P$:

$$\begin{aligned}\left|\frac{\partial S^1}{\partial P}\right| &= \left|\frac{\partial J}{\partial P}\frac{\partial S^1}{\partial J} + \frac{\partial\varphi}{\partial P}\frac{\partial S^1}{\partial\varphi}\right| < \rho\left(k_5 + \frac{m_3}{\rho}\right)M_i + \left|\frac{q}{q^2+P^2}\right||g_i| \\ &< (\rho k_5 + m_3)M_i + \left|\frac{q}{q^2+P^2}\right|\left|\frac{\partial g_i}{\partial q}q + \frac{\partial g_i}{\partial P}P\right| \\ &< (\rho k_5 + m_3)M_i + 2M_i < m_4M_i.\end{aligned} \tag{25}$$

Estimates for $\partial S^1/\partial q$ are obtained in a similar way:

$$\left|\frac{\partial S^1}{\partial q}\right| < m_5M_i\,; \tag{26}$$

The derivatives $\partial S^1/\partial y_1$ and $\partial S^1/\partial z_2$ can easily be estimated by differentiating g_i in the definition of the solution for S:

$$\left|\frac{\partial S^1}{\partial y_1}\right| < m_6M_i, \qquad \left|\frac{\partial S^1}{\partial z_2}\right| < m_7M_i. \tag{27}$$

Define $k_6 = \max(m_4, m_5, m_6, m_7)$. The function g_{i+1} we can defined as follows:

$$\begin{aligned}|g_{i+1}| = \bigg|&\frac{\varepsilon}{2}\left(\frac{\partial^2\mathcal{H}_i}{\partial P^2}\right)_\theta\left(\frac{\partial S^1}{\partial q}\right)^2 - \frac{\varepsilon}{2}\left(\frac{\partial^2\mathcal{H}_i}{\partial q^2}\right)_\theta\left(\frac{\partial S^1}{\partial P}\right)^2 + \left(\frac{\partial\mathcal{H}_i}{\partial z_2}\right)_\theta\frac{\partial S^1}{\partial y_1} - \left(\frac{\partial\mathcal{H}_i}{\partial y_1}\right)_\theta\frac{\partial S^1}{\partial z_2} \\ &+ \varepsilon\left(\frac{\partial g_i}{\partial P}\right)_\theta\frac{\partial S^1}{\partial q} - \varepsilon\left(\frac{\partial\bar g_i}{\partial q}\right)_\theta\frac{\partial S^1}{\partial P} + \varepsilon\left(\frac{\partial g_i}{\partial z_2}\right)_\theta\frac{\partial S^1}{\partial y_1} - \varepsilon\left(\frac{\partial\bar g_i}{\partial y_1}\right)_\theta\frac{\partial S^1}{\partial z_2}\bigg|,\end{aligned}$$

where $(\,)_\theta$ denotes the derivative at the midpoint. Then g_{i+1} can be estimated using (22), (24)–(27):

$$|g_{i+1}| < \left|2\frac{\varepsilon k_5k_6M_i^2}{2} + 2\varepsilon\left(k_4 + \frac{2k_1}{K}\right)M_i + 4\varepsilon k_6M_i^2\right| < k_7M_i\varepsilon.$$

Using Cauchy estimates [4], we obtain estimates for ∇g_{i+1}:

$$|\nabla g_{i+1}| < \left| 4\frac{\varepsilon k_5 k_6 M_i^2}{2K\varepsilon} + 2\varepsilon\left(k_4 + \frac{2k_1}{K}\right)\frac{M_i}{K\varepsilon} + 8\varepsilon k_6 \frac{M_i^2}{K\varepsilon}\right| < k_8 M_i\left(\varepsilon + \frac{1}{K}\right).$$

Choosing K sufficiently large and ε sufficiently small, we obtain $k_7\varepsilon < 1/4$ and $k_8(\varepsilon+1/K) < 1/4$. In that case

$$|g_{i+1}| + |\nabla g_{i+1}| < \frac{M_i}{2} = M_{i+1}$$

and the other inductive inequalities in (22) are valid for $i = r+1$. Thus we can make the required changes of variables with the chosen K and ε as long as D_r is not empty. After the substitutions we have $r = (\delta/(4K\varepsilon)) > k_9/\varepsilon$, obtaining

$$|g_r| + |\nabla g_r| < \frac{k_1\varepsilon}{2^{r-1}} < c_2 \exp\left(-\frac{1}{c_1\varepsilon}\right),$$

and inequalities (4) are also satisfied.

The theorem is now proved.

5. ADDENDUM

When this paper was prepared for publication, at the conference dedicated to the 100th anniversary of I. G. Petrovskii we came across similar results obtained in parallel by V. Gelfreich and L. Lerman [6].

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