ON THE SPECTRAL THEORY OF SURFACES WITH CUSPS

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Abstract. We are interested in the spectral properties of Dirac operators on noncompact surfaces. Under the assumption that 1) the ends of the given surface M are cusps as in the case of finite area surfaces of negative curvature and 2) the geometry of the Dirac bundle in question is closely related to the geometry of M we investigate the essential spectrum of the corresponding Dirac operator D and discuss its Fredholm index.

Introduction

Let M be a noncompact surface with a complete Riemannian metric. Assume that M has finitely many ends and that each end U of M is a cusp. By this we mean that U is diffeomorphic to $S \times (0, \infty)$, where $S = \mathbb{R}/\mathbb{Z}$ is the circle, the metric on U is of the form

$$f^2 ds^2 + dt^2$$

and has finite area, and the Gauss curvature K satisfies

$$-b^2 \leq K = \partial_t^2 f / f \leq -a^2$$

on U, where a, b are some appropriate positive constants. These assumptions imply that M is conformally equivalent to a closed surface $\bar{M}$ with finitely many points deleted.

The most important examples are complete surfaces with finite area and globally pinched negative Gauss curvature. For that class to be included in our discussion, we have to be somewhat modest in the regularity assumption since then the above diffeomorphism need only be C^2, cf. [Eb], [HIH]. More precisely, we assume that f is C^1 and that the second partial derivative of f in the t–direction exists and is continuous. When necessary we label the ends by an index k.

Let E be a graded Dirac bundle over M in the sense of Gromov and Lawson, see [LM]. That is, E is a bundle of left modules over the Clifford bundle $\mathrm{Cl}\, M$ of M with compatible Hermitian metric and connection together with a parallel field α of unitary involutions which anticommutes with Clifford multiplication by vector fields. These data determine a Dirac operator D and a decomposition

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$E = E^+ \oplus E^-$ into the eigenspaces of α with eigenvalue ± 1. We are interested in the spectral properties of D as an unbounded and essentially self–adjoint operator in the space $L^2(M, E)$ of square integrable sections of E.

There is no reason to expect close relations between the geometry of M and the spectral properties of D if the underlying Dirac bundle is not closely related to the geometry of M. To establish such a relation we propose the following notion. We say that a vector bundle E over M with Hermitian metric and connection is *geometric* if there is a parallel twoform S with values in the bundle of skew Hermitian endomorphisms of E such that the curvature R of E satisfies

$$R(X, Y)u = KS(X, Y)u$$

for all vector fields X, Y of M and sections u of E. The class of geometric bundles over M admits all the standard operations on vector bundles and contains all flat bundles and all bundles which are associated to the Riemannian structure or a spin structure of M.

Now let E be a fixed graded Dirac bundle over M and assume that E is geometric. Then for any local orthonormal frame X, Y of TM, $C := -XYS(X, Y)$ is a parallel field of Hermitian endomorphisms of E independent of the choice of X and Y and hence is globally defined. We have

$$XY \cdot R(X, Y)u = KCu$$

for any section u of E. Now the field α of involutions of E is parallel and anticommutes with Clifford multiplication by vector fields, hence $C(E^\pm) \subset E^\pm$. We set

$$C^\pm = C|E^\pm : E^\pm \to E^\pm \,.$$

Let $U = S \times (0, \infty)$ be an end of M. Let $T = \partial_t$ and $X = f^{-1}\partial_s$ and orient U by this frame; if M is oriented, then we asume that the orientations of U and M coincide. The oriented orthonormal frame T, X is a section of the $SO(2)$–principal bundle $SO(U)$ of oriented orthonormal frames on U and determines a trivialization $SO(U) = U \times SO(2)$. Let $\widetilde{SO}(U) = U \times \mathbb{R} \to U \times SO(2)$ be the corresponding lift, where we map $r \in \mathbb{R}$ to the counterclockwise rotation by the angle $2\pi r$. Then $\widetilde{SO}(U)$ is an $\mathbb{R}$–principal bundle over U and any bundle associated to $\widetilde{SO}(U)$ via a unitary representation of $\mathbb{R}$ comes with a Hermitian metric and connection. Now for $c \in \mathbb{R}$ and $w \in \mathbb{C}$ with $|w| = 1$ we let $L(c, w)$ be the complex line bundle associated to $\widetilde{SO}(U)$ via the unitary representation $\exp(2\pi icr)$ of $\widetilde{SO}(2)$, and with atlas given by the holonomy requirement $[s + 1, t, 0, u] = [s, t, 0, wu]$ along the circles $S \times \{t\}$. Note that $L(1, 1)$ is the tangent bundle of U with its natural complex structure determined by the chosen orientation.

The complex area element $\omega_{\mathbb{C}} = iTX$ defines a parallel field of unitary involutions of E over U which commutes with α and C. We set

$$E^\pm(c, \varepsilon) = \{u \in E^\pm|U \mid Cu = cu \,, \omega_{\mathbb{C}} u = \varepsilon u\} \,,$$

where $c \in \mathbb{R}$ and $\varepsilon \in \{+1,-1\}$. We will show that $E^+(c,\varepsilon)$ is isomorphic to an orthogonal sum of line bundles $L(-\varepsilon c, w)$, and we denote by $m(c,\varepsilon,w)$ the number of times $L(-\varepsilon c, w)$ occurs in this sum. The spectral characteristics which we investigate in this paper are given in terms of C^+ and the integers $m(c,\varepsilon,w)$. All data associated to an end $U = U_k$ will be indexed by k if necessary.

Suppose $E = \mathrm{Cl}\, M$ is the Clifford bundle over M with the even–odd decomposition. Then $E^+|U = E^+(0,1) \oplus E^+(0,-1)$, $m(0,\varepsilon,1) = 1$ and $m(0,\varepsilon,w) = 0$ for $w \neq 1$. If M is oriented and E is the spinor bundle associated to the corresponding spin^c structure, then $E^+|U = E^+(0,1)$, $m(0,1,1) = 1$ and $m(0,1,w) = 0$ for $w \neq 1$. If E is a spinor bundle associated to a spin structure of M, then $E^+|U = E^+(1/2,1)$. If the spin structure is the restriction of a spin structure of the closed surface $\bar{M}$, then $m(1/2,1,-1) = 1$ and $m(1/2,1,w) = 0$ for $w \neq -1$.

We first analyze the essential spectrum $\operatorname{spec}_e D$ of D which is localized on the ends of M by the decomposition principle. Recall that D is Fredholm if and only if $0 \notin \operatorname{spec}_e D$.

0.1. Theorem. *1) If $m_k(c,\varepsilon,1) = 0$ for all k, c, ε, then* $\operatorname{spec}_e D = \emptyset$.
2) If $m_k(1/2,\varepsilon,1) = 0$ for all k and ε, then D is Fredholm.
3) If $m_k(1/2,\varepsilon,1) \neq 0$ and the directional derivative $X(K)$ is uniformly bounded on U_k for some k, then $\operatorname{spec}_e D = \mathbb{R}$.

Note that the assumption on $X(K)$ in the third assertion holds if the end is warped, that is, if f does not depend on s.

Set $D^+ = D|L^2(M,E^+) \cap C^1(M,E^+)$. In the case when D is Fredholm, we also determine the index of D^+ explicitly.

0.2. Theorem. *Let $m = \operatorname{rank} E^+$ and suppose that $m_k(1/2,\varepsilon,1) = 0$ for all k and ε. Then*

$$\operatorname{ind} D^+ = (m/2 - \operatorname{tr} C^+)\,\chi(M) + \frac{1}{2}\sum_{k,c,\varepsilon} \operatorname{sign}(1/2 - c)\, m_k(c,\varepsilon,1) - \frac{1}{2}\sum_{k,c,\varepsilon,w} \varepsilon\, \eta(w)\, m_k(c,\varepsilon,w)\,,$$

where $\eta(1) = 0$ and $\eta(\exp(2\pi i\rho)) = 1 - 2\rho$ for $0 < \rho < 1$.

We emphasize that in our theorems, we do not need assumptions on derivatives of the curvature or, respectively, assumptions on the asymptotic behaviour of third or higher derivatives of the metric. Special cases of the above theorems were obtained in [DX], [St], [Br], and [Bä].

In the case when E is the Clifford bundle with the even–odd decomposition, then D^+ is called the *Gauss–Bonnet operator*. We get

$$\operatorname{ind} D^+ = \chi(M) + \#\{\text{ends}\} = \chi(\bar{M})\,.$$

When M is oriented and E is the spinor bundles associated to the spin^c structure of M, then D^+ is called the *Riemann–Roch operator*. In this case

$$\operatorname{ind} D^+ = \big(\chi(M) + \#\{\text{ends}\}\big)/2 = \chi(\bar M)/2\,.$$

When M is oriented and E is the spinor bundle associated to a a spin structure which extends to a spin structure on $\bar M$, then $\operatorname{ind} D^+ = 0$.

Recall that M is conformally equivalent to a closed surface $\bar M$ with finitely many points deleted. Hence the Riemannian metric on M is conformally equivalent to a Riemannian metric with cylindrical ends. The latter were discussed in the work of Atiyah, Patodi and Singer [APS], and therefore it is interesting to know whether the dimension of the space $\mathcal{H}^2(M,E)$ of square integrable harmonic sections of E is a conformal invariant. Now it is a rather straightforward consequence of our description of geometric bundles in the text that $\dim \mathcal{H}^2(M,E^+(1))$ and $\dim \mathcal{H}^2(M,E^-(0))$ are conformally invariant, where

$$E^\pm(c) = \{u \in E^\pm \mid Cu = cu\}\,,$$

but in other cases conformal invariance may fail. Using the the conformal invariance of $\dim \mathcal{H}^2(M,E^-(0))$ the Gauss–Bonnet and Riemann–Roch formulas above are easy consequences of the results in [APS], but also follow easily by a direct argument for surfaces with cylindrical ends as in Section 4 of [APS].

The plan of the paper is as follows. In Section 1 we discuss the geometry of the ends in more detail. In Section 2 we characterize geometric bundles and determine their structure over the ends of M. In Section 3 we introduce operators which model the Dirac operator along the ends. In Section 4 we prove our main analytical lemmas. In Section 5 we investigate the essential spectrum and prove Theorem 0.1. In Section 6 we compute the index formula from Theorem 0.2.

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1. The Ends

By assumption each end U is diffeomorphic to $S \times (0, \infty)$, where $S = \mathbb{R}/\mathbb{Z}$ is the circle, and the metric is of the form $f^2 ds^2 + dt^2$, where $f = f(s,t)$ is continuously differentiable with respect to s and twice continuously differentiable with respect to t. We recall that we assume that the area of U is finite. We have $f'' + Kf = 0$, where the *Gauss curvature* $K = K(s,t)$ is continuous and bounded by $-b^2 \le K \le -a^2 < 0$ for appropriate positive constants a, b. Here and below the prime $'$ denotes differentiation with respect to the variable t.

The curves $c_s(.) = (s, .)$ are unit speed geodesics. The circles $S_t = S \times \{t\}$ are perpendicular to the geodesics c_s. The length and geodesic curvature of S_t are given by

$$\bar{f}(t) = \int_0^1 f(s,t) ds \tag{1.1}$$

and

$$\kappa(s,t) = -f'(s,t)/f(s,t)\,, \tag{1.2}$$

respectively. Recall that κ satisfies the *Riccati equation*

$$\kappa' = \kappa^2 + K\,. \tag{1.3}$$

By the assumption on finite area f is the *stable solution* of the Jacobi equation. Hence

$$\begin{gathered} f(s,0)\exp(-bt) \le f(s,t) \le f(s,0)\exp(-at)\,, \\ a \le \kappa(s,t) \le b\,. \end{gathered} \tag{1.4}$$

We also introduce the *averaged geodesic curvature*

$$\bar{\kappa}(t) = \frac{1}{\bar{f}(t)} \int_0^1 \kappa(s,t) f(s,t) ds\,. \tag{1.5}$$

By (1.2),

$$\bar{\kappa}(t) = -\bar{f}'(t)/\bar{f}(t)\,. \tag{1.6}$$

Furthermore,

$$\begin{gathered} \bar{f}(0)\exp(-bt) \le \bar{f}(t) \le \bar{f}(0)\exp(-at)\,, \\ a \le \bar{\kappa}(t) \le b\,. \end{gathered} \tag{1.7}$$

Later it will be important to consider the *arc length* along S_t,

$$r(s,t) = \frac{1}{\bar{f}(t)} \int_0^s f(s,t) ds\,. \tag{1.8}$$

Then

$$r(0,t) = 0 \quad \text{and} \quad r(1,t) = 1\,. \tag{1.9}$$

Let ∇ be the Levi–Civita connection and define the orthonormal frame T, X as in the introduction. This frame is parallel in the T–direction and we have

$$\nabla_X T = -\kappa X \quad \text{and} \quad \nabla_X X = \kappa T\,. \tag{1.10}$$

Denote by $\operatorname{var}\kappa_t$ the *variance* of κ on S_t,

$$\operatorname{var}\kappa_t = \frac{1}{\bar f(t)} \int_S (\kappa(s,t) - \bar\kappa(t))^2 f(s,t)ds\,. \tag{1.11}$$

The following lemma is well known, compare &7 in [Ho], and gives a sufficient criterion for the eventual vanishing of the variance of κ.

1.12. Lemma. *If the metric on U is C^3 and $X(K)$ is uniformly bounded, then*

$$\lim_{t\to\infty} \operatorname{var}\kappa_t = 0\,.$$

2. The Bundles

We fix an end $U = S \times (0, \infty)$ of M and let $SO(U)$ be the $SO(2)$–principal bundle of oriented orthonormal frames over U. Our preferred frame $F = (T, X)$ is a global section of $SO(U)$ and determines a trivialization

$$U \times SO(2) \to SO(U)\,, (s,t,A) \mapsto F(s,t)A\,,$$

and a lift

$$\widetilde{SO}(U) := U \times \mathbb{R} \xrightarrow[\mathrm{id}\times\pi]{} U \times SO(2) = SO(U)$$

of $SO(U)$ to an $\mathbb{R}$–principal bundle over U, where $\pi(r)$ is counterclockwise rotation by the angle $2\pi r$.

The tangent bundle with its natural parallel complex structure — counterclockwise rotation by a right angle — is the complex line bundle associated to the representation $\exp(2\pi i r)$ of $\mathbb{R}$ on $\mathbb{C}$.

The section $\tilde{F}(s,t) = (s,t,0)$ is a lift of the frame $F = F(s,t)$ and determines lifts $\tilde{T} = \tilde{F}_* T$ of T and $\tilde{X} = \tilde{F}_* X$ of X along the image of $\tilde{F}$. In terms of $\tilde{T}$ and $\tilde{X}$, the connection form ω and curvature form Ω of the Levi-Civita connection are given by

$$\omega(\tilde{T}) = 0\,, \quad \omega(\tilde{X}) = -\kappa i \quad \text{and} \quad \Omega(\tilde{T}, \tilde{X}) = -Ki\,.$$

Fix $c \in \mathbb{R}$ and $w \in \mathbb{C}$ with $|w| = 1$. Let $L(c,w)$ be the complex line bundle associated to $\widetilde{SO}(U)$ via the representation $\exp(2\pi i c r)$ of $\mathbb{R}$ on $\mathbb{C}$ with an atlas given by the holonomy relation $[s+1, t, 0, u] = [s, t, 0, wu]$. Since the representation of $\mathbb{R}$ is unitary, $L(c,w)$ has a natural Hermitian metric. Furthermore, ω determines a unitary connection on $L(c,w)$. It is easy to see that $L(c,w)$ is a geometric bundle; we prove a more precise statement in the lemma below. We also have

$$L(c,w) \otimes L(c',w') = L(c+c', ww')\,. \tag{2.1}$$

2.2. Lemma. *The line bundle $L(c,w)$ has a section Φ of norm 1 such that*

$$\Phi(s+1,t) = w\Phi(s,t)\,, \quad \nabla_T \Phi = 0\,, \quad \nabla_X \Phi = -\kappa i c \Phi\,, \quad R(T,X)\Phi = -Kic\Phi\,.$$

Proof. In the standard notation for bundles associated to principal bundles let $\Phi = \Phi(s,t) = [s,t,0,1]$. Then Φ has norm 1 and the asserted holonomy. Now under the representation $\exp(2\pi i c r)$, the connection form ω and curvature form Ω are multiplied by c. Hence Φ also satisfies the remaining equations. □

We now describe the structure of geometric bundles over U. A similar description is also valid over all of M. However, for ease of presentation we stick to geometric bundles over U.

2.3. Lemma. *Let E be a geometric bundle over U. Then E is isomorphic to an orthogonal sum of line bundles $L(c,w)$.*

Proof. For $c \in \mathbb{R}$ let $E(c) = \{u \in E \mid Cu = cu\}$. Then E is the direct sum of the parallel and pairwise perpendicular subbundles $E(c)$. Hence we may assume that $E = E(c)$ for some fixed c.

Let $t > 0$ and Φ, Ψ be sections of $E|S_t$ solving

$$\nabla_X \Phi = -\kappa i c \Phi \,, \quad \nabla_X \Psi = -\kappa i c \Psi \,.$$

Then

$$\begin{aligned} X\langle \Phi, \Psi \rangle &= \langle \nabla_X \Phi, \Psi \rangle + \langle \Phi, \nabla_X \Psi \rangle \\ &= \langle -\kappa i c \Phi, \Psi \rangle + \langle \Phi, -\kappa i c \Psi \rangle = 0 \,. \end{aligned}$$

Now extend Φ by parallel translation in the T-direction and recall that $[T, X] = \kappa X$, see (1.10). We get

$$\nabla_T(\nabla_X \Phi) = \nabla_{[T,X]} \Phi + R(T, X)\Phi = -\kappa^2 i c \Phi - K i c \Phi \,.$$

Using (1.3) we also have

$$\nabla_T(-\kappa i c \Phi) = -\kappa' i c \Phi = -(\kappa^2 + K) i c \Phi \,.$$

Hence $\nabla_X \Phi$ and $-\kappa i c \Phi$ solve the same differential equation. Hence $\nabla_X \Phi = -\kappa i c \Phi$ on all of U.

We conclude that there is an orthonormal frame $\Phi_1, \dots, \Phi_m$ of E, where $m = \operatorname{rank} E$, such that $\nabla_T \Phi_j = 0$ and $\nabla_X \Phi_j = -\kappa i c \Phi_j$ for all j. It follows that the holonomy $H = (h_{ij})$, defined by

$$\Phi_j(s+1, t) = \sum h_{ij} \Phi_i(s, t) \,,$$

is independent of (s, t) and unitary. Now an appropriate unitary change of frame with constant coefficients diagonalizes H. □

We now turn to geometric Dirac bundles. Fix $c \in \mathbb{R}$, $\varepsilon \in \{-1, +1\}$ and $w \in \mathbb{C}$ with $|w| = 1$. Let

$$L(c, \varepsilon, w) = L(-\varepsilon c, w) \oplus L(\varepsilon(1 - c), w) \,.$$

Viewing the Clifford bundle as the bundle associated to $\widetilde{SO}(U)$ via the representation $\exp(2\pi i r)$ on $\operatorname{Cl} \mathbb{R}^2$, we describe Clifford multiplication on $L(c, \varepsilon, w)$ by the following representation of $\operatorname{Cl} \mathbb{R}^2$ on $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \begin{pmatrix} 0 & \varepsilon i \\ \varepsilon i & 0 \end{pmatrix} , \quad \begin{pmatrix} -\varepsilon i & 0 \\ 0 & \varepsilon i \end{pmatrix} .$$

Here the matrices describe Clifford multiplication by $e_1, e_2, e_1 e_2 \in \operatorname{Cl} \mathbb{R}^2$ respectively. Furthermore, let $\alpha = \varepsilon \omega_{\mathbb{C}}$, where $\omega_{\mathbb{C}} = iTX$ is the complex area element. It is easy to see that $L(c, \varepsilon, w)$ is a graded Dirac bundle with $L^+(c, \varepsilon, w) = L(-\varepsilon c, w)$ and $L^-(c, \varepsilon, w) = L(\varepsilon(1 - c), w)$. Moreover, it is immediate that

$$L(c, \varepsilon, w) \otimes L(c', w') = L(c - \varepsilon c', \varepsilon, w w') \,. \tag{2.4}$$

2.5. Lemma. *There are sections* Φ^+ *of* $L^+(c,\varepsilon,w)$ *and* Φ^- *of* $L^-(c,\varepsilon,w)$ *of norm* 1 *such that*

$$\Phi^{\pm}(s+1,t) = w\Phi^{\pm}(s,t)\,, \quad T\Phi^{\pm} = \pm\Phi^{\mp}\,, \quad X\Phi^{\pm} = \varepsilon i\Phi^{\mp}\,,$$
$$\nabla_T\Phi^{\pm} = 0\,, \quad X\nabla_X\Phi^+ = -\kappa c\Phi^-\,, \quad X\nabla_X\Phi^- = \kappa(1-c)\Phi^+\,,$$
$$TXR(T,X)\Phi^+ = Kc\Phi^+\,, \quad TXR(T,X)\Phi^- = K(1-c)\Phi^-\,.$$

Proof. Let Φ^+ be a section of $L(-\varepsilon c,w)$ as in Lemma 2.2 and set $\Phi^- = T\Phi^+$. It is immediate from the definition of Clifford multiplication that these sections have the asserted properties. □

We now describe the structure of graded geometric Dirac bundles over U. As in the case of geometric bundles, there is a similar description for graded geometric Dirac bundles over all of M.

2.6. Lemma. *Let* E *be a graded geometric Dirac bundle over* U *with involution* α*. Then* E *is isomorphic to an orthogonal sum of bundles* $L(c,\varepsilon,w)$*.*

It is clear that the numbers $m(c,\varepsilon,w)$ from the introduction are equal to the number of times that the bundles $L(c,\varepsilon,w)$ occur in the decomposition of E over U in Lemma 2.6.

Proof of Lemma 2.6. As in the introduction, let $TXR(T,X) = KC$. We have

$$C(v\cdot u) = v\cdot(1-C)u\,,$$

for any $v \in TM$ and $u \in E$ with the same foot point [1]. Note that E^+ and E^- are invariant under R and Clifford multiplication by TX, hence $C(E^{\pm}) \subset E^{\pm}$.

The complex area element $\omega_{\mathbb{C}} = iTX$ commutes with the involution α of E and C and hence $E^{\pm}$ is the direct sum of the pairwise perpendicular and parallel subbundles

$$E^{\pm}(c,\varepsilon) = \{u \in E^{\pm} \mid C(u) = cu,\, \omega_{\mathbb{C}}u = \varepsilon u\}\,,$$

where $c \in \mathbb{R}$ and $\varepsilon \in \{+1,-1\}$. Hence E is the direct sum of the pairwise perpendicular and parallel graded Dirac subbundles

$$E(c,\varepsilon) = E^+(c,\varepsilon) \oplus E^-(1-c,-\varepsilon)\,.$$

Hence we may assume $E = E(c,\varepsilon)$. But then by Lemma 2.3, E^+ is isomorphic to the orthogonal sum of line bundles $L(-\varepsilon c,w)$. It follows easily that E is isomorphic to the orthogonal sum of the corresponding graded Dirac bundles $L(c,\varepsilon,w)$. □

2.7. Examples. 1) The Clifford bundle $\mathrm{Cl}\,U$ together with the even–odd decomposition is equal to $L(0,1,1) \oplus L(0,-1,1)$.

2) The chosen orientation determines a spin^c structure on U. The corresponding spinor bundle is equal to $L(0,1,1)$.

[1] For this conclusion we need that $K(s,t) \neq 0$ for some $(s,t) \in U$.

3) There are two spin structures on U. The *trivial spin structure* is $\widetilde{SO}(U)/2\mathbb{Z}$. The spinor bundle associated to the trivial spin structure is $L(1/2, 1, 1)$.

The correspondence $(s,t) \leftrightarrow \exp(-(t+is))$ identifies U with the unit ball B with midpoint removed. The restriction of the unique spin structure of B to U is called the *nontrivial spin structure*. The spinor bundle associated to the nontrivial spin structure of U is equal to $L(1/2, 1, -1)$.

It follows from the above description of spinor bundles and (2.4) that we can factor graded geometric Dirac bundles with involution $\alpha = \omega_{\mathbb{C}}$ by spinor bundles,

$$L(c, 1, w) = L(1/2, 1, \pm 1) \otimes L(1/2 - c, \pm w)\,. \tag{2.8}$$

3. The Model Operators

Lemma 2.6 implies that the properties of the Dirac operator on a graded Dirac bundle E over an end $U = S \times (0, \infty)$ are determined by the properties of the Dirac operators on the bundles $L(c, \varepsilon, w)$.

For $t > 0$ write

$$C^1(S_t)_w = \{u \in C^1([0,1] \times \{t\}) \mid u(1,t) = u(0,t)/w\}\,.$$

For an interval $I \subset (0, \infty)$ write correspondingly

$$C^1(S \times I)_w = \{u \in C^1([0,1] \times I) \mid u(1,t) = u(0,t)/w\}\,.$$

We let $C^1_0(S \times I)_w$ be the subspace of $u \in C^1(S \times I)_w$ which have compact support in $[0,1] \times I$.

We now transform the Dirac operators on the bundles $L(c, \varepsilon, w)$ to operators which are convenient for our purposes. Define matrices

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}$$

Then the map

$$\begin{aligned} \Psi : C^1_0\big(S \times (0,\infty)\big)_w \otimes \mathbb{C}^2 &\to C^1_0(U, L(c,\varepsilon,w))\,, \\ u = (u^+, u^-) &\mapsto u^+\Phi^+ + u^-\Phi^-\,, \end{aligned}$$

transforms the Dirac operator to the *model operator*

$$\begin{aligned} D_\infty := \Psi^{-1} D \Psi &= \gamma\big((\partial_t - \kappa/2) + (\varepsilon i X + \kappa(1/2 - c))\tau\big) \\ &= \gamma\big((\partial_t - \kappa/2) + A + \kappa B\big)\,, \end{aligned} \tag{3.2}$$

where

$$A = A(t) = \begin{pmatrix} A^+(t) & 0 \\ 0 & -A^+(t) \end{pmatrix} = \begin{pmatrix} \varepsilon i X & 0 \\ 0 & -\varepsilon i X \end{pmatrix} \tag{3.3}$$

and

$$B = (1/2 - c)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 - c & 0 \\ 0 & c - 1/2 \end{pmatrix}. \tag{3.4}$$

The operator D_∞ depends on the Riemannian metric on $U = U_k$ and on the parameters c, ε and w.

We recall that the area element on U is $f ds dt$. For $t > 0$ and functions $u, v \in C^1(S_t)_w \otimes \mathbb{C}^2$ we let

$$(u, v)_t = \int_S \langle u(s,t), v(s,t)\rangle f(s,t) ds\,, \tag{3.5}$$

where $\langle .\,,. \rangle$ denotes the inner product in $\mathbb{C}^2$. For an interval $I \subset (0,\infty)$ and functions $u, v \in C^1(S \times I)_w \otimes \mathbb{C}^2$ we let

$$(u,v)_I = \int_I (u,v)_t dt = \int_I \int_S \langle u(s,t), v(s,t) \rangle f(s,t) ds dt\,, \tag{3.6}$$

denote the L^2–inner product of u and v. We use analogous notation for the corresponding norms. Note that Ψ extends to a unitary map

$$\Psi : L^2\big((S \times (0,\infty); f ds dt\big) \otimes \mathbb{C}^2 \to L^2(U, L(c,\varepsilon,w))\,.$$

We now turn to the family of operators $A = A(t)$ from (3.2) which splits into operators $A^+(t) = \varepsilon i X$ and $A^-(t) = -\varepsilon i X$ according to (3.3). We recall that the length of S_t is $\bar{f}(t)$ and that X has norm one. Hence we have the following lemma.

3.7. Proposition. *The operator iX is essentially self–adjoint as an operator in $L^2(S_t)$ with domain $C^1(S_t)_w$. Its spectrum is discrete and consists of the eigenvalues*

$$-2\pi(\rho + k)/\bar{f}(t)\,, \quad k \in \mathbb{Z}\,,$$

where we write $w = \exp(2\pi i \rho)$. The corresponding eigenspaces are spanned by the functions $\exp(2\pi i(\rho + k) r/\bar{f}(t))$, where $r = r(s,t)$ denotes arc length along S_t as in (1.8).

We note that 0 is in the spectrum of $A(t)$ if and only if $w = 1$ and that the nonzero eigenvalues grow like $1/\bar{f}(t) \to \infty$. Both these observations will be important below.

We let $P_0 = P_0(t)$ be the orthogonal projection onto the kernel of $A(t)$. If $w \neq 1$, then $P_0 = 0$ and if $w = 1$, then

$$P_0 u(s,t) = \bar{u}(t) = \frac{1}{\bar{f}(t)} \int_S u(s,t) f(s,t) ds\,. \tag{3.8}$$

Note that $\bar{u}$ is continuously differentiable if u is and that

$$\operatorname{supp} \bar{u} \subset \{t \mid \operatorname{supp} u \cap S_t \neq \emptyset\}\,. \tag{3.9}$$

In the case $w = 1$ we will also consider the following symmetric *model operator*

$$D_{\infty 0} := P_0 D_\infty P_0 = \gamma\big((\partial_t - \bar{\kappa}/2) + \bar{\kappa}(1/2 - c)\tau\big)\,, \tag{3.10}$$

with domain $C_0^1(0,\infty) \otimes \mathbb{C}^2$ in $L^2\big((0,\infty), \bar{f} dt\big) \otimes \mathbb{C}^2$. Here we identify functions in $C_0^1(0,\infty) \otimes \mathbb{C}^2$ with functions on $S \times (0,\infty)$ which do not depend on s, compare (3.8). Note that the substitution $\tilde{u} = \sqrt{\bar{f}} u$ defines a unitary equivalence of $L^2\big((0,\infty), \bar{f} dt\big) \otimes \mathbb{C}^2$ and $L^2\big((0,\infty), dt\big) \otimes \mathbb{C}^2$ and transforms $D_{\infty 0}$ into the Dirac system

$$\gamma\big(\partial_t + \bar{\kappa}(1/2 - c)\tau\big) \tag{3.11}$$

with domain $C_0^1(0,\infty) \otimes \mathbb{C}^2$.

4. The Weapons

We fix an end $U = S\times(0,\infty)$ and consider a model operator D_∞ as in (3.2). We recall that the area element on U is $f\,dsdt$. The following lemma is now immediate from the definition of the L^2–inner product (3.6) and partial integration.

4.1. Lemma. *Let $I = [t_0, t_1] \subset (0,\infty)$ and assume that $u, v \in C^1(S\times I)_w \otimes \mathbb{C}^2$. Then*

$$(\partial_t u, v)_I = (u, -\partial_t v + \kappa v)_I + \{(u,v)_{t_1} - (u,v)_{t_0}\}\,.$$

In particular, $\gamma(\partial_t - \kappa/2)$ is symmetric on $C^1_0\big(S\times(0,\infty)\big)_w \otimes \mathbb{C}^2$.

Now P_0 is self–adjoint, see (3.8). Hence the following formulas are immediate from Lemma 4.1.

4.2. Lemma. *On $C^1\big(S\times(0,\infty)\big)_w \otimes \mathbb{C}^2$ we have*

$$P_0\partial_t P_0 = \partial_t P_0 \quad \text{and} \quad P_0\kappa P_0 = \bar\kappa P_0\,.$$

Therefore we also get

$$(1-P_0)\partial_t P_0 = 0\,, \qquad (1-P_0)\kappa P_0 = (\kappa-\bar\kappa)P_0\,,$$
$$P_0\partial_t(1-P_0) = P_0(\kappa-\bar\kappa)\,, \qquad P_0\kappa(1-P_0) = P_0(\kappa-\bar\kappa)\,.$$

Our first weapon is the following lemma.

4.3. Lemma. *On $C^1\big(S\times(0,\infty)\big)_w \otimes \mathbb{C}^2$ we have*

$$\begin{aligned} P_0 D_\infty P_0 &= \gamma\big((\partial_t - \bar\kappa/2) + \bar\kappa B\big)P_0\,,\\ (1-P_0)D_\infty P_0 &= \gamma(\kappa-\bar\kappa)(B-1/2)P_0\,,\\ P_0 D_\infty(1-P_0) &= \gamma P_0(\kappa-\bar\kappa)(B+1/2)(1-P_0)\,. \end{aligned}$$

Proof. By definition, $AP_0 = P_0A = 0$. Furthermore, $[P_0,\gamma] = [P_0,\tau] = 0$ and $B\gamma + \gamma B = 0$. The formulas are now immediate from Lemma 4.2. $\square$

By $P_>(t)$ and $P_<(t)$ we denote the spectral projection in $L^2(S_t)_w \otimes \mathbb{C}^2$ corresponding to the eigenvalues of $A(t)$ which are > 0 and < 0, respectively. The following lemma is our second weapon.

4.4. Lemma. *Let $I = [t_0, t_1] \subset (0,\infty)$ be a compact interval and suppose $u \in C^1(S\times I)_w \otimes \mathbb{C}^2$ satisfies $P_>(t_0)u_{t_0} = 0$ and $P_<(t_1)u_{t_1} = 0$. Then*

$$\|Au\|_I \le \|D_\infty u\|_I + C_1\,\|u\|_I\,,$$

where $C_1 = b\max(|c|, |1-c|)$.

Proof. By approximation we may assume that u is C^2. Now by (1.10), $[\partial_t, X] = \kappa X$. Therefore $\partial_t Au = A\partial_t u + \kappa Au$ and hence, by Lemma 4.1,

$$(\partial_t u, Au)_I = (u, -A\partial_t u)_I + \{(u,Au)_{t_1} - (u,Au)_{t_0}\}\,.$$

Now A is symmetric. Therefore our assumption on u implies

$$2\,\mathrm{Re}(\partial_t u, Au)_I = \{(u, Au)_{t_1} - (u, Au)_{t_0}\} \geq 0\,.$$

Hence

$$\begin{aligned}\|D_\infty u\|_I \|Au\|_I &\geq \mathrm{Re}(D_\infty u, \gamma A u)_I \\ &= \mathrm{Re}(\partial_t u, Au)_I + \|Au\|_I^2 + \mathrm{Re}(\kappa(B - 1/2)u, Au)_I \\ &\geq \|Au\|_I^2 - b\max(|c|, |1-c|)\|u\|_I\|Au\|_I\,.\end{aligned}$$

□

Recall that $P_0 = 0$ if $w \neq 1$. We assume now that $w = 1$. Then for $u = (u^+, u^-) \in C_0^1\big(S \times (0,\infty)\big) \otimes \mathbb{C}^2$ we have $P_0 u = (\bar{u}^+, \bar{u}^-)$, where $\bar{u}^\pm$ is defined as in (3.8). We set $P_0^+ u = (\bar{u}^+, 0)$ and $P_0^- u = (0, \bar{u}^-)$ so that $P_0 = P_0^+ + P_0^-$. Now our third and last weapon reads as follows.

4.5. Lemma. *Assume $w = 1$ and $c \neq 1/2$. Let $I = [t_0, t_1] \subset (0,\infty)$ be a compact interval and suppose $u \in C^1(S \times I) \otimes \mathbb{C}^2$ satisfies*

$$\begin{aligned}P_0^-(t_0)u_{t_0} = 0\,, P_0^+(t_1)u_{t_1} = 0 \quad \text{if } 1/2 < c\,,\\ P_0^+(t_0)u_{t_0} = 0\,, P_0^-(t_1)u_{t_1} = 0 \quad \text{if } 1/2 > c\,.\end{aligned}$$

Then

$$a|1/2 - c|\,\|P_0 u\|_I \leq \|P_0 D_\infty u\|_I + C_2\,\|(1 - P_0)u\|_I\,,$$

where $C_2 = b|1/2 - c| + (b-a)/2$.

Proof. We set $\tilde\tau = \mathrm{sign}(1/2 - c)\,\tau$, with τ from (3.1). Now we have $P_0 A = AP_0 = 0$ by the definition of P_0. Hence

$$\begin{aligned}(P_0 D_\infty u, \gamma\tilde\tau P_0 u)_I &= ((\partial_t - \kappa/2)P_0 u, \tilde\tau P_0 u)_I + ((\partial_t - \kappa/2)(1 - P_0)u, \tilde\tau P_0 u)_I \\ &\quad + (\kappa B P_0 u, \tilde\tau P_0 u)_I + (\kappa B(1 - P_0)u, \tilde\tau P_0 u)_I\,.\end{aligned}$$

Next we estimate the four terms on the right hand side. By Lemma 4.1, the first term is

$$\begin{aligned}((\partial_t - \kappa/2)P_0 u, \tilde\tau P_0 u)_I &= -(P_0 u, (\partial_t - \kappa/2)\tilde\tau P_0 u)_I \\ &\quad + (P_0 u, \tilde\tau P_0 u)_{t_1} - (P_0 u, \tilde\tau P_0 u)_{t_0}\,.\end{aligned}$$

By our assumption

$$(P_0 u, \tilde\tau P_0 u)_{t_1} - (P_0 u, \tilde\tau P_0 u)_{t_0} = \begin{cases}\|\bar u^-\|_{t_1} + \|\bar u^+\|_{t_0} & \text{if } 1/2 < c\,,\\ \|\bar u^+\|_{t_1} + \|\bar u^-\|_{t_0} & \text{if } 1/2 > c\,,\end{cases}$$

so the above computation gives

$$\mathrm{Re}((\partial_t - \kappa/2)P_0 u, \tilde\tau P_0 u)_I \geq 0\,.$$

By Lemma 4.2, the second term can be estimated as follows,

$$
\begin{aligned}
|((\partial_t - \kappa/2)(1-P_0)u, \tilde{\tau} P_0 u)_I| &= |(P_0 \frac{\kappa - \bar{\kappa}}{2}(1-P_0)u, \tilde{\tau} P_0 u)_I| \\
&\leq \frac{b-a}{2} \|(1-P_0)u\|_I \|P_0 u\|_I \,.
\end{aligned}
$$

Recall that $B = (1/2 - c)\tau$. Hence the absolute value of the fourth term can be estimated by

$$|(\kappa B(1-P_0)u, \tilde{\tau} P_0 u)_I| \leq b|1/2 - c| \|(1-P_0)u\|_I \|P_0 u\|_I \,.$$

The crucial estimate involves the third term, but this is immediate from the definition of B and $\tilde{\tau}$:

$$(\kappa B P_0 u, \tilde{\tau} P_0 u)_I \geq a|1/2 - c| \|P_0 u\|_I^2 \,.$$

We conclude that

$$
\begin{aligned}
&\|P_0 D_\infty u\|_I \|P_0 u\|_I \geq \\
&\qquad a|1/2 - c| \|P_0 u\|_I^2 - (b|1/2 - c| + (b-a)/2) \|(1-P_0)u\|_I \|P_0 u\|_I \,.
\end{aligned}
$$

This finishes the proof of the lemma. □

5. The Essential Spectrum

Since M is complete, the Dirac operator D with domain $C_0^1(M,E)$ is essentially self–adjoint in $L^2(M,E)$, see [LM, p.117]. We denote by $\bar{D}$ the closure of D and recall that the *spectrum* $\operatorname{spec}\bar{D}$ of $\bar{D}$ is contained in $\mathbb{R}$.

5.1. Definition. A real number λ belongs to the *discrete spectrum* $\operatorname{spec}_d \bar{D}$ of $\bar{D}$ if λ is an eigenvalue of finite multiplicity of $\bar{D}$ and an isolated point of $\operatorname{spec}\bar{D}$. The complement $\operatorname{spec}_e \bar{D} = \operatorname{spec}\bar{D} \setminus \operatorname{spec}_d \bar{D}$ is called the *essential spectrum* of $\bar{D}$.

In particular, $\bar{D}$ is Fredholm if and only if $0 \notin \operatorname{spec}_e \bar{D}$. An easy consequence of Weyl's Criterion and the Rellich Lemma, the so–called Decomposition Principle, says that $\operatorname{spec}_e \bar{D}$ does not depend on compact parts of M, see below.

We say that a sequence (u_n) in $C_0^1(M,E)$ is a *special Weyl sequence (for E)* if

$$\lim_{n\to\infty} \|u_n\| = 1 \quad \text{and} \quad \operatorname{supp} u_n \to \infty \quad \text{as } n \to \infty\,. \tag{5.2}$$

Now the following well known characterization of $\operatorname{spec}_e \bar{D}$ by special Weyl sequences is a version of the Decomposition Principle.

5.3. Lemma. *$\lambda \in \operatorname{spec}_e \bar{D}$ if and only if there is a special Weyl sequence (u_n) with*

$$\lim_{n\to\infty} \|(D-\lambda)u_n\| = 0\,.$$

5.4. Corollary. *Assume that there is a compact subset K of M and a constant $c_K > 0$ such that $\|Du\| \geq c_K\|u\|$ for all $u \in C_0^1(M\setminus K, E)$. Then*

$$\operatorname{spec}_e \bar{D} \cap (-c_K, c_K) = \emptyset\,.$$

It is convenient to introduce the following notation. Let $\tilde{E}$ be a Hermitian vector bundle over a complete Riemannian manifold $\tilde{M}$ with compact boundary. Let $\tilde{D}$ be a symmetric first order differential operator on $C_0^1(\tilde{M},\tilde{E})$. Set

$$\begin{aligned} \operatorname{spec}_\infty \tilde{D} = \{\lambda \in \mathbb{R} \,|\, &\text{there is a special Weyl sequence } (u_n) \\ &\text{for } \tilde{E} \text{ with } \lim\nolimits_{n\to\infty} \|(\tilde{D}-\lambda)u_n\| = 0\}\,. \end{aligned} \tag{5.5}$$

Then $\operatorname{spec}_e \bar{D} = \operatorname{spec}_\infty D$ by Lemma 5.3, but, in general, (5.5) ignores possible contributions from the boundary. The following result is now obvious from Lemma 2.6.

5.6. Lemma. *We have $\operatorname{spec}_e \bar{D} = \cup \operatorname{spec}_\infty D_\infty$, where the union is over all k, c, ε, and w with $m_k(c,\varepsilon,w) > 0$.*

It remains to discuss $\operatorname{spec}_\infty D_\infty$. In the case $w = 1$, the model operator $D_{\infty 0}$ from (3.10) comes into play.

5.7. Theorem. *Fix an end U and constants c, ε, and w. Then we have:*
1) If $w \neq 1$ then $\operatorname{spec}_\infty D_\infty = \emptyset$.
2) If $w = 1$ and $\lambda \in \operatorname{spec}_\infty D_\infty$ then $|\lambda| \geq a|1/2 - c|$. Furthermore, $\operatorname{spec}_\infty D_\infty \subset \operatorname{spec}_\infty D_{\infty 0}$ with equality if $\limsup \operatorname{var}(\kappa_t) = 0$.

Proof. Let $\lambda \in \operatorname{spec}_\infty D_\infty$ and (u_n) be a special Weyl sequence for D_∞ with $(D_\infty - \lambda)u_n \to 0$. Now for any $N > 0$ there is a constant $t_0 > 0$ such that the nonzero eigenvalues of $A(t)$ are of absolute value $\geq N$ for all $t \geq t_0$. Hence $(1 - P_0)u_n \to 0$, by Lemma 4.4. This completes the proof of the first assertion.

Assume now that $w = 1$. By what we just said we have $(1 - P_0)u_n \to 0$, hence $\|P_0 u_n\| \to 1$ and therefore, by Lemma 4.5,

$$\liminf_{t\to\infty} \|D_\infty u_n\| \geq \liminf_{t\to\infty} \|P_0 D_\infty u_n\| \geq a|1/2 - c|\,.$$

This establishes the inequality in the second assertion. Furthermore, by Lemma 4.3,

$$P_0(D_\infty - \lambda)(1 - P_0)u_n = \gamma P_0(\kappa - \bar{\kappa})(B + 1/2)(1 - P_0)u_n \to 0\,.$$

Therefore

$$(P_0 D_\infty P_0 - \lambda)P_0 u_n = P_0(D_\infty - \lambda)P_0 u_n \to 0\,.$$

But then $(P_0 u_n)$ is a special Weyl sequence with $(D_{\infty 0} - \lambda)P_0 u_n \to 0$. Hence λ is in $\operatorname{spec}_\infty D_{\infty 0}$, hence $\operatorname{spec}_\infty D_\infty \subset \operatorname{spec}_\infty D_{\infty 0}$.

Suppose now that $w = 1$ and $\limsup_{t\to\infty} \operatorname{var}(\kappa_t) = 0$. Let $\lambda \in \operatorname{spec}_\infty D_{\infty 0}$ and (u_n) be a special Weyl sequence for $D_{\infty 0}$ with $(D_{\infty 0} - \lambda)u_n \to 0$. Then $u_n = P_0 u_n$ and hence

$$\begin{aligned}(D_\infty - \lambda)u_n &= (D_\infty - \lambda)P_0 u_n \\ &= P_0(D_\infty - \lambda)P_0 u_n + (1 - P_0)(D_\infty - \lambda)P_0 u_n \\ &= (D_{\infty 0} - \lambda)P_0 u_n + \gamma(\kappa - \bar{\kappa})(B - 1/2)P_0 u_n\,,\end{aligned}$$

by Lemma 4.3. Now by assumption, the first term on the right hand side tends to zero as n tends to infinity. The second term tends to zero since $\|P_0 u_n\|$ is uniformly bounded, $\operatorname{supp} u_n \to \infty$, and $\limsup_{t\to\infty} \operatorname{var}(\kappa_t) = 0$. Hence λ is in $\operatorname{spec}_\infty D_\infty$, hence $\operatorname{spec}_\infty D_\infty = \operatorname{spec}_\infty D_{\infty 0}$. $\square$

We recall that $\bar{D}$ is Fredholm if and only if $0 \notin \operatorname{spec}_e \bar{D}$.

5.8. Corollary. *If $m_k(1/2, \varepsilon, 1) = 0$ for all k and ε, then $\bar{D}$ is Fredholm. On the other hand, if for some k and ε we have $m_k(1/2, \varepsilon, 1) > 0$ and $\operatorname{var}(\kappa_t) \to 0$ as $t \to \infty$, then $\operatorname{spec}_e \bar{D} = \mathbb{R}$.*

Theorem 5.7 and Corollary 5.8 imply Theorem 0.1 from the introduction.

6. The Index

We are all set for the discussion of the index of D^+. As usual, the ends of M come with an index k. For each end $U = U_k$, we also enumerate the line bundles $L(c, \varepsilon, w)$ which occur in the decomposition of $E|U$ by an additional index j, $L_{kj} = L(c_{kj}, \varepsilon_{kj}, w_{kj})$, where $1 \le j \le m = \dim E^+$. These indices will also be used for other objects attached to L_{kj} if necessary.

Throughout this section we assume that $m_k(1/2, \varepsilon, 1) = 0$ for all k. In other words, we assume that $w_{kj} \neq 1$ whenever $c_{kj} = 1/2$. Then the Dirac operator $\bar{D}$ is Fredholm, by Theorem 0.1 or Corollary 5.8 respectively.

Consider an end $U = U_k = S \times (0, \infty)$ and set

$$U_t = U_{k,t} = S \times (t, \infty)\,, \quad S_t = S_{k,t} = S \times \{t\}\,. \tag{6.1}$$

Furthermore set

$$V_t = \cup_k U_{k,t}\,, \quad M_t = M \setminus V_t\,. \tag{6.2}$$

Then M_t is a compact surface with smooth boundary ∂M_t, consisting of the pairwise disjoint circles $S_{k,t}$. We let $\bar{U}_{k,t}$ and $\bar{V}_t$ denote the closures of $U_{k,t}$ and V_t, respectively.

Our aim is to reduce the computation of the index of D^+ to a boundary value problem on M_t, where $t > 0$ is sufficiently large.

By an *elliptic boundary value problem* we mean a pseudodifferential boundary condition which is well–posed in the sense of Seeley [Se].

6.3. Lemma. *Let $t > 0$. Then any self–adjoint extension of D over M_t or $\bar{V}_t$, defined by an elliptic boundary value problem at ∂M_t is Fredholm.*

Proof. In Theorem 4.1 of [BL] Brüning and Lesch characterize elliptic boundary value problems in terms of orthogonal projections and show that they are Fredholm. This implies the result for M_t.

The local analysis used for the result just mentioned implies a 'decomposition principle' for self–adjoint extensions of D over $\bar{V}_t$, hence the second assertion since $\bar{D}$ is Fredholm, by Corollary 5.8. □

We choose $t_0 >> 0$ and introduce boundary conditions at t_0. To that end we write a section u over U_k as

$$u = \sum_j \left(u_{kj}^+ \Phi_{kj}^+ + u_{kj}^- \Phi_{kj}^- \right), \tag{6.4}$$

where Φ_{kj}^+, Φ_{kj}^- is a frame of L_{kj} as in Lemma 2.5. We recall that $P_{kj,>}(t_0)$ and $P_{kj,<}(t_0)$ denote the spectral projections in $L^2(S_{k,t_0})_{w_{kj}} \otimes \mathbb{C}^2$ corresponding to the eigenvalues of $A_{kj}(t_0)$ which are > 0 and < 0, respectively. Since A_{kj} decomposes into two operators according to the decomposition $u = (u^+, u^-)$, we have a corresponding decomposition $P_{kj,>} = P_{kj,>}^+ + P_{kj,>}^-$ and similarly for $P_{kj,<}$.

We also recall the orthogonal projection $P_{kj,0}(t_0)$ onto the kernel of $A_{kj}(t_0)$ in $L^2(S_{k,t_0})_{w_{kj}} \otimes \mathbb{C}^2$ and the decomposition $P_{kj,0} = P^+_{kj,0} + P^-_{kj,0}$ as in Lemma 4.5.

In their discussion of the index problem for Dirac operators on compact manifolds with boundary, Atiyah, Patodi and Singer used the projections

$$P_{kj,APS} = P_{kj,<} + P^+_{kj,0} \tag{6.5}$$

on the component S_{kt_0} of ∂M_{t_0}, assuming in addition that the metric is a product near the boundary, i.e., $f(s,t) = \bar{f}(t_0)$ for $s \in [0,1]$ and t near t_0, see [APS]. In our discussion we need a variation of this. We set

$$P_{kj}(t_0) = \begin{cases} P_{kj,<}(t_0) + P^+_{kj,0}(t_0) & \text{if } 1/2 < c\,, \\ P_{kj,<}(t_0) + P^-_{kj,0}(t_0) & \text{if } 1/2 > c\,. \end{cases} \tag{6.6}$$

The identity

$$\gamma P_{kj}(t_0) = (1 - P_{kj}(t_0))\gamma \tag{6.7}$$

where γ is as in (3.1), implies that D with each of the following two domains

$$\mathcal{D}_{\text{int}} = \{u \in C^1(M_{t_0}, E) \mid P_{kj}(t_0)(u_{kj}|S_{k,t_0}) = 0 \text{ for all } k, j\ \}$$
$$\mathcal{D}_{\text{ext}} = \{u \in C^1_0(\bar{V}_{t_0}, E) \mid (1 - P_{kj}(t_0))(u_{kj}|S_{k,t_0}) = 0 \text{ for all } k, j\ \}$$

is symmetric. We denote the corresponding operators by D_{int} and D_{ext}, respectively. Then the boundary conditions are elliptic. To see this, we invoke again [BL, Theorem 4.1]. This requires a localization in the standard form used in [BL, (4.2)]: the map

$$u \mapsto (f_{t_0}/f_t)^{1/2} u$$

transforms D_{kj} into the operator

$$D_{kj} = \gamma(\partial_t + \tilde{A}(t))\,, \tag{6.8}$$

on the Hilbert space $L^2\big(\big(S \times (0,\infty), f_{t_0} ds dt\big)$. Here

$$\tilde{A}(t_0) = A(t_0) + \kappa(t_0)B =: \tilde{A} = \begin{pmatrix} \tilde{A}_+ & 0 \\ 0 & -\tilde{A}_+ \end{pmatrix}.$$

Next we need to determine a projection $P_+(\tilde{A})$ with the properties listed in (3.10) of [BL], and we have to show that $(P_+(\tilde{A}), P(t_0))$ forms a Fredholm pair. Now it is clear that the projection

$$P_+(\tilde{A}) = \begin{pmatrix} P_{\geq 0}(\tilde{A}_+) & 0 \\ 0 & P_{<0}(\tilde{A}_+) \end{pmatrix}$$

of Atiyah, Singer and Patodi is an admissible choice. Since B is bounded and $P(t_0)$ differs by a finite rank projection from the analogous projection $P_+(A(t_0))$, the Fredholm property follows from standard perturbation theory. It follows also from Theorem 4.1 in [BL] (with an obvious modifiction for D_{ext}) that both D_{int} and D_{ext} are essentially self–adjoint. By Lemma 6.3, they are both Fredholm.

The boundary projections respect the decomposition $E = E^+ \oplus E^-$, hence we have the subdomains $\mathcal{D}^+_{\mathrm{int}}$ and $\mathcal{D}^+_{\mathrm{ext}}$ and the corresponding restrictions

$$D^+_{\mathrm{int}} : \mathcal{D}^+_{\mathrm{int}} \to L^2(M_{t_0}, E^-)\,,$$
$$D^+_{\mathrm{ext}} : \mathcal{D}^+_{\mathrm{ext}} \to L^2(\bar{V}_{t_0}, E^-)\,.$$

The proof of our index formula below rests on the following result.

6.9. Theorem (Brüning–Lesch, [BL]). $\operatorname{ind} D^+ = \operatorname{ind} D^+_{\mathrm{int}} + \operatorname{ind} D^+_{\mathrm{ext}}$.

The proof of this formula consists in showing that the transmission condition at ∂M_{t_0} for sections in $H^1(M, E^+)$ can be deformed into the above boundary conditions in the definition of D_{int} and D_{ext} without affecting the index. The important point is that the interior boundary condition above is independent of the exterior one, so that they can be handled separately. Another important feature of our choice of boundary condition is that the exterior contribution to the index can be computed explicitly: it is zero if t_0 is sufficiently large.

6.10. Lemma. *For t_0 sufficiently large,* $\ker D_{\mathrm{ext}} = 0$. *In particular,* $\operatorname{ind} D^+_{\mathrm{ext}} = 0$.

Proof. It suffices to consider the (self–adjoint) closure of the operator $D_{kj,\infty} =: D_{kj}$ on $\mathcal{D}_{kj,\mathrm{ext}}$, that is, on $C^1_0\big(S \times [t_0, \infty)\big)_{w_{kj}} \otimes \mathbb{C}^2$ with the boundary condition $(1 - P_{kj}(t_0))$. Let $C_3 > 0$ be a lower bound for the absolute value of the nonzero eigenvalues of the operator $A_{kj}(1)$. Then $C_3\bar{f}(1)/\bar{f}(t_0)$ is a lower bound for the nonzero eigenvalues of $A_{kj}(t)$ for all $t \geq t_0$, by Proposition 3.7, hence for any $u \in C^1_0\big([t_0, \infty)\big)_{w_{kj}} \otimes \mathbb{C}^2$ satisfying the boundary condition we have

$$\begin{aligned} C_3 \frac{\bar{f}(1)}{\bar{f}(t_0)} \|(1 - P_0)u\|_{[t_0,\infty)} &\leq \|Au\|_{[t_0,\infty)} \\ &\leq C_4\big(\|D_{kj}u\|_{[t_0,\infty)} + \|(1 - P_0)u\|_{[t_0,\infty)}\big)\,, \end{aligned}$$

by Lemmas 4.4 and 4.5, where C_4 is a constant independent of k, j, and t_0. Since $\mathcal{D}_{kj,\mathrm{ext}}$ is dense in the domain of the closure of D_{kj}, the above inequality and the inequality in Lemma 4.5 persist to hold in that domain. On the other hand, $\bar{f}(t) \to 0$ as $t \to \infty$ so

$$C_3\bar{f}(1) > C_4\bar{f}(t_0)$$

if t_0 is sufficiently large. Thus if $u \in \ker D_{kj}$, then $(1 - P_0)u = 0$ and so, by Lemma 4.5, also $u = 0$. $\square$

We now come to the proof of the asserted index formula. By Theorem 6.9 and Lemma 6.10 we have

$$\operatorname{ind} D^+ = \operatorname{ind} D^+_{\mathrm{int}}\,, \tag{6.11}$$

and hence it remains to compute $\operatorname{ind} D^+_{\mathrm{int}}$. Now in the standard versions of the index theorem for manifolds with boundary, it is assumed that near the boundary the manifold is a product. To arrive at such a situation, we deform the given metric in a small neighborhood of t_0 by replacing f with

$$f^\alpha = (1-\alpha)f + \alpha(\phi \bar{f}(t_0) + (1-\phi)f)\,, \quad \alpha \in [0,1]\,, \tag{6.12}$$

where ϕ is a suitable cut–off function near t_0. Using the representation (6.8), we see that the family D^α_{int} is graph continuous on $\mathcal{D}_{\mathrm{int}}$, hence has constant index. Hence we can, and will, assume from now on that near the boundary, the metric of M_{t_0} is a product. Thus we are finally in the situation considered in [APS], except for a variation in the boundary condition. This is taken care of by a result of Agranovich-Dynin, see Theorem 23.1 in [BW] or Theorem 4.2 in [BL].

6.13. THEOREM. *Denote by ω_{ind} the index form of D. Then*

$$\begin{aligned}\operatorname{ind} D^+_{\mathrm{int}} = \int_{M_{t_0}} \omega_{\mathrm{ind}} - \frac{1}{2}\sum_{k,j} \dim\ker A^+_{kj}(t_0) + \frac{1}{2}\sum_{k,j} \eta_{kj}(t_0) \\ + \sum_{k,j} \operatorname{ind}(P^+_{kj}(t_0) : \operatorname{im} P^+_{APS}(t_0) \to \operatorname{im} P^+_{kj}(t_0))\,,\end{aligned}$$

where $\eta_{kj}(t_0)$ denotes the η–invariant of $A^+_{kj}(t_0)$ and where P_{APS} denotes the Atiyah–Singer–Patodi spectral projection.

It remains to explain and evaluate the different terms on the right hand side of this formula.

We start with the index form ω_{ind}. We could compute ω_{ind} by using the local structure of graded geometric Dirac bundles as in Section 2. However, there is also the following way: By the Local Index Theorem, we have

$$\omega_{\mathrm{ind}}(p) = \lim_{t\to 0} \operatorname{tr}_E[\omega_{\mathbb{C}} e^{-tD^2}(p,p)]\,,$$

hence the computation of ω_{ind} is a local problem. Therefore we may consider an open contractible subset $W \subset M$, for which we choose an orientation. If M is oriented, we assume that the orientations of W and M coincide. The field C is parallel, hence the subbundles $E^\pm$ split over W as a direct sum of the pairwise orthogonal and parallel subbundles

$$E^\pm(c,\varepsilon) = \{u \in E^\pm \mid Cu = cu\,,\ \omega_{\mathbb{C}} u = \varepsilon u\}\,,$$

where $c \in \mathbb{R}$ and $\varepsilon \in \{-1,+1\}$. Now the subbundles

$$E(c,\varepsilon) = E^+(c,\varepsilon) \oplus E^-(1-c,-\varepsilon)\,.$$

are graded Dirac subbundles of E, hence we may assume that $E = E(c,\varepsilon)$. We discuss the case $\varepsilon = +1$ first. Then

$$E^+ = \{u \in E \mid \omega_{\mathbb{C}} u = u\}\,.$$

Let Σ be the spinor bundle associated to the spin structure of W. Then we have

$$E = \Sigma \otimes F \quad \text{with} \quad F = \mathrm{Hom}_{\mathrm{Cliff}}(\Sigma, E)\,,$$

where $\mathrm{Hom}_{\mathrm{Cliff}}(\Sigma, E)$ is the bundle of homomorphisms from Σ to E which are linear over Clifford multiplication, a parallel subbundle of $\mathrm{Hom}(\Sigma, E)$ with the canonical connection. Now the curvature endomorphism C_Σ on $\Sigma^\pm$ is multiplication by $K/2$. By our assumption on E, the curvature endomorphism C_E on $E^\pm$ is multiplication by Kc and $K(1-c)$, respectively. It follows easily that for any oriented orthonormal frame X, Y of W, $R_F(X,Y)$ is multiplication by $Ki(c-1/2)$. Hence

$$\omega_{\mathrm{ind}} = \frac{1}{2\pi}(1/2-c)\dim F\, KdA = \frac{1}{2\pi}(1/2-c)\dim E^+ KdA\,,$$

by the formula for the index form of twisted Dirac operators as explained in [APS].

In the case where $\varepsilon = -1$ we reverse the roles of E^+ and E^-, then we are back in the previous case but ω_{ind} changes sign. We get

$$-\omega_{\mathrm{ind}} = \frac{1}{2\pi}(1/2-(1-c))\dim E^- KdA = -\frac{1}{2\pi}(1/2-c)\dim E^+ KdA\,.$$

In conclusion,

$$\omega_{\mathrm{ind}} = \frac{1}{2\pi}\big\{m/2 - \mathrm{tr}(C^+)\big\}KdA\,,$$

and this gives the first term of the claimed index formula.

Now the second term in the index formula 6.13 is obviously equal to $-1/2$ times $\#\{(k,j) \mid w_{kj} = 1\}$. Furthermore, note that for each pair (k,j) the projection $P^+_{kj} : \mathrm{im}\, P^+_{APS} \to \mathrm{im}\, P^+_{kj}$ is surjective, by (6.5) and (6.6). It has trivial kernel if $w_{kj} \neq 1$ or if $w_{kj} = 1$ and $c_{kj} > 1/2$. The kernel is of dimension 1 if $w_{kj} = 1$ and $c_{kj} < 1/2$. This gives the second term of the claimed index formula.

If $w_{kj} = 1$, then the spectrum of $A^+_{kj}(t_0)$ is symmetric about 0 and hence $\eta_{kj}(t_0) = 0$ in this case. If $w_{kj} \neq 1$, we write $w_{kj} = \exp(2\pi i\rho)$ with $0 < \rho = \rho_{kj} < 1$. Then if $\varepsilon_{kj} = 1$, we get from Proposition 3.7

$$\eta_{kj}(s) = -\rho^{-s} - \sum_{k\geq 1}\{(k+\rho)^{-s} - (k-\rho)^{-s}\}\,.$$

If $\varepsilon_{kj} = -1$, we get the corresponding negative of the right hand side. Now

$$\lim_{s\to 0}\sum_{k\geq 1}\{(k+\rho)^{-s} - (k-\rho)^{-s}\} = -2\rho\,.$$

This concludes the proof of Theorem 0.2.

6.14. Remark. The proof of Theorem 0.2 gives a somewhat more general result. We only need to assume that the given graded Dirac bundle E is geometric along the ends of M. Then the same index formula holds, except that the first term on the right hand side has to be replaced by the integral over the corresponding index form.

References

[APS] M. Atiyah, V. Patodi and I. Singer. Spectral asymmetry and Riemannian Geometry. I. *Math. Proc. Cambridge Philos. Soc.* 77 (1975), 43–69.

[Bä] C. Bär. The Dirac Operator on Hyperbolic Manifolds of Finite Volume. *Preprint 566*, SFB256, Bonn 1998.

[BW] B. Booß–Bavnbek and K. Wojciechowski. *Elliptic Boundary Problems for Dirac Operators.* Birkhäuser 1993.

[Br] J. Brüning. L^2–Index Theorems on Certain Complete Manifolds. *J. Diff. Geometry 32* (1990), 491–532.

[BL] J. Brüning and M. Lesch. Spectral Theory of boundary value problems for Dirac type operators. *Contemporary Math.*, to appear.

[DX] H. Donnelly and F. Xavier. On the Differential Form Spectrum of Negatively Curved Riemannian Manifolds. *Amer. J. Math.* 106 (1984), 169–185.

[Eb] P. Eberlein. Lattices in Manifolds of Nonpositive Curvature. *Annals of Math.* 111 (1980), 435–476.

[HIH] E. Heintze and H. C. Im Hof. Geometry of Horospheres. *J. Diff. Geometry* 12 (1977), 481–491.

[Ho] E. Hopf. Statistik der Lösungen geodätischer Probleme vom unstabilen Typus. II. *Math. Ann.* 117 (1940), 590–608.

[LM] H. B. Lawson and M. L. Michelsohn. *Spin Geometry.* Princeton University Press 1989.

[Se] R. T. Seeley. Topics in pseudo–differential operators *C. I. M. E., Conference on pseudo–differntial opertors 1968*, Edizione Cremonese, Roma 1969, pp. 169–305.

[St] M. Stern. L^2–index theorems on warped products. *Thesis*, Princeton University 1984.

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