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THE ASYMPTOTIC FORM OF THE LOWER LANDAU BANDS IN A STRONG MAGNETIC FIELD

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The asymptotic form of the bottom part of the spectrum of the two-dimensional magnetic Schrödinger operator with a periodic potential in a strong magnetic field is studied in the semiclassical approximation. Averaging methods permit reducing the corresponding classical problem to a one-dimensional problem on the torus; we thus show the "almost integrability" of the original problem. Using elementary corollaries from the topological theory of Hamiltonian systems, we classify the almost invariant manifolds of the classical Hamiltonian. The manifolds corresponding to the bottom part of the spectrum are closed or nonclosed curves and points. Their geometric and topological characteristics determine the asymptotic form of parts of the spectrum (spectral series). We construct this asymptotic form using the methods of the semiclassical approximation with complex phases. We discuss the relation of the asymptotic form obtained to the magneto-Bloch conditions and asymptotics of the band spectrum.

Keywords: magnetic Schrödinger operator, Landau bands, semiclassical approximation, Reeb graph

1. Introduction

The motion of a charged quantum particle in a uniform magnetic and periodic electric field is described by the Hamiltonian $\widehat{H}_{B,w}$ [1],

$$\widehat{H}_{B,w} = \frac{1}{2m}\left(-i\hbar\nabla - \frac{eA(z)}{c}\right)^2 + w(z_1, z_2),$$

acting in $L^2(\mathbb{R}^2_z)$, $z = (z_1, z_2)$, where $A(z) = (-Bz_2, 0)$ is the vector potential of the magnetic field (we use the so-called Landau gauge), B is the strength of the magnetic field, and w is the electric potential. The function w is periodic with respect to some lattice Γ spanned by two linearly independent vectors $l_1 = (l_{11}, l_{12})$ and $l_2 = (l_{21}, l_{22})$. Because $\widehat{H}_{B,w}$ is invariant under gauge transformations, we assume, without loss of generality, that $l_{12} = 0$. Also, without loss of generality, we assume that $B > 0$. Introducing the new coordinates $x = 2\pi z/L_0$, where $L_0 = l_{11}$ is the so-called characteristic size of the lattice, we can rewrite the Hamiltonian as

$$\widehat{H}_{B,w} = \frac{(eBL_0)^2}{4\pi^2 mc^2}\widehat{H}, \qquad \widehat{H} = \frac{1}{2}\left(-ih\frac{\partial}{\partial x_1} + x_2\right)^2 + \frac{1}{2}\left(-ih\frac{\partial}{\partial x_2}\right)^2 + \varepsilon v(x_1, x_2),$$

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where

$$h = (2\pi)^2 \left(\frac{l_M}{L_0}\right)^2, \qquad \varepsilon = h\frac{W}{\hbar\omega_c}, \qquad l_M = \sqrt{\frac{\hbar}{m\omega_c}},$$

$$\omega_c = \frac{|eB|}{cm}, \qquad W = \max|w|, \qquad v(x) = \frac{1}{W} w\left(\frac{L_0 x}{2\pi}\right),$$

(l_M is the magnetic length and ω_c is the cyclotron frequency). The spectra of $\widehat{H}_{B,w}$ and $\widehat{H}$ are related by

$$\sigma(\widehat{H}_{B,w}) = \frac{(eBL_0)^2}{(2\pi c)^2 m}\sigma(\widehat{H}).$$

We study the asymptotic behavior of the spectrum of $\widehat{H}_{B,w}$ under the assumption that the parameters h and ε are small. The smallness of h means that the magnetic length is small compared with the characteristic size of the lattice. Under this condition, ε is small if the potential energy W of the lattice confinement is small relative to the cyclotron energy or comparable to it. Such a situation is realized in superlattices and in arrays of quantum dots and antidots [2]. This assumption about the smallness of the parameters h and ε is essential; some other possible situations leading to different results were considered, for example, in [3]–[5].

The function v is assumed to be real-analytic in $\mathbb{R}^2$ and periodic with respect to the two linearly independent vectors $a_1 = (2\pi/L_0)l_1$ and $a_2 = (2\pi/L_0)l_2$, i.e., $a_1 = (2\pi, 0)$ and $a_2 = (a_{21}, a_{22})$, $a_{22} \neq 0$.

It is well known [1] that for $v = 0$, the spectrum of $\widehat{H}$ consists of infinitely degenerate eigenvalues (*Landau levels*) $E_\mu = (\mu + 1/2)h$, where $\mu \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ is the level index. The appearance of the potential v leads to a "broadening" of these levels into sets, which are called *Landau bands*. It is also known that if the number $\eta = a_{22}/h$ (the flux of the magnetic field) is rational, then the spectrum of $\widehat{H}$ has a band structure and does not contain a singular component [6].

Based on the fundamental principle of correspondence between classical and quantum mechanics, one can expect that the asymptotic properties of the operator $\widehat{H}$ for small h can be described in terms of the corresponding classical dynamic system given in this case by the classical Hamiltonian H,

$$H(p, x, \varepsilon) = \frac{1}{2}(p_1 + x_2)^2 + \frac{1}{2}p_2^2 + \varepsilon v(x_1, x_2).$$

For problems with a discrete spectrum, such a relation has long been known. More precisely, knowing some family of invariant manifolds of the classical Hamiltonian allows describing the asymptotic behavior of the spectrum of the corresponding quantum Hamiltonian near the corresponding energy levels using the Bohr–Sommerfeld quantization rules [7].

Such an approach cannot be applied directly to the operator $\widehat{H}$, because, first, as mentioned above, its spectrum is not discrete (moreover, the semiclassical approximation is rarely used in problems with a continuous spectrum) and, second, the Hamiltonian system for H is nonintegrable, and we cannot obtain the desired invariant manifolds explicitly.

The presence of the small parameter ε allows at least avoiding the second of these difficulties. In the new canonical variables $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Y}_1$, and $\mathcal{Y}_2$, the Hamiltonian H becomes

$$H = \mathcal{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \varepsilon) + e^{-C/\varepsilon}\mathcal{G}(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \varepsilon), \qquad \mathcal{I}_1 = \frac{1}{2}(\mathcal{P}^2 + \mathcal{Q}^2)$$

(see Sec. 2), and the corresponding Hamiltonian system is reduced modulo $e^{-C/\varepsilon}$ to a system with one degree of freedom on a torus and is hence "almost integrable." The Hamiltonian H is a Morse function on

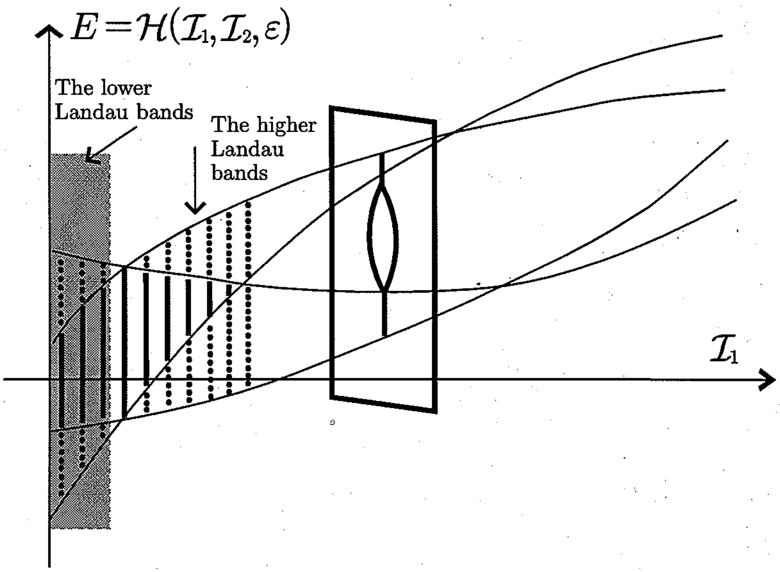

Fig. 1. Global description of the spectrum.

this torus for almost all fixed $\mathcal{I}_1$. The trajectories of the obtained Hamiltonian system are classified using the Reeb graph [8] for $\mathcal{H}$, and the points of each edge of the Reeb graph are parameterized by the variable $\mathcal{I}_2$ (see Secs. 3 and 4) such that $\mathcal{H}$ is a function of only $\mathcal{I}_1$ and $\mathcal{I}_2$. As $\mathcal{I}_1$ runs through its domain, the moving Reeb graph forms a surface (the "Reeb surface"), and this surface gives a global classification of the classical motion [9], [10], while the "quantization" of this surface (i.e., the choice of a discrete subset of the values of the variables $\mathcal{I}_1$ and $\mathcal{I}_2$) leads to the general description of the asymptotic behavior of the spectrum. In this case, we have a picture similar to that shown in Fig. 1 (see [10] for more detail). This figure shows the projection of the "Reeb surface" on the plane $(E, \mathcal{I}_1)$. The section of this surface by the plane $\mathcal{I}_1 = \text{const}$ coincides with the Reeb graph of the function $\mathcal{H}$. The variable $\mathcal{I}_1$ takes the values $(1/2+\mu)h$, $\mu \in \mathbb{Z}_+$, $\mathcal{I}_1 \geq \alpha > 0$ as $h \to 0$. The variable $\mathcal{I}_2$ takes the values $(1/2+\nu)h$, $\nu \in \mathbb{Z}$, for the end edges and all possible values for the interior edges. The unions of the values of $\mathcal{H}$ at all these points gives the asymptotic behavior of the spectrum (see Sec. 5).

In the present paper, we concentrate our attention on the lower Landau bands (i.e., on those with indices $\mu = O(1)$, in contrast to the higher Landau bands with indices $\mu = O(1/h)$; the corresponding domain is emphasized in Fig. 1). We have some reasons for such attention. Studying the lower Landau bands in the context of our problem leads to a semiclassical asymptotic behavior with complex phases in contrast to the "classical" method of the real canonical operator used in [9], [10] (we explain the difference below). Moreover, as is shown below, the lower Landau bands admit an asymptotic description by simpler formulas (see Sec. 5). *Formally*, these formulas can be obtained from the corresponding formulas for the high-energy part of the spectrum by passing to the limit $\mathcal{I}_1 \to 0$, but the corresponding formulas for asymptotic eigenfunctions are different. Studying the lower Landau bands also plays an important role in investigating semiconductor structures [11].

Using the formulas for asymptotic eigenvalues and eigenfunctions (with respect to h and ε), by analogy with the so-called Lifshits–Gelfand–Zak representation [12] (see formula (22) below), we can construct asymptotic eigenfunctions satisfying the magnetic Bloch conditions [13], [14] in the case of a rational flux. Such an approach gives the asymptotic behavior of the band spectrum of the operator $\widehat{H}$ on a heuristic level (Sec. 6).

2. Averaging the classical Hamiltonian

2.1. The averaged Hamiltonian. We introduce new canonical variables, the generalized momenta I_1, y_1 (or P, y_1) and the generalized positions φ_1, y_2 (or Q, y_2), as follows:

$$p_1 = -y_2, \qquad p_2 = -Q, \qquad x_1 = Q + y_1, \qquad x_2 = P + y_2,$$
$$Q = \sqrt{2I_1}\,\sin\varphi_1, \qquad P = \sqrt{2I_1}\,\cos\varphi_1. \tag{1}$$

In these coordinates, the Hamiltonian H becomes

$$H = I_1 + \varepsilon v\left(\sqrt{2I_1}\,\sin\varphi_1 + y_1, \sqrt{2I_1}\,\cos\varphi_1 + y_2\right) = \frac{1}{2}(P^2 + Q^2) + \varepsilon v(Q + y_1, P + y_2).$$

The variables P, Q (or I_1, φ_1) correspond to the cyclotron motion around a guiding center described by the variables y_1, y_2 [15], [16].

The idea of averaging methods is to remove the phase φ_1 from the expression for H, at least with some discrepancy. For the Hamiltonian under consideration, such a procedure was previously performed, for example, in [16], but usually in other variables. Using the variables P, Q, and y makes the formulas significantly simpler.

Generally speaking, the classical averaging methods are usually applied in the domain $I_1 > \kappa > 0$ (i.e., in the domain of analyticity). Nevertheless, for the problem under consideration, a neighborhood of the boundary $I_1 = 0$ is particular interesting. A rigorous averaging procedure for such problems has appeared only recently [17].

Proposition 1. *For any $\kappa > 0$, there exists $\varepsilon_0 > 0$ such that for $I_1 \le \kappa$, there exists a canonical change of variables,*

$$P = \mathcal{P} + \varepsilon U_1(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \varepsilon), \qquad Q = \mathcal{Q} + \varepsilon U_2(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \varepsilon),$$
$$y_1 = \mathcal{Y}_1 + \varepsilon U_3(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \varepsilon), \qquad y_2 = \mathcal{Y}_2 + \varepsilon U_4(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \varepsilon), \tag{2}$$

for which the Hamiltonian becomes

$$H = \mathcal{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \varepsilon) + e^{-C/\varepsilon}\mathcal{G}(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \varepsilon), \tag{3}$$

where

$$\mathcal{H}(\mathcal{I}_1, \mathcal{Y}, \varepsilon) = \mathcal{I}_1 + \varepsilon J_0\big(\sqrt{-2\mathcal{I}_1\Delta_{\mathcal{Y}}}\,\big)v(\mathcal{Y}) + O(\varepsilon^2) \tag{4}$$

for $0 < \varepsilon < \varepsilon_0$. Here, J_0 is the zeroth-order Bessel function, $U_{1,2,3,4}$ and $\mathcal{G}$ are real-analytic functions of $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{Y}_{1,2}$, $\mathcal{I}_1 = (\mathcal{P}^2 + \mathcal{Q}^2)/2$, $\mathcal{H}$ is a real-analytic function of $\mathcal{I}_1$ and $\mathcal{Y}_{1,2}$, $|\mathcal{G}| + |\nabla_{\mathcal{Y}}\mathcal{G}| \le G$, C and G are positive constants, and all the functions $U_{1,2,3,4}$, $\mathcal{H}$, and $\mathcal{G}$ are periodic with respect to $\mathcal{Y}$ with the periods a_1 and a_2.

The complete proof can be found in [17]; it uses the method in [18]. Here, we describe only the averaging procedure without estimating.

2.2. The averaging procedure. The required change of variables is constructed as a composition of subsequently determined transformations $(P^m, Q^m, y^m) \mapsto (P^{m+1}, Q^{m+1}, y^{m+1})$, $m = 0, 1, 2, \dots$, $(P^0, Q^0, y^0) = (P, Q, y)$, using the so-called generating function S^m [19]. We assume that at the mth step, the Hamiltonian H has the form

$$H(P, Q, y, \varepsilon) = H^m(I_1^m, y^m, \varepsilon) + \varepsilon^{m+1}g^m(P^m, Q^m, y^m, \varepsilon),$$

where

$$I_1^m = \frac{1}{2}((P^m)^2 + (Q^m)^2).$$

We set

$$\bar{g}^m(I, y, \varepsilon) = \int_0^{2\pi} g^m(\sqrt{2I}\,\cos\varphi, \sqrt{2I}\,\sin\varphi, y, \varepsilon)\,d\varphi$$

and

$$\tilde{g}^m(P, Q, y, \varepsilon) = \bar{g}^m\left(\frac{1}{2}(P^2 + Q^2), y, \varepsilon\right) - g^m(P, Q, y, \varepsilon).$$

We introduce the function

$$\sigma^m(P, Q, y, \varepsilon) = \frac{1}{2}\biggl(\int_0^{\varphi} \tilde{g}^m(\sqrt{2I}\,\cos\psi, \sqrt{2I}\,\sin\psi, y, \varepsilon)\,d\psi + \\ + \int_{\pi}^{\varphi} \tilde{g}^m(\sqrt{2I}\,\cos\psi, \sqrt{2I}\,\sin\psi, y, \varepsilon)\,d\psi\biggr)\bigg|_{\substack{P=\sqrt{2I}\,\cos\varphi,\\ Q=\sqrt{2I}\,\sin\varphi}} \tag{5}$$

and define the *generating function*

$$S^m(P, Q, y, \varepsilon) = PQ + y_1 y_2 + \varepsilon^{m+1}\sigma^m(P, Q, y, \varepsilon).$$

Solving the system

$$\begin{aligned} P^m &= \frac{\partial S^m}{\partial Q}(P^{m+1}, Q^m, y_1^{m+1}, y_2^m, \varepsilon), & Q^{m+1} &= \frac{\partial S^m}{\partial P}(P^{m+1}, Q^m, y_1^{m+1}, y_2^m, \varepsilon), \\ y_1^m &= \frac{\partial S^m}{\partial y_2}(P^{m+1}, Q^m, y_1^{m+1}, y_2^m, \varepsilon), & y_2^{m+1} &= \frac{\partial S^m}{\partial y_1}(P^{m+1}, Q^m, y_1^{m+1}, y_2^m, \varepsilon) \end{aligned} \tag{6}$$

with respect to P^{m+1}, Q^{m+1}, and y^{m+1}, for sufficiently small ε, we obtain a canonical change of variables of form (2) that reduces the Hamiltonian H to the form

$$H(P, Q, y, \varepsilon) = H^{m+1}(I_1^{m+1}, y^{m+1}, \varepsilon) + \varepsilon^{m+2} g^{m+1}(P^{m+1}, Q^{m+1}, y^{m+1}, \varepsilon),$$

where

$$H^{m+1}(I_1^{m+1}, y^{m+1}, \varepsilon) = H^m(I_1^{m+1}, y^{m+1}, \varepsilon) + \varepsilon^{m+1}\bar{g}^m(I^{m+1}, y^{m+1}, \varepsilon).$$

Fine estimates [17], similar to those in [18], show that such a procedure indeed gives the necessary exponentially small discrepancy for $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$.

We must emphasize that the analyticity of the Hamiltonian obtained and of the corresponding change of variables is ensured by the special form of (5).

2.3. The averaging with an accuracy of $O(\varepsilon^2)$. For clarity, we average with an accuracy of $O(\varepsilon^2)$, i.e., we reduce the Hamiltonian to the form

$$H = \overline{H}(\mathcal{I}_1, \mathcal{Y}, \varepsilon) + \varepsilon^2 g(\mathcal{P}, \mathcal{Q}, \mathcal{Y}, \varepsilon)$$

by a change of variables of form (2). Obviously, this corresponds to the first step of the procedure described above. The averaged Hamiltonian is given by the expression

$$\overline{H}(\mathcal{I}_1, y, \varepsilon) = \mathcal{I}_1 + \varepsilon\mathcal{V}(\mathcal{I}_1, \mathcal{Y}), \tag{7}$$

where

$$\mathcal{V}(\mathcal{I}_1, \mathcal{Y}) = \frac{1}{2\pi} \int_0^{2\pi} v\big(\sqrt{2\mathcal{I}_1} \sin\varphi + \mathcal{Y}_1, \sqrt{2\mathcal{I}_1} \cos\varphi + \mathcal{Y}_2\big)\, d\varphi\, v(\mathcal{Y}). \tag{8}$$

Relation (8) can also be rewritten as $\mathcal{V}(\mathcal{I}_1, \mathcal{Y}) = J_0\big(\sqrt{-2\mathcal{I}_1\Delta_{\mathcal{Y}}}\big)v(\mathcal{Y})$. Estimate (4) is now obvious. The properties of the Bessel functions imply the estimate

$$\mathcal{V}(\mathcal{I}_1, \mathcal{Y}) = v(\mathcal{Y}) + \frac{1}{2}\mathcal{I}_1 \Delta v(\mathcal{Y}) + O(\mathcal{I}_1^2) \quad \text{for } \mathcal{I}_1 \to 0.$$

The corresponding change of variables is defined by (6), where

$$(P^m, Q^m, y^m) = (P, Q, y), \qquad (P^{m+1}, Q^{m+1}, y^{m+1}) = (\mathcal{P}, \mathcal{Q}, \mathcal{Y})$$

and

$$S^m(P, Q, y, \varepsilon) = PQ + y_1 y_2 + \frac{1}{2}\varepsilon\bigg(\int_0^{\varphi} \tilde{v}\big(\sqrt{2I}\, \cos\psi, \sqrt{2I}\, \sin\psi, y\big) d\psi + \\ + \int_{\pi}^{\varphi} \tilde{v}\big(\sqrt{2I}\, \cos\psi, \sqrt{2I}\, \sin\psi, y\big)\, d\psi\bigg)\bigg|_{\substack{P=\sqrt{2I}\cos\varphi, \\ Q=\sqrt{2I}\sin\varphi}} \tag{9}$$

with

$$\tilde{v}(P, Q, y) = \mathcal{V}\left(\frac{1}{2}(P^2 + Q^2), y\right) - v(Q + y_1, P + y_2).$$

2.4. The Hamiltonian system for the averaged Hamiltonian. Obviously, the Hamiltonian system for $\mathcal{H}$ is integrable, and its invariant manifolds can be described as

$$\begin{cases} \mathcal{I}_1 = \text{const} \geq 0, \\ \mathcal{Y} = \mathcal{Y}(\mathcal{I}_1, \tau, \varepsilon), \quad \tau \in \mathbb{R}, \end{cases}$$

where $\mathcal{Y}(\mathcal{I}_1, \tau, \varepsilon)$ are solutions of the system

$$\frac{d\mathcal{Y}}{d\tau} = J\nabla_{\mathcal{Y}}\mathcal{H}(\mathcal{I}_1, \mathcal{Y}, \varepsilon), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{10}$$

3. Almost invariant trajectories

As follows from the above, the classical energy of the system admits the estimate

$$H = \mathcal{I}_1 + O(\varepsilon).$$

To study the bottom part of the spectrum, we set $\mathcal{I}_1 = 0$. The corresponding invariant manifolds are given as

$$\mathcal{P} = 0, \qquad \mathcal{Q} = 0, \qquad \mathcal{Y} = \mathcal{Y}^0(\tau, \varepsilon), \tag{11}$$

where $\mathcal{Y}^0(\tau, \varepsilon) = \mathcal{Y}(0, \tau, \varepsilon)$ (see (10)) are curves and points. Substituting (11) in (2) and (1), we obtain a family of curves and points in the original space (p, x).

We introduce a useful definition.

Definition 1. A curve $\gamma^1(\varepsilon) = \{p = P(t,\varepsilon), x = X(t,\varepsilon), t \in \mathbb{R}\}$ is called an *almost invariant curve of the Hamiltonian* $H(p,x,\varepsilon)$ with an accuracy of $O(e^{-C/\varepsilon})$ if $H|_{\gamma^1(\varepsilon)} = \text{const} \mod O(e^{-C/\varepsilon})$ and the function $(P(t,\varepsilon), X(t,\varepsilon))$ satisfies the Hamiltonian system for H uniformly in $t \in \mathbb{R}$ with an accuracy of $O(e^{-C/\varepsilon})$.

Proposition 2. *Let $\mathcal{Y}(0,\tau,\varepsilon)$ be a nonconstant solution of system* (10) *for $\mathcal{I}_1 = 0$. Then the curves given by* (11), (2), *and* (1) *are almost invariant curves of the Hamiltonian H with an accuracy of $O(e^{-C/\varepsilon})$.*

We now describe the rest points of $\mathcal{H}$. We first note that if $X = (X_1, X_2)$ is a critical point of the function v, then $(-X_2, 0, X_1, X_2)$ is a rest point of the Hamiltonian system for H.

We now consider the constant solutions of system (10) for $\mathcal{I}_1 = 0$. The corresponding trajectories in the phase space $\mathbb{R}^4_{p,x}$ are points. If $\gamma^0(\varepsilon)$ is such a point, then we at least have

$$\frac{\partial H}{\partial x}\bigg|_{\gamma^0(\varepsilon)} = O(e^{-C/\varepsilon}), \qquad \frac{\partial H}{\partial p}\bigg|_{\gamma^0(\varepsilon)} = O(e^{-C/\varepsilon}).$$

We assume that v is a *Morse function* (this means that all its critical points are nondegenerate). Then we can show that all these points $\gamma^0(\varepsilon)$ lie in a $O(e^{-C/\varepsilon})$ neighborhood of the set of the rest points of the Hamiltonian H described above; we therefore consider only these "exact" rest points.

4. Classification of the trajectories

In this section, we classify the almost invariant curves constructed in the preceding section. We fix $\mathcal{I}_1 \geq 0$. Because of the periodicity of $\mathcal{H}$ relative to $\mathcal{Y}$, system (10) can be considered as a Hamiltonian system on the torus $\mathbb{T}^2 = \mathbb{R}^2/(a_1, a_2)$. Such an approach immediately gives a classification of its trajectories [8], [20].

Proposition 3. *Hamiltonian system* (10) *on the torus $\mathbb{T}^2$ can have trajectories of the following types:*

1. *the critical points of the function $\mathcal{H}$,*
2. *contractible (homotopic to a point) smooth closed curves on $\mathbb{T}^2$,*
3. *noncontractible (nonhomotopic to a point) smooth closed curves on $\mathbb{T}^2$, and*
4. *separatrices.*

Obviously, there is a correspondence between the trajectories of the system on the torus and those in the plane $\mathbb{R}^2_{\mathcal{Y}}$: contractible curves on the torus correspond to closed curves in the plane, and noncontractible curves on the torus correspond to nonclosed trajectories in $\mathbb{R}^2_{\mathcal{Y}}$. Each trajectory lies in some level set of the function $\mathcal{H}$.

Because all the trajectories on the plane have periodic preimages on the torus, for each trajectory $\mathcal{Y}$, there is a two-dimensional vector $d(\mathcal{Y}) = (d_1, d_2)$ with coprime integer components d_1 and d_2 such that

$$\mathcal{Y}(\tau + T) = \mathcal{Y}(\tau) + d \cdot a, \tag{12}$$

where $d \cdot a = d_1 a_1 + d_2 a_2$ and T is the period of the corresponding trajectory on the torus (see Fig. 2). For closed trajectories, we obviously have $d = 0$. The vector d is called the *drift vector* (cf. [3]). The ratio d_1/d_2 is called the *rotation number* [8]. It is easy to see that only two nonzero drift vectors with opposite directions can exist for a given $\mathcal{I}_1$.

A visual classification of the trajectories on the torus can be given using the *Reeb graph* of the function $\mathcal{H}$. The Reeb graph is constructed as follows. Each connected component of any level set $\mathcal{H} = E$ corresponds to a vertex of the graph. As E runs through $[\min \mathcal{H}, \max \mathcal{H}]$, we obtain edges whose combination forms the whole graph. Examples of the construction of the Reeb graph are shown in Figs. 3 and 4.

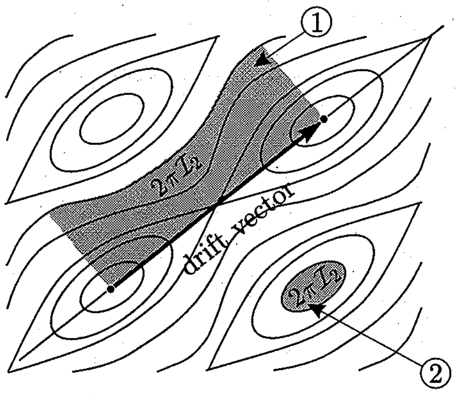

Fig. 2. The drift vector and the action for nonclosed (1) and closed (2) trajectories.

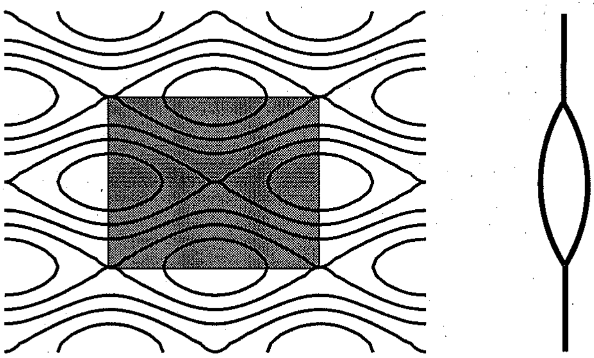

Fig. 3. The level curves and the Reeb graph of the function $A\cos x_1 + B\cos x_2$.

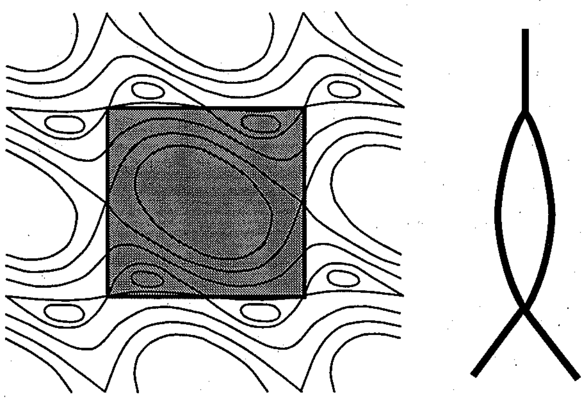

Fig. 4. The level curves and the Reeb graph of the function $A\cos x_1 + B\cos x_2 + C\cos(x_1 + x_2)$.

Remark. The function $\mathcal{H}$ can be considered as a function on any torus $\mathbb{T}^2_{n_1,n_2} = \mathbb{R}^2/(n_1 a_1, n_2 a_2)$, which leads to a Reeb graph with a more complicated structure. We discuss this case later.

Each point of the Reeb graph corresponds to a trajectory of the Hamiltonian $\mathcal{H}$ on the torus and to a set of trajectories in the plane. It useful to parameterize the points of the Reeb graph using the action

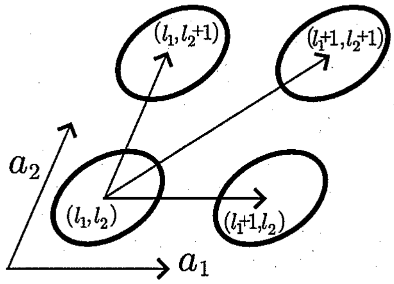

Fig. 5. Numbering of closed trajectories.

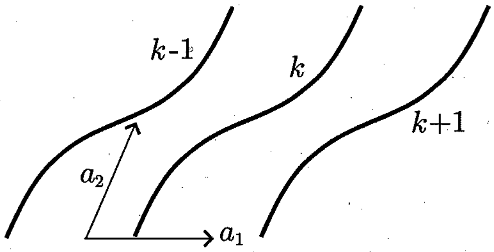

Fig. 6. Numbering of nonclosed trajectories.

variable $\mathcal{I}_2$ (see [8]):

$$\mathcal{I}_2(\mathcal{Y}) = \frac{1}{2\pi}\int_{\sigma}^{\sigma+T} \mathcal{Y}_1(\mathcal{I}_1,\tau,\varepsilon)\,d\mathcal{Y}_2(\mathcal{I}_1,\tau,\varepsilon) - \frac{\mathcal{Y}_2(\mathcal{I}_1,\sigma,\varepsilon)(d\cdot a)_1}{2\pi} - \frac{(d\cdot a)_1(d\cdot a)_2}{4\pi},$$

where $\mathcal{Y}(\mathcal{I}_1,\tau,\varepsilon)$ is the corresponding trajectory on the plane, T is the period of the corresponding trajectory on the torus, and d is the corresponding drift vector.

The variable $\mathcal{I}_2$ admits a simple geometrical interpretation that is well known for closed trajectories in the plane (see, e.g., [8], [19]): $2\pi\mathcal{I}_2$ is the oriented area of the domain bounded by the trajectory (see Fig. 2). The interpretation for nonclosed trajectories is different. Let $\mathcal{Y}(\mathcal{I}_1,\tau,\varepsilon)$ be such an nonclosed curve with the drift vector d, and let L_d be the straight line given by $L_d = \{\tau d\cdot a, \tau\in\mathbb{R}\}$. We fix some points $\mathcal{Y}(\mathcal{I}_1,\sigma,\varepsilon)$ and $\mathcal{Y}(\mathcal{I}_1,\sigma+T,\varepsilon)$ and project them on L_d. Then $2\pi\mathcal{I}_2$ is the oriented area of the obtained curvilinear trapezium (see Fig. 2). It is easy to see that the definition of $\mathcal{I}_2$ for nonclosed curves depends on their concrete representation (i.e., on a parallel transport in the plane). Nevertheless, $\mathcal{I}_2$ is defined uniquely up to $\big(|d_1|+|d_2|\big)a_{22}$, and this arbitrariness in the choice is inessential in what follows. On each edge of the Reeb graph, the Hamiltonian $\mathcal{H}$ is a function of $\mathcal{I}_1$ and $\mathcal{I}_2$: $\mathcal{H}=\mathcal{H}(\mathcal{I}_1,\mathcal{I}_2,\varepsilon)$.

It is obvious that if $\mathcal{Y}$ is a solution of the Hamiltonian system for $\mathcal{H}$, then the function $\mathcal{Y}+m\cdot a$, where $m=(m_1,m_2)\in\mathbb{Z}^2$ and $m\cdot a = m_1a_1+m_2a_2$, is also a solution of the same system. We introduce a numbering in the set of all such solutions and the corresponding trajectories on the plane $\mathbb{R}^2$. Closed trajectories and points are numbered by the multi-index $l=(l_1,l_2)\in\mathbb{Z}^2$ as follows. We fix some solution $\mathcal{Y}$ and assign the index $(0,0)$ to it. Then the trajectory with index l is given as $\mathcal{Y}+l\cdot a$ (see Fig. 5). In addition, we assume that the family of $\mathcal{Y}$ depends on $\mathcal{I}_2$ continuously. This numbering cannot be used for nonclosed trajectories because of equality (12). Let $f=(f_1,f_2)\in\mathbb{Z}^2$ be a vector adjoint to d, i.e., let $d_1f_1+d_2f_2=1$ (obviously, the choice of the vector f is rather arbitrary). We now number the nonclosed trajectories as follows. We fix a certain trajectory $\mathcal{Y}$ with the index 0, then the trajectory with the index $k\in\mathbb{Z}$ is given by the relation $\mathcal{Y}-k(Jf)\cdot a$ (see Fig. 6). We also assume that the family of $\mathcal{Y}$ depends on $\mathcal{I}_2$ continuously.

We return to the rest points and the almost invariant curves constructed in the preceding section. Their numbering is inherited from the numbering of the trajectories of the Hamiltonian $\mathcal{H}$. If the closed curve γ_0^{1c} (point γ_0^0) is given by

$$p_{1,2} = P_{1,2}^0(\tau), \qquad x_{1,2} = X_{1,2}^0(\tau), \tag{13}$$

then the curve γ_l^{1c} (point γ_l^0) with the index $l = (l_1, l_2) \in \mathbb{Z}^2$ is given by

$$p_1 = P_1^0(\tau) - (l \cdot a)_2, \qquad p_2 = P_2^0(\tau), \qquad x_{1,2} = X_{1,2}^0(\tau) + (l \cdot a)_{1,2}.$$

If the nonclosed curve γ_0^{1o} is given by (13), then the corresponding curve γ_k^{1o} with the index $k \in \mathbb{Z}$ is given by

$$p_1 = P_1^0(\tau) + \big(k(Jf) \cdot a\big)_2, \qquad p_2 = P_2^0(\tau), \qquad x_{1,2} = X_{1,2}^0(\tau) - \big(k(Jf) \cdot a\big)_{1,2}.$$

5. Spectral series

5.1. Heuristic considerations. There is a very attractive idea to find asymptotic formulas for the lower Landau bands as the limit of the corresponding formulas for the higher parts of the spectrum [9]. We describe them for clarity. For simplicity, we omit some details that are not important at the moment.

We quantize the action $\mathcal{I}_1$,

$$\mathcal{I}_1 = \mathcal{I}_1^\mu = \left(\frac{1}{2} + \mu\right) h, \quad \mu \sim O\left(\frac{1}{h}\right), \quad \mu \in \mathbb{Z}_+,$$

and choose the subset of the trajectories of the Hamiltonian $\mathcal{H}(\mathcal{I}_1^\mu, \cdot\,)$ satisfying the condition

$$\mathcal{I}_2 = \mathcal{I}_2^\nu = \left(\frac{1}{2} + \nu\right) h, \quad \nu \in \mathbb{Z}.$$

The set of the points $\mathcal{H}(\mathcal{I}_1^\mu, \mathcal{I}_2^\nu, \varepsilon)$ for closed trajectories and of the points $\mathcal{H}(\mathcal{I}_1^\mu, \mathcal{I}_2, \varepsilon)$ for nonclosed ones gives the asymptotic behavior of the μth Landau band with an accuracy of $O(h^2) + O(e^{-C/\varepsilon})$.

Strictly speaking, these considerations are valid only for the higher parts of the spectrum of $\widehat{H}$. Nevertheless, as we see below, the corresponding asymptotic eigenvalues for the lower Landau bands are

$$E = \mathcal{H}(0, \mathcal{I}_2, \varepsilon) + \frac{\partial \mathcal{H}}{\partial \mathcal{I}_1}(0, \mathcal{I}_2, \varepsilon)\mathcal{I}_1^\mu + O(h^2), \tag{14}$$

where $\mathcal{I}_2$ is quantized in the same way. It is easy to see that (14) is the Taylor expansion of the formulas for the higher bands at the point $\mathcal{I}_1 = 0$, i.e., we have a "uniform passage" in the formulas for asymptotic eigenvalues.

We note that the corresponding formulas for asymptotic eigenfunctions do not admit such a passage to the limit and to justify formulas (14), we must use the methods of the semiclassical approximation with complex phases (these methods are also known as the complex WKB method or the Maslov complex germ theory). A description of this method can be found in [21]; some explicit formulas are also given in [22]–[24].

We fix some positive numbers K and L. In the remaining parts of this section, the properties of the asymptotic eigenvalues and asymptotic eigenfunctions of the operator $\widehat{H}$ are described with an accuracy of $O(h^L) + O(\varepsilon^K)$. More detailed formulas are contained in Appendices A and B.

5.2. Spectral series corresponding to points. The procedure of assigning spectral series to the rest points of the classical Hamiltonian is known as the *oscillatory approximation method*; this procedure is a particular case of the Maslov complex canonical operator. Only the points corresponding to the extremum points of the potential v can be used for such an assignment.

The oscillatory approximation gives the expression

$$E^0_{\mu,\nu} = H\big|_{\gamma_l^0} + h\left(\frac{1}{2}+\mu\right)\frac{\partial\mathcal{H}}{\partial\mathcal{I}_1}\bigg|_{\gamma_l^0} + h\left(\frac{1}{2}+\nu\right)\frac{\partial\mathcal{H}}{\partial\mathcal{I}_2}\bigg|_{\gamma_l^0} + O(h^2), \quad \mu,\nu = O(1)\in\mathbb{Z}_+,$$

for asymptotic eigenvalues of the operator $\widehat{H}$. We justify this formula in Appendix A.

To each of these asymptotic eigenvalues, we assign a set of asymptotic eigenfunctions $\psi_l^{0,\mu,\nu}$ also defined by the oscillatory approximation method and satisfying the conditions

1. $\|\psi_l^{0,\mu,\nu}\|_{L^2(\mathbb{R}^2_x)} \ge c > 0$ for $h\to 0$,
2. $\big\|(\widehat{H} - E^0_{\mu,\nu})\psi_l^{0,\mu,\nu}\big\|_{L^2(\mathbb{R}^2_x)} = O(h^L)$, and
3. $\psi_l^{0,\mu,\nu}$ have compact supports and are localized near the projections of the corresponding points (P_l, X_l) on the plane $\mathbb{R}^2_x$, i.e., $\lim_{h\to 0}\psi_l^{0,\mu,\nu}(x,\varepsilon,h) = 0$ for all $x \ne X_l$.

A more detailed construction of these functions is given in Appendices A and B. We only note now that all the functions $\psi_l^{0,\mu,\nu}$ can be expressed through the single function $\psi_0^{0,\mu,\nu}$ by the formula

$$\psi_l^{0,\mu,\nu}(x,\varepsilon,h) = \psi_0^{0,\mu,\nu}(x - l\cdot a,\varepsilon,h)e^{-ia_{22}l_2x_1/h}. \tag{15}$$

We now use the well-known inequality

$$\operatorname{dist}(E,\sigma(A)) \le \frac{\|(A-E)f\|}{\|f\|}, \tag{16}$$

which holds for any self-adjoint operator A acting in an arbitrary Hilbert space (see, e.g., [7]). This gives the estimate

$$\operatorname{dist}(E^0_{\mu,\nu},\sigma(\widehat{H})) = O(h^L).$$

5.3. Spectral series corresponding to almost invariant closed curves. We let $\Gamma^{1c}(\varepsilon,h)$ denote the set of curves γ_l^{1c} satisfying the quantization condition

$$\mathcal{I}_2 = \mathcal{I}_2^\nu = \left(\nu+\frac{1}{2}\right)h, \quad \nu\in\mathbb{Z}, \quad \alpha_1 \le |h\nu| \le \alpha_2 \text{ as } h\to 0. \tag{17}$$

Proposition 4. *For each γ_l^{1c} in $\Gamma^{1c}(\varepsilon,h)$, there is a set of functions $\psi_l^{1c,\mu}(x,\mathcal{I}_2^\nu,\varepsilon,h)\in L^2(\mathbb{R}^2_x)$ and numbers (asymptotic eigenvalues)*

$$E^{1c}_\mu(\mathcal{I}_2^\nu,\varepsilon,h) = \mathcal{H}\big|_{\gamma_l^{1c}} + h\left(\frac{1}{2}+\mu\right)\frac{\partial\mathcal{H}}{\partial\mathcal{I}_1}\bigg|_{\gamma_l^{1c}} + O(h^2), \quad \mu\in\mathbb{Z}_+, \quad \mu = O(1), \quad l\in\mathbb{Z}^2, \tag{17a}$$

defined by the complex canonical operator method and satisfying the conditions

1. $\big\|(H - E^{1c}_\mu)\psi_l^{1c,\mu}\big\|_{L^2(\mathbb{R}^2_x)} = O(h^L) + O(\varepsilon^K)$ *and*
2. $\|\psi_l^{1c,\mu}\|_{L^2(\mathbb{R}^2_x)} \ge c > 0$ *as* $h\to 0$.

The functions $\psi_l^{1c,\mu}$ have compact supports and are asymptotically localized near the projections $\pi_x\gamma_l^{1c}$ of the corresponding curves γ_l^{1c} on the plane $\mathbb{R}_x^2$, i.e., $\lim_{h\to 0}\psi_l^{1c,\mu}(x,\mathcal{I}_2^\nu,\varepsilon,h)=0$ for all $x\notin\pi_x\gamma_l^{1c}$.

Proposition 4 follows immediately from the definition and the properties of the canonical operator. We show how to obtain formula (17a) for asymptotic eigenvalues in Appendix A.

As in the preceding subsection, we obtain the estimate

$$\operatorname{dist}\big(E_\mu^{1c}(\mathcal{I}_2^\nu,\varepsilon,h),\sigma(\widehat{H})\big)=O(h^L)+O(\varepsilon^K).$$

Another useful fact, which follows from the definition of the canonical operator and is used in what follows, is that all the functions $\psi_l^{1c,\mu}(x,\mathcal{I}_2,\varepsilon,h)$ can be expressed through $\psi_0^{1c,\mu}(x,\mathcal{I}_2^\nu,\varepsilon,h)$ by the formulas

$$\psi_l^{1c,\mu}(x,\mathcal{I}_2^\nu,\varepsilon,h)=\psi_0^{1c,\mu}(x-l\cdot a,\mathcal{I}_2^\nu,\varepsilon,h)e^{-ia_{22}l_2x_1/h}. \tag{18}$$

5.4. Spectral series corresponding to almost invariant nonclosed curves. We now consider the set $\Gamma^{1o}(\varepsilon)$ of almost invariant nonclosed curves of H.

Proposition 5. *For any curve $\gamma^{1o}\in\Gamma^{1o}(\varepsilon)$, there is a set of functions $\psi_k^{1o,\mu}(x,\mathcal{I}_2,\varepsilon,h)$ and numbers (asymptotic eigenvalues)*

$$E_\mu^{1o}(\mathcal{I}_2,\varepsilon,h)=\mathcal{H}|_{\gamma_k^{1c}}+h\left(\frac{1}{2}+\mu\right)\frac{\partial\mathcal{H}}{\partial\mathcal{I}_1}\bigg|_{\gamma_k^{1o}}+O(h^2),\quad \mu\in\mathbb{Z}_+,\quad \mu=O(1),$$

defined by the complex canonical operator method and satisfying the conditions

1. $\big\|(H-E_\mu^{1o})\psi_k^{1o,\mu}\big\|_{L^2(\Pi_d)}=O(h^L)+O(\varepsilon^K)$,
2. $\|\psi_k^{1o}\|_{L^2(\Pi_d)}\ge c>0$ *as* $h\to 0$, *and*
3. $\psi_k^{1o,\mu}(x+d\cdot a,\mathcal{I}_2,\varepsilon,h)=e^{i(2\pi\mathcal{I}_2-(d\cdot a)_2x_1-(d\cdot a)_1(d\cdot a)_2/2)/h}\psi_k^{1o,\mu}(x,\mathcal{I}_2,\varepsilon,h)$.

Here, d is the drift vector of the corresponding trajectory, and

$$\Pi_d=\big\{\tau_1(d\cdot a)+\tau_2J(d\cdot a):\tau_1\in[-1,1],\ \tau_2\in(-\infty,+\infty)\big\}.$$

Property 3 means that $\psi_k^{1o,\mu}$ do not belong to $L^2(\mathbb{R}^2)$ and we cannot use inequality (16) directly. We act as follows.

Proposition 6. *Let a number E_μ^{1o} and a function $\psi_k^{1o,\mu}$ satisfy properties 1–3. Then*

$$\operatorname{dist}\big(E_\mu^{1o},\sigma(\widehat{H})\big)=O(h^L)+O(\varepsilon^K).$$

Proof. For brevity, we set $\Psi=\psi_k^{1o,\mu}$ and $E=E_\mu^{1o}$. We introduce new coordinates y_1, y_2:

$$\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}\alpha&\beta\\-\beta&\alpha\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix},\quad\text{where}\quad\begin{pmatrix}\alpha\\\beta\end{pmatrix}=\frac{d\cdot a}{|d\cdot a|}.$$

In these coordinates, Ψ satisfies the condition

$$\Psi(y_1+|d\cdot a|,y_2)=\exp\left\{\frac{i}{h}\left(2\pi\mathcal{I}_2-(d\cdot a)_2(\alpha y_1-\beta y_2)+\frac{1}{2}(d\cdot a)_1(d\cdot a)_2\right)\right\}\Psi(y_1,y_2),$$

and the operator $\widehat{H}$ has the form

$$\widehat{H} = -\frac{1}{2}h^2\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\right) - ih(\beta y_1+\alpha y_2)\left(\alpha\frac{\partial}{\partial y_1}-\beta\frac{\partial}{\partial y_2}\right)+$$
$$+\frac{1}{2}(\beta y_1+\alpha y_2)^2+\varepsilon w(y_1,y_2),$$

where $w(y_1,y_2)=v(\alpha y_1-\beta y_2,\beta y_1+\alpha y_2)$.

We set

$$S(y_1,y_2)=\frac{1}{2}(-\alpha\beta y_1^2+\alpha\beta y_2^2+2\beta^2 y_1 y_2)$$

and introduce the operator $U=e^{iS/h}$. This operator is unitary, and the spectra of the operators $\widehat{H}$ and $\widetilde{H}=U^{-1}\widehat{H}U$ therefore coincide. We note that

$$\widetilde{H}=\frac{1}{2}\left(-ih\frac{\partial}{\partial y_1}+y_2\right)^2+\frac{1}{2}\left(-ih\frac{\partial}{\partial y_2}\right)^2+\varepsilon w(y_1,y_2).$$

We also note that the function $\Phi=U^{-1}\Psi=e^{-iS/h}\Psi$ satisfies the condition

$$\Phi\big(y_1+|d\cdot a|,y_2\big)=e^{2\pi i\mathcal{I}_2/h}\Phi(y_1,y_2). \tag{19}$$

We set $\widetilde{\Pi}=\big\{(y_1,y_2):-|d\cdot a|\le y_1\le |d\cdot a|\big\}$ and $\widetilde{\Pi}_m=\big\{(y_1,y_2):-|d\cdot a|m\le y_1\le |d\cdot a|m\big\}$. We choose a smooth function $e(z)$, $0\le |e|\le 1$, such that $e(z)=0$ for $z\notin\big(-2|d\cdot a|,2|d\cdot a|\big)$ and $e(z)=1$ for $z\in\big(-|d\cdot a|,|d\cdot a|\big)$, and we introduce a constant c_1 such that $|e|+|e'|+|e''|\le c_1$. We set $e_m(y_1,y_2)=e(y_1/m)$.

We note that $\|f\|_{\widetilde{\Pi}_m}=\sqrt{m}\,\|f\|_{L^2(\widetilde{\Pi})}$ holds for any function f satisfying condition (19), in particular, for the function $\phi=(\widetilde{H}-E)\Phi$.

We now have the system of equalities and inequalities

$$\begin{aligned}\sqrt{m}\,\operatorname{dist}\big(\sigma(\widetilde{H}),E\big)\|\Phi\|_{L^2(\widetilde{\Pi})} &\le \operatorname{dist}\big(\sigma(\widetilde{H}),E\big)\|e_m\Phi\|\le\\ &\le \|(\widetilde{H}-E)(e_m\Phi)\|=\\ &=\left\|e_m\phi-\frac{h^2}{2}\Delta e_m\Phi-h^2\nabla e\nabla\Phi-ihx_2\frac{\partial e_m}{\partial x_1}\right\|\le\\ &\le\|e_m\phi\|+\frac{h^2}{2}\|\Delta e_m\Phi\|+h^2\|\nabla e\nabla\Phi\|+h\left\|x_2\frac{\partial e}{\partial x_1}\Phi\right\|\le\\ &\le c_1\sqrt{2m}\,\|\phi\|_{L^2(\widetilde{\Pi})}+\frac{h^2c_1\sqrt{m}}{2m^2}\|\Phi\|_{L^2(\widetilde{\Pi})}+\\ &\quad+\frac{h^2c_1\sqrt{m}}{m}\|\nabla\Phi\|_{L^2(\widetilde{\Pi})}+\frac{hc_1\sqrt{m}}{m}\|x_2\Phi\|_{L^2(\widetilde{\Pi})}.\end{aligned}$$

We take the first and the last expression in this system, divide them by $\sqrt{m}$, and let m tend to ∞. We then obtain

$$\operatorname{dist}\big(\sigma(\widetilde{H}),E\big)\|\Phi\|_{L^2(\widetilde{\Pi})}\le\big\|(\widetilde{H}-E)\Phi\big\|_{L^2(\widetilde{\Pi})}.$$

Because $\|\Phi\|_{L^2(\widetilde{\Pi})}\ge c>0$ as $h\to 0$, we obtain

$$\operatorname{dist}\big(\sigma(\widetilde{H}),E\big)=O(h^L)+O(\varepsilon^K).$$

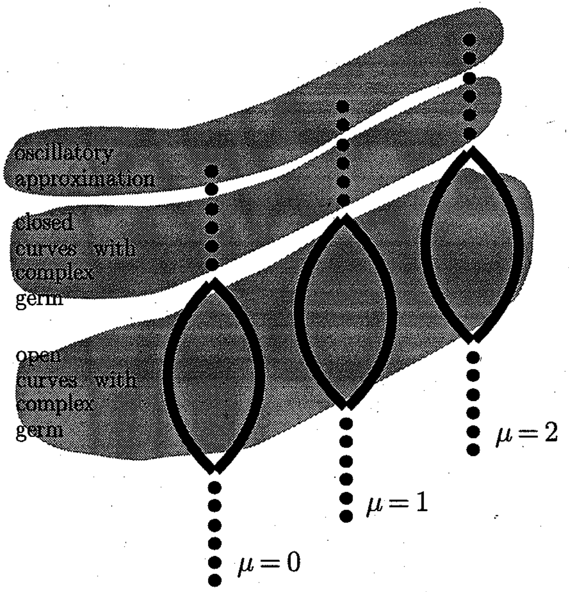

Fig. 7. Structure of the semiclassical asymptotic form for the lower Landau bands.

As in the finite motion cases, all the functions $\psi_k^{1o,\mu}$ have locally compact supports and are localized near the projections of the corresponding trajectories on the plane $\mathbb{R}_x^2$. They can all be expressed through the single function $\psi_0^{1o,\mu}$ as

$$\psi_k^{1o,\mu}(x) = \psi_0^{1o,\mu}(x + k(Jf) \cdot a)e^{ikf_1 a_{22} x_1/h}. \tag{20}$$

5.5. General structure of the spectrum. The procedure for constructing asymptotic eigenvalues described above leads to selecting a discrete subset of points on each edge of the Reeb graph corresponding to finite motion and to including all the edges corresponding to infinite motion as a whole in the spectrum. We therefore have sets of the numbers $E^0_{\mu,\nu}$, $E^{1c}_\mu(\mathcal{I}_2^\nu)$, and $E^{1o}_\mu(\mathcal{I}_2)$.

Definition 2. For a fixed $\mu \in \mathbb{Z}_+$, the union Σ_μ of the values $E^0_{\mu,\nu}$, $E^{1c}_\mu(\mathcal{I}_2^\nu)$, and $E^{1o}_\mu(\mathcal{I}_2)$ is called the *semiclassical asymptotic form* of the μth Landau band.

As follows from the preceding, the semiclassical asymptotic form of a Landau band has the following structure. We consider the Reeb graph of the function $\mathcal{H}(\mathcal{I}_1^\mu, \cdot)$ for a *fixed* μ. We quantize the variable $\mathcal{I}_2$ on the edges corresponding to finite motion by rule (17), i.e., we assume that all these values are admissible. We note that although this formula holds for all μ, its justification is realized by different methods. Namely, the oscillatory approximation method (see Sec. 5.2) is used for $\mu = O(1)$ and the method of the complex germ for closed curves (see Proposition 4) is used for $\mu = O(1/h)$. On the edges corresponding to infinite motion, the variable $\mathcal{I}_2$ is not quantized, and all the values are admissible (this is justified by the method of the complex germ for nonclosed curves; see Proposition 5). The set of values of $\mathcal{H}(\mathcal{I}_1^\mu, \mathcal{I}_2, \varepsilon)$ now coincides at all admissible points with the semiclassical asymptotic form of the μth Landau band with an accuracy of $O(h^2)$. This semiclassical asymptotic form is thus determined by the Reeb graph of the function $\mathcal{H}(\mathcal{I}_1^\mu, \cdot)$. We note that because h is small, the structure of all these graphs coincides with the structure of the Reeb graph of the function $\mathcal{H}(0, \cdot)$. The semiclassical asymptotic forms therefore have the same structure up to $O(h^2)$ (see Fig. 7).

Strictly speaking, we can only prove that there are points of the spectrum of $\widehat{H}$ in a $\big(O(h^L) + O(\varepsilon^K)\big)$ neighborhood of Σ_μ. Nevertheless, this seems the most complete result that can be obtained using the above asymptotic methods, which are based on the simplest *additive* (with respect to the parameter h) asymptotic approximation; this asymptotic approximation does not take the *tunneling effects* into account. It is important to emphasize that the above considerations are valid for both rational and irrational values of the magnetic field flux.

6. The band structure of the spectrum: Heuristic considerations

6.1. The Bloch conditions. The number $\eta = a_{22}/h$ has the meaning of the *number of the flux quanta of the magnetic field through an elementary cell* (and is sometimes called *flux*). It is well known that if the number η is rational, $\eta = N/M$, where N and M are coprime integers, then the spectrum of $\widehat{H}$ has the band structure. It is convenient to parameterize the points in a band by the *quasimomentum* $q = (q_1, q_2) \in [0, 1/M) \times [0, 1)$, $E = E_n(q, \varepsilon, h)$, where n is the index of the corresponding band. For each band, there is also a set of *Bloch functions* $\Psi_n^j(x, q, \varepsilon, h)$, $j = \overline{0, M-1}$. For a fixed q, these functions are the generalized eigenfunctions of $\widehat{H}$ corresponding to the spectral value $E_n(q, \varepsilon, h)$. They also satisfy the following (*magneto-Bloch*) conditions:

$$\begin{aligned} &\Psi_n^j(x + a_1, q, \varepsilon, h) = \Psi_n^j(x, q, \varepsilon, h)e^{-2\pi i(q_1 - \eta j)}, && j = \overline{0, M-1}, \\ &\Psi_n^j(x + a_2, q, \varepsilon, h) = \Psi_n^{j+1}(x, q, \varepsilon, h)e^{-i\eta(x_1 + a_{21}/2)}, && j = \overline{0, M-2}, \\ &\Psi_n^{M-1}(x + a_2, q, \varepsilon, h) = \Psi_n^0(x, q, \varepsilon, h)e^{-i\eta(x_1 + a_{21}/2) - 2\pi i q_2}. \end{aligned} \tag{21}$$

Using the asymptotic eigenfunctions constructed in Sec. 5, we try to construct a family of functions satisfying (21). Our further considerations depend on the structure of the Reeb graph of $\mathcal{H}$, and we restrict ourselves to the simplest case where $\mathcal{H}(0, \cdot)$ is a minimal Morse function, i.e., where $\mathcal{H}(0, \cdot)$ has one minimum and one maximum in the unit cell. The corresponding graph is shown in Fig. 3.

6.2. Finite motion. We first consider the Reeb graph edges corresponding to the finite classical motion. By analogy with the so-called Lifshits–Gelfand–Zak representation, we seek the functions satisfying the Bloch conditions in the form

$$\Psi^j(x, q, \varepsilon, h) = \sum_{l \in \mathbb{Z}^2} C_l^j(q, h)\psi_l(x, \varepsilon, h), \tag{22}$$

where ψ_l denotes $\psi_l^{0,\mu,\nu}(x, \varepsilon, h)$ or $\psi_l^{1c,\mu}(x, \mathcal{I}_2^\nu, \varepsilon, h)$.

Proposition 7. *For each μ and ν, there are M^2 linearly independent functions of form* (22) *forming M sets satisfying* (21). *The function $\Psi_n^{s,j}$ (the jth member of the sth set) can be given by the coefficients $C_l^{s,j}(q)$ of the form*

$$C_{l_1,l_2}^{s,j}(q, h) = \begin{cases} \exp\left\{-2\pi i(q_1 l_1 + q_2 n) + 2\pi i \eta l_1 j - \dfrac{i\eta l_2 a_{21}}{2}\right\} & \text{if } l_2 + j - s + nM = 0,\ n \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Proof. For simplicity, we omit the dependence of all the functions on h and ε. We use relations (15) and (18). We then obtain

$$\Psi^j = \sum_{(l_1, l_2) \in \mathbb{Z}^2} C_{l_1,l_2}^j(q)\psi_0(x - l_1 a_1 - l_2 a_2)e^{-ia_{22}l_2x_1/h}.$$

Because the asymptotic supports of ψ_l do not intersect, Bloch conditions (21) immediately imply

$$\begin{cases} C^j_{l_1+1,l_2}(q) = C^j_{l_1,l_2}(q)e^{-2\pi i(q_1-j\eta)}, & j=\overline{0,M-1}, \\ C^j_{l_1,l_2+1} = C^{j+1}_{l_1,l_2}e^{-i\eta a_{21}/2}, & j=\overline{0,M-2}, \\ C^{M-1}_{l_1,l_2+1} = C^0_{l_1,l_2}e^{-i\eta a_{21}/2-2\pi i q_2}. \end{cases} \tag{24}$$

We see that all the numbers $C^j_{l_1,l_2}$ are defined through the numbers $C^j_{0,0}$, $j=\overline{0,M-1}$, which can be chosen arbitrarily. Therefore, there are at most M^2 linearly independent solutions. If we set $C^{s,j}_{0,0}=\delta_{sj}$, $s=\overline{0,M-1}$, for the sth solution, then we obtain formulas (23). We show that all these solutions are indeed linearly independent. For this, it suffices to prove the linear independence of M^2 vectors

$$A^{s,j} = (C^{s,j}_{l_1,l_2})_{l_1,l_2=\overline{0,M-1}}, \quad s,j=\overline{0,M-1}.$$

The Gram matrix of this system of vectors $G_{(s_1,j_1),(s_2,j_2)} = \langle A^{s_1,j_1} \mid A^{s_2,j_2}\rangle$ can be easily calculated:

$$G_{(s_1,j_1),(s_2,j_2)} = \begin{cases} M & \text{if } s_1=s_2 \text{ and } j_1=j_2, \\ 0 & \text{otherwise.} \end{cases}$$

Because this matrix is nondegenerate, the linear independence is proved.

Proposition 7 shows that for each quantization point $E^0_{\mu,\nu}$ or $E^{1c}_\mu(\mathcal{I}^\nu_2)$, there are M exponentially small spectral zones contained in some neighborhood of it that is exponentially small (with respect to h). Our approximation cannot give a rigorous justification; this requires applying other, finer methods based on multiplicative asymptotic approximations and taking the tunneling effects into account. Nevertheless, it seems possible to give some additional arguments.

If we enlarge the elementary cell by the rule $a_1 \to Ma_1$, then we obtain the integer flux case, and only one Bloch function exists for each band. On each level set of $\mathcal{H}$ corresponding to finite motion, there are M connected components, i.e., the corresponding Reeb graph has M end edges on each side, and for each of these edges, we can use the above considerations to construct some functions satisfying the Bloch conditions. Obviously, all these functions are linearly independent. In the well-known problems (such as the double-well problem, for example), a similar situation actually corresponds to the existence of M asymptotic eigenvalues with an exponentially small distance between them.

It also follows from (24) that on the space of constructed asymptotic eigenfunctions, a representation of the magnetic translation group for the flux N/M is realized. It is well known that irreducible representations of this group are M-dimensional [13] and a small variation of the parameters splits the eigenvalue into M numbers [6].

It is interesting to consider the representation of the magnetic translation group realized by (24) in the context of the so-called Langbein duality [25], [26]. On one hand, it is a subrepresentation of the standard representation of this group, which dictates splitting the Landau band into N magnetic subbands [6] (see below). On the other hand, it is analogous to the representation of the magnetic translation group in a discrete space [27], where its decomposition into irreducible representations dictates splitting the band into M magnetic minibands.

6.3. Infinite motion. We now try to construct functions satisfying the Bloch conditions and corresponding to the infinite-motion edges of the Reeb graph. Each level set of $\mathcal{H}$ consists of two families of nonclosed trajectories whose drift vectors have opposite directions. These vectors are denoted by $\pm d$. The corresponding asymptotic eigenfunctions are denoted by $\psi^{1o,\mu,\pm}_k(x,\mathcal{I}^\pm_2,\varepsilon,h)$. We use the explicit dependence of the corresponding action variable $\mathcal{I}^\pm_2$ on the index k, i.e., the actions corresponding to the trajectories with the index k are $\mathcal{I}^\pm_2 + ka_{22}$.

To simplify the calculations, we assume that $d = (1, 0)$ (if this is not the case, we can rotate the coordinates by a gauge transformation; in fact, we already applied this procedure in the proof of Proposition 6). Then the corresponding adjoint vectors are $\pm f = (\pm 1, 0)$.

It follows from condition 3 (see Proposition 5) and (20) that the functions $\psi_k^{1o,\mu,\pm}(x, \mathcal{I}_2, \varepsilon, h)$ satisfy the conditions

$$\begin{aligned} \psi_k^{1o,\mu,\pm}(x + a_1, \mathcal{I}_2^\pm + ka_{22}, \varepsilon, h) &= \psi_k^{1o,\mu,\pm}(x, \mathcal{I}_2^\pm + ka_{22}, \varepsilon, h)e^{\pm 2\pi i(\mathcal{I}_2^\pm + ka_{22})/2}, \\ \psi_k^{1o,\mu,\pm}(x + a_2, \mathcal{I}_2^\pm + ka_{22}, \varepsilon, h) &= \psi_{k\pm 1}^{1o,\mu,\pm}(x, \mathcal{I}_2^\pm + ka_{22}, \varepsilon, h)e^{-i\eta x_1 \pm i\eta k a_{21}}. \end{aligned} \tag{25}$$

We omit the dependence on μ and again seek the functions satisfying the Bloch conditions in the form

$$\Psi^{j,\pm}(x, q, \varepsilon, h) = \sum_{k\in\mathbb{Z}} C_k^{j,\pm}(q, h)\psi_k^{1o,\pm}(x, \mathcal{I}_2^\pm + ka_{22}, \varepsilon, h). \tag{26}$$

Proposition 8. *The functions $\Psi^{j,\pm}$ given by (26) satisfy the Bloch conditions only if*

$$\mathcal{I}_2^\pm = \mp h\left(q_1 + \frac{n^\pm}{M}\right). \tag{27}$$

The coefficients in (26) can be chosen as

$$C_k^{j,\pm} = \begin{cases} \exp\left\{\pm i\eta\left(k|k-1| \mp \dfrac{|k|}{2}\right)a_{21} + 2\pi i n q_2\right\} & \text{for } k + j = nM,\ n \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Proof. We substitute (26) and (25) in (21). Because the asymptotic supports of ψ_k^{1o} do not intersect, we obtain the relations

$$\begin{aligned} &C_k^{j,\pm}(q, h)e^{\pm 2\pi i(\mathcal{I}_2^\pm + ka_{22})/h} = C_k^{j,\pm}(q, h)e^{-2\pi i(q_1 - \eta j)}, && j = \overline{0, M-1}, \\ &C_{k\mp 1}^{j,\pm}(q, h)e^{-i\eta x_1 \pm i\eta(k\mp 1)a_{21}} = C_k^{j+1,\pm}(q, h)e^{-i\eta(x_1 + a_{21}/2)}, && j = \overline{0, M-2}, \\ &C_{k\mp 1}^{M-1,\pm}(q, h)e^{-i\eta x_1 \pm i\eta(k\mp 1)a_{21}} = C_k^{0,\pm}(q, h)e^{-i\eta(x_1 + a_{21}/2) - 2\pi i q_2}. \end{aligned}$$

The first relation implies

$$\mathcal{I}_2^\pm + ka_{22} = h\big(m^\pm \mp q_1 \pm \eta(j-1)\big),$$

and we obtain (27). In addition, we have the conditions for the coefficients $C_k^{j,\pm}$

$$\begin{gathered} C_{k\pm 1}^{(j+1) \bmod M,\pm}(q, h) = \sigma_j(q)C_k^{j,\pm}(q, h)e^{\pm i a_{21}(k\pm 1)/2}, \\ j = \overline{0, M-1}, \qquad k \in \mathbb{Z}, \qquad \sigma_{0,\dots,M-2}(q) = 1, \qquad \sigma_{M-1}(q) = e^{2\pi i q_2}. \end{gathered}$$

All these coefficients are uniquely determined by $C_0^{j,\pm}$, and we therefore have at most M families of functions $\Psi^{s,j,\pm}$ determined by $C_0^{s,j,\pm} = \delta_{sj}$. The coefficients $C_k^{s,j}$ are then determined by formulas (28) with j replaced with $j - s$. It is easy to see that the equalities $\Psi_{(s+1) \bmod M}^{(j+1) \bmod M} = \Psi_s^j$ hold for any s and j. This means that the sets Ψ_s^j, $j = \overline{0, M-1}$, can be obtained from each other by renumbering. We therefore have only one set of functions satisfying the Bloch conditions.

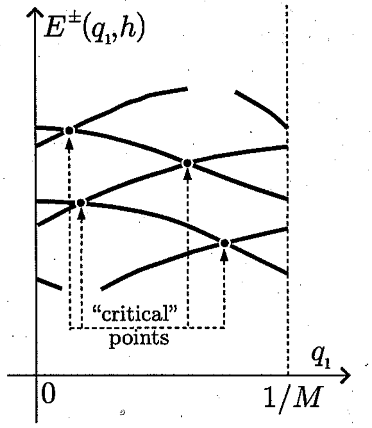

Fig. 8. Heuristic dispersion relations.

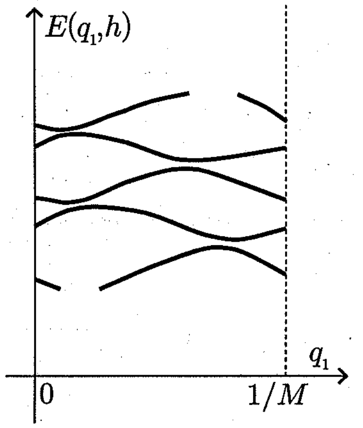

Fig. 9. True dispersion relation (conjecture).

The dependence of $\mathcal{I}_2^{\pm}$ on q_1 implies some relations between energy and quasimomenta (the heuristic dispersion relations):

$$E = E^{\pm}(q_1, h) = \mathcal{H}\big(0, \mathcal{I}_2^{\pm}(q_1, h)\big)$$

(the dependence on q_2 is absent up to $O(h^L) + O(\varepsilon^K)$). At certain "critical" points q_1^*, we have $E^+(q_1^*, h) = E^-(q_1^*, h)$ (see Fig. 8). We think that in certain neighborhoods of these points, which are exponentially small with respect to h, there are gaps and these gaps are also exponentially small with respect to h. More precisely, we can expect that near "critical" points, there are points from different bands, and the asymptotic approximations of the corresponding Bloch functions are described by a nontrivial linear superposition of the functions $\Psi^{j,\pm}$. In this case, the true dispersion relations have the form shown in Fig. 9. A rigorous proof of these facts must be based on the study of tunneling effects (cf. [20]).

We can verify that all the coefficients $C_k^{j,\pm}$ are nonzero for $d \neq (1,0)$. We can also expect that the dispersion relations $E(q,h)$ are functions of some linear combination q_1 and q_2 with some accuracy.

6.4. Conclusions. Based on the above heuristic considerations, we give a general description of the asymptotic form of the band spectrum.

We have different asymptotic forms for the spectrum of $\widehat{H}$ on different edges of the Reeb graph. In the simplest case, we have exponentially small bands on the edges corresponding to finite motion. These bands are grouped into M-tuples (M is the denominator of the flux), and the distances between the bands of the same group are also exponentially small. These groups are separated by gaps of length $O(\varepsilon h)$. On the edges corresponding to infinite motion, we have bands of length $O(\varepsilon h)$ separated by exponentially small gaps. We note that our approximation does not give any asymptotic form of the spectrum near the energy levels corresponding to separatrices; special methods must be used here also.

Returning to the operator $\widehat{H}_{B,w}$, we obtain bands that are exponentially small with respect to B and gaps of length of the order $1/B$ on the edges corresponding to finite motion, as well as gaps that are exponentially small with respect to B and bands of length of the order $1/B$ on the edges corresponding to infinite motion. We note that the diameter D_μ of the semiclassical asymptotics of the μth Landau band for the operator $\widehat{H}$ has the form

$$D_\mu = \max \mathcal{H}(\mathcal{I}_1^\mu, \cdot) - \min \mathcal{H}(\mathcal{I}_1^\mu, \cdot) \approx \varepsilon(\max \overset{\circ}{v} - \min v)$$

with an accuracy of $O(h^2)$. Taking the relation between ε and B, as well as between $\widehat{H}$ and $\widehat{H}_{B,w}$, into account (see the introduction), we reach the following important conclusion: the order of the leading term of the diameter in the semiclassical asymptotic form of the Landau band of the operator $\widehat{H}_{B,w}$ is independent of B.

Each Landau band is then split into approximately N subbands (N is the numerator of the flux). To verify this, we need only count the quantization points on all edges with their degeneracy taken into account. As the value of the flux η approaches an irrational value, both of the numbers N and M vary irregularly; a picture similar to the *Hofstadter butterfly* appears [28]. From the above formulas for the asymptotic eigenfunctions satisfying the Bloch conditions, it is also easy to see that each of them is localized (under a suitable choice of coordinates) in strips of elementary cells separated by $M-1$ "empty" strips. The number M becomes infinitely large as the value of the flux tends to an irrational value, and this probably means that in the limit, each of these functions is localized in a neighborhood of a single strip.

If the Reeb graph has a more complicated structure, then the corresponding asymptotic form of the band spectrum becomes more complicated, too. In particular, some tunneling effects between the edges corresponding to finite and infinite motion can arise. Problems with different Reeb graphs need a separate study.

Appendix A: Justification of formulas for asymptotic eigenvalues

A.1. Asymptotic eigenvalues corresponding to points. We give the following definition.

Definition 3. A rest point γ^0 of the Hamiltonian H is said to be *stable in the linear approximation* if the matrix

$$H'' = \begin{pmatrix} -H_{xp} & -H_{xx} \\ H_{pp} & H_{px} \end{pmatrix}\Bigg|_{\gamma^0} \tag{29}$$

is diagonalizable and has a purely imaginary spectrum.

We let $\langle \cdot \,|\, \cdot \rangle$ denote the bilinear product in $\mathbb{C}^2$,

$$\big\langle (w_1, w_2) | (z_1, z_2) \big\rangle = w_1 z_1 + w_2 z_2, \quad w_{1,2}, z_{1,2} \in \mathbb{C},$$

and let $[\cdot \,|\, \cdot]$ denote the induced skew-linear product in $\mathbb{C}^4$,

$$\big[(w^1, z^1) | (w^2, z^2)\big] = \langle w^1 | z^2 \rangle - \langle w^2 | z^1 \rangle, \quad w^{1,2}, z^{1,2} \in \mathbb{C}^2.$$

The stability of a point γ^0 is equivalent to the existence of a two-dimensional complex plane r in $\mathbb{C}^4$ satisfying the conditions

1. for any vectors $\lambda', \lambda'' \in r$, we have $[\lambda'|\lambda''] = 0$,
2. for any nonzero vector $\lambda \in r$, we have $[\lambda \mid \bar{\lambda}]/(2i) > 0$, and
3. $H''r \subset r$.

Such a plane is a particular case of the *Maslov complex germ*.

Let $i\beta_1$ and $i\beta_2$ be eigenvalues of H'' such that the corresponding eigenvectors lie in r. The oscillatory approximation then gives the formula

$$E^0_{\mu,\nu} = H|_{\gamma^0} + \beta_1\left(\mu + \frac{1}{2}\right)h + \beta_2\left(\nu + \frac{1}{2}\right)h + O(h^2) \tag{30}$$

for asymptotic eigenvalues of the operator $\widehat{H}$ [21].

We now return to the matrix H''. The following simple fact holds.

Proposition 9. *Let a be an eigenvector of the matrix H'' with the eigenvalue α, let $(\tilde{p}, \tilde{x})$ be new canonical variables in which the Hamiltonian H has the form $\widetilde{H}(\tilde{p}, \tilde{x})$, and let*

$$\widetilde{H}'' = \begin{pmatrix} -\widetilde{H}_{\tilde{x}\tilde{p}} & -\widetilde{H}_{\tilde{x}\tilde{x}} \\ \widetilde{H}_{\tilde{p}\tilde{p}} & \widetilde{H}_{\tilde{p}\tilde{x}} \end{pmatrix}\Bigg|_{\gamma^0}.$$

Then the vector $\partial(p,x)/\partial(\tilde{p},\tilde{x})a$ is an eigenvector of $\widetilde{H}''$ with the same eigenvalue α.

We set $\tilde{p} = (\mathcal{P}, \mathcal{Y}_1)$ and $\tilde{x} = (\mathcal{Q}, \mathcal{Y}_2)$. We then obtain the matrix of the second derivatives

$$\widetilde{H}'' = \begin{pmatrix} 0 & 0 & -\omega & 0 \\ 0 & -\mathcal{H}_{12} & 0 & -\mathcal{H}_{22} \\ \omega & 0 & 0 & 0 \\ 0 & \mathcal{H}_{11} & 0 & \mathcal{H}_{12} \end{pmatrix} + O(e^{-C/\varepsilon}), \qquad \omega = \frac{\partial H}{\partial \mathcal{I}_1}\bigg|_{\gamma^0}, \qquad \mathcal{H}_{ij} = \frac{\partial^2 \mathcal{H}}{\partial \mathcal{Y}_i \partial \mathcal{Y}_j}\bigg|_{\gamma^0}.$$

The eigenvalues of $\widetilde{H}''$ with an accuracy of $O(e^{-C/\varepsilon})$ can be found explicitly: they are $i\beta_1^\pm = \pm i\omega$ and $i\beta_2^\pm = \pm i\sqrt{\det(\mathcal{H}_{ij})}$. Constructing the complex germ, we see that we should have $\beta_1 = \omega$ and $\beta_2 = \beta_2^\pm$ in (30), where the sign $+/-$ corresponds to the minimum/maximum of v. It is also easy to show that $\pm\sqrt{\det(\mathcal{H}_{ij})} = \partial\mathcal{H}/\partial\mathcal{I}_2$ in this case.

A.2. Asymptotic eigenvalues corresponding to curves. According to the canonical operator method, each point $\big(P(\tau,\varepsilon), X(\tau,\varepsilon)\big)$ of a curve $\gamma^1(\varepsilon)$ is assigned a two-dimensional complex plane $r(\tau,\varepsilon) \subset \mathbb{C}^4$ spanned by the vectors $a_1(\tau,\varepsilon) = \big(w(\tau,\varepsilon), z(\tau,\varepsilon)\big) = (P_\tau, X_\tau)$ and $a_2(\tau,\varepsilon) = \big(W(\tau,\varepsilon), Z(\tau,\varepsilon)\big)$, $W, Z \in \mathbb{C}^2$, satisfying the conditions

1. $[a_1|a_2] = 0$,
2. $[a_2 \mid \bar{a}_2]/(2i) > 0$,
3. the vector a_2 is invariant under the linearized Hamiltonian system

$$\dot{\xi} = H''\xi, \quad \text{where } H'' = \begin{pmatrix} -H_{xp} & -H_{xx} \\ H_{pp} & H_{px} \end{pmatrix}\Bigg|_{\gamma^1}, \tag{31}$$

4. the vector a_2 is an eigenvector of the monodromy operator of system (31).

Such a family of planes is called the *Maslov complex germ* over the curve γ^1. The existence of the complex germ guarantees that the absolute value of the eigenvalue α of the monodromy operator corresponding to a_2 is equal to one. We write this value as $\alpha = e^{i\beta T}$. We then have the formula

$$E = \mathcal{H}|_{\gamma^1} + h\left(\mu + \frac{1}{2}\right)\beta$$

for asymptotic eigenvalues [22]–[24] (we refer to [24] for answering the question why we prefer β to other numbers $\beta + 2\pi n/T$; this is a question of normalization).

It is difficult to find the eigenvalues of the monodromy operator directly, and we therefore proceed as follows. Let $(\tilde{p}, \tilde{x})$ be new canonical variables in which the Hamiltonian H has the form $\widetilde{H}(\tilde{p}, \tilde{x})$. We consider the linear system

$$\dot{\eta} = \widetilde{H}''\eta, \quad \text{where } \widetilde{H}'' = \begin{pmatrix} -\widetilde{H}_{\tilde{x}\tilde{p}} & -\widetilde{H}_{\tilde{x}\tilde{x}} \\ \widetilde{H}_{\tilde{p}\tilde{p}} & \widetilde{H}_{\tilde{p}\tilde{x}} \end{pmatrix}\Bigg|_{\gamma^1}. \tag{32}$$

Proposition 10 (see [23]). *The linear transformation* $\xi = \partial(p,x)/\partial(\tilde{p},\tilde{x})\eta$ *transforms system* (31) *into system* (32) *and preserves the skew-linear product of any two solutions.*

Setting $\tilde{p} = (\mathcal{P}, \mathcal{Y}_1)$ and $\tilde{x} = (\mathcal{Q}, \mathcal{Y}_2)$, we obtain

$$\widetilde{H}'' = \begin{pmatrix} 0 & 0 & -\omega & 0 \\ 0 & -\mathcal{H}_{12} & 0 & -\mathcal{H}_{22} \\ \omega & 0 & 0 & 0 \\ 0 & \mathcal{H}_{11} & 0 & \mathcal{H}_{12} \end{pmatrix} + O(e^{-C/\varepsilon}), \qquad \omega = \frac{\partial H}{\partial \mathcal{I}_1}\Big|_{\gamma^1}, \qquad \mathcal{H}_{ij} = \frac{\partial^2 \mathcal{H}}{\partial \mathcal{Y}_i \partial \mathcal{Y}_j}\Big|_{\gamma^1}.$$

Because the matrix $\partial(p,x)/\partial(\mathcal{P},\mathcal{Y}_1,\mathcal{Q},\mathcal{Y}_2)$ is periodic, the monodromy operators corresponding to systems (31) and (32) have equal eigenvalues. The monodromy matrix of system (32) has the eigenvalues $e^{\pm i\omega T}$ and the twice degenerate eigenvalue 1 (with an accuracy of $O(e^{-C/\varepsilon})$). Returning to the original coordinates (p, x), we see that the eigenvalue corresponding to the vector a_2 is $e^{i\omega T}$.

Appendix B: Formulas for spectral series with an accuracy of $O(\varepsilon^2)$

B.1. Almost invariant manifolds. In this appendix, we construct spectral series for the lower Landau bands with an accuracy of $O(\varepsilon^2)$. For this, we must first reduce the Hamiltonian H to form (7) with change of variables (1). Obviously, it suffices to construct this change only for $\mathcal{I}_1 = 0$. By simple calculations, we obtain $\mathcal{H}(0, \mathcal{Y}, \varepsilon) = \varepsilon v(\mathcal{Y})$ from (7) and (8), and we can find the desired change of variables

$$P = -\varepsilon\frac{\partial v}{\partial x_2}(\mathcal{Y}^0), \qquad Q = -\varepsilon\frac{\partial v}{\partial x_1}(\mathcal{Y}^0), \qquad y = \mathcal{Y},$$

from (6) and (9). The construction of almost invariant curves and rest points is now obvious. Let $\mathcal{Y}^0$ denote a solution of the Hamiltonian system for v:

$$\frac{d\mathcal{Y}_1^0}{d\tau} = -\varepsilon\frac{\partial v}{\partial x_2}(\mathcal{Y}^0), \qquad \frac{d\mathcal{Y}_2^0}{d\tau} = \varepsilon\frac{\partial v}{\partial x_1}(\mathcal{Y}^0).$$

We can verify that the curves and the points given by

$$p_1 = -\mathcal{Y}_2^0, \qquad p_2 = \varepsilon\frac{\partial v}{\partial x_1}(\mathcal{Y}^0), \qquad x_1 = \mathcal{Y}_1^0 - \varepsilon\frac{\partial v}{\partial x_1}(\mathcal{Y}^0), \qquad x_2 = \mathcal{Y}_2^0 - \varepsilon\frac{\partial v}{\partial x_2}(\mathcal{Y}^0)$$

are almost invariant manifolds of the Hamiltonian H with an accuracy of $O(\varepsilon^2)$. Moreover, the points induced by the rest points of v are actually exact invariant points of H.

B.2. Spectral series corresponding to points. In accordance with the oscillatory approximation method, only the local extremums of v can be used to construct spectral series. Let $\mathcal{Y}^0$ be a nondegenerate local extremum of the function v. The corresponding (zero-dimensional) invariant manifolds are given by the formulas

$$\gamma_l^0 = (p = P_l, x = X_l) = \big(-\mathcal{Y}_2^0 - (l\cdot a)_2, 0, \mathcal{Y}_1^0 + (l\cdot a)_1, \mathcal{Y}_2^0 + (l\cdot a)_2\big).$$

The formulas for the asymptotic eigenvalues are

$$E_{\mu,\nu}^{0,\pm} = \varepsilon v + h\left[\left(1+\frac{1}{2}\varepsilon\Delta v\right)\left(\mu+\frac{1}{2}\right) \pm \left(\varepsilon\sqrt{\det(v_{ij})}\right)\left(\nu+\frac{1}{2}\right)\right] + O(h\varepsilon^2), \tag{33}$$

where

$$(v_{ij}) = \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)_{i,j=1,2}, \qquad \mu,\nu = O(1),$$

the sign $+/-$ corresponds to the minimum/maximum point of v, and all the functions are calculated at the point $\mathcal{Y}^0$.

The basis of a complex germ is given by the vectors a_1 and $a_{2\pm}$,

$$a_1 = \begin{pmatrix} w_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ i \\ -i \\ 1 \end{pmatrix} + O(\varepsilon), \qquad a_{2\pm} = \begin{pmatrix} w_{2\pm} \\ z_{2\pm} \end{pmatrix} = \begin{pmatrix} -v_{12} \pm i\sqrt{\det(v_{ij})} \\ 0 \\ v_{22} \\ v_{12} \mp i\sqrt{\det(v_{ij})} \end{pmatrix} + O(\varepsilon),$$

where the sign $+/-$ corresponds to the minimum/maximum point of v. We also introduce the matrices

$$B^\pm = (w_1, w_{2\pm}) = \begin{pmatrix} 0 & -v_{12} \pm i\sqrt{\det(v_{ij})} \\ i & 0 \end{pmatrix} + O(\varepsilon),$$

$$C^\pm = (z_1, z_{2\pm}) = \begin{pmatrix} -i & v_{22} \\ 1 & v_{12} \mp i\sqrt{\det(v_{ij})} \end{pmatrix} + O(\varepsilon),$$

$$Q^\pm = B^\pm (C^\pm)^{-1} = \frac{1}{-v_{22} \mp \sqrt{\det(v_{ij})} - iv_{12}} \begin{pmatrix} v_{12} \mp \sqrt{\det(v_{ij})} - iv_{12} & \pm\sqrt{\det(v_{ij})} + iv_{12} \\ \pm\sqrt{\det(v_{ij})} + iv_{12} & -iv_{22} \end{pmatrix} + O(\varepsilon)$$

and the creation operators

$$\hat{a}_{1,l} = \frac{1}{\sqrt{h}}\left[\langle \overline{w}_1 \mid x - X_l\rangle - \left\langle \bar{z}_1 \,\middle|\, -ih\frac{\partial}{\partial x} - P_l\right\rangle\right],$$

$$\hat{a}_{2,l\pm} = \frac{1}{\sqrt{h}}\left[\langle \overline{w}_{2\pm} \mid x - X_l\rangle - \left\langle \bar{z}_{2\pm} \,\middle|\, -ih\frac{\partial}{\partial x} - P_l\right\rangle\right].$$

We also set

$$S_l^\pm(x) = \langle P_l | x - X_l\rangle + \frac{1}{2}\langle x - X_l | Q^\pm (x - X_l)\rangle.$$

The asymptotic eigenfunction corresponding to the asymptotic eigenvalue $E_{\mu,\nu}^{0,\pm}$ in some neighborhood of X_l can now be given by the formula

$$\psi_l^{0,\mu,\nu,\pm}(x) = \frac{1}{\sqrt{h}}(\hat{a}_{1,l})^\mu (\hat{a}_{2,l\pm})^\nu e^{iS_l^\pm(x)/h}.$$

B.3. Spectral series corresponding to curves. Let a curve $\gamma^1 = \big(P(\tau,\varepsilon), X(\varepsilon)\big)$ be given by the equations

$$P_1(\tau,\varepsilon) = -\mathcal{Y}_2^0(\tau,\varepsilon), \qquad P_2(\tau,\varepsilon) = \varepsilon\frac{\partial v}{\partial x_1}(\mathcal{Y}^0(\tau,\varepsilon)),$$

$$X_1(\tau,\varepsilon) = \mathcal{Y}_1^0(\tau,\varepsilon) - \varepsilon\frac{\partial v}{\partial x_1}(\mathcal{Y}^0(\tau,\varepsilon)), \qquad X_2(\tau,\varepsilon) = \mathcal{Y}_2^0(\tau,\varepsilon) - \varepsilon\frac{\partial v}{\partial x_2}(\mathcal{Y}^0(\tau,\varepsilon)),$$

where $\mathcal{Y}^0$ is either a closed trajectory of (10) satisfying quantization condition (17) or any nonclosed trajectory of (10). It follows from the preceding that we can easily obtain the formula

$$E = H|_{\gamma^1} + \omega\left(\mu + \frac{1}{2}\right)h + O(h^2), \qquad \omega = 1 + \frac{1}{2}\varepsilon\Delta v|_{\gamma^1}$$

for asymptotic eigenvalues.

We now describe the scheme for constructing the corresponding asymptotic eigenfunctions. The basis of the corresponding complex germ consists of the vectors $a_1(\tau)$ and $a_2(\tau)$:

$$a_1(\tau) = \begin{pmatrix} w_1(\tau) \\ z_1(\tau) \end{pmatrix} = \frac{1}{\varepsilon}\begin{pmatrix} -\dot{\mathcal{Y}}_2^0(\tau) \\ 0 \\ \dot{\mathcal{Y}}_1^0(\tau) \\ \dot{\mathcal{Y}}_2^0(\tau) \end{pmatrix} + O(\varepsilon), \qquad a_2(\tau) = \begin{pmatrix} w_2(\tau) \\ z_2(\tau) \end{pmatrix} = e^{i\omega\tau}\begin{pmatrix} 0 \\ i \\ -i \\ 1 \end{pmatrix} + O(\varepsilon).$$

We introduce the matrices

$$B(\tau,\varepsilon) = \big(w_1(\tau,\varepsilon), w_2(\tau,\varepsilon)\big) = \frac{1}{\varepsilon}\begin{pmatrix} -\dot{\mathcal{Y}}_2^0(\tau,\varepsilon) & 0 \\ 0 & i\varepsilon e^{i\omega\tau} \end{pmatrix} + O(\varepsilon),$$

$$C(\tau,\varepsilon) = \big(z_1(\tau,\varepsilon), z_2(\tau,\varepsilon)\big) = \frac{1}{\varepsilon}\begin{pmatrix} \dot{\mathcal{Y}}_1^0(\tau,\varepsilon) & -i\varepsilon e^{i\omega\tau} \\ \dot{\mathcal{Y}}_2^0(\tau,\varepsilon) & \varepsilon e^{i\omega\tau} \end{pmatrix} + O(\varepsilon),$$

$$Q(\tau,\varepsilon) = B(\tau,\varepsilon)C^{-1}(\tau,\varepsilon) = \frac{\dot{\mathcal{Y}}_2^0(\tau,\varepsilon)}{\dot{\mathcal{Y}}_1^0(\tau,\varepsilon) + i\dot{\mathcal{Y}}_2^0(\tau,\varepsilon)}\begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} + O(\varepsilon).$$

In a certain tubular neighborhood $\Omega(\gamma^1)$ of the projection of the curve γ^1 on the x plane, the equation $\langle \dot{X}(\tau) | x - X(\tau)\rangle = 0$ is uniquely solvable (up to T for closed curves) with respect to τ. We let $\tau(x,\varepsilon)$ denote this solution and introduce the creation operators

$$\hat{a}(x,\varepsilon,h) = \frac{1}{\sqrt{h}}\left\{\langle \overline{w}_2 | x - X(\tau,\varepsilon)\rangle - \left\langle \overline{z}_2 \,\middle|\, -ih\frac{\partial}{\partial x} - P(\tau,\varepsilon)\right\rangle\right\}\bigg|_{\tau=\tau(x,\varepsilon)}$$

and the phase

$$S(x,\varepsilon) = \left\{\int_0^\tau P(\tau,\varepsilon)\dot{X}(\tau,\varepsilon)\,d\tau + \langle P(\tau,\varepsilon) | x - X(\tau,\varepsilon)\rangle + \right.$$
$$\left. + \frac{1}{2}\langle x - X(\tau,\varepsilon) | Q(\tau,\varepsilon)(x - X(\tau,\varepsilon))\rangle\right\}\bigg|_{\tau=\tau(x,\varepsilon)}.$$

In a neighborhood of $\Omega(\gamma^1)$, the asymptotic eigenfunction corresponding to the asymptotic eigenvalue E can now be written as

$$\psi(x,\varepsilon,h) = \big(\hat{a}(x,\varepsilon,h)\big)^{\mu} e^{iS(x,\varepsilon)/h}.$$

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