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BRIEF COMMUNICATIONS

Integral Representation of Analytical Solutions of the Equation $yf'_x - xf'_y = g(x, y)$

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Equations of the form

$$\frac{\partial f}{\partial \varphi} \equiv y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} = g(x, y) \tag{1}$$

often arise in different averaging problems for systems of differential equations [1–4]. Suppose that $g(x, y)$ is an analytic function of two variables on the plane $\mathbb{R}^2$, (ρ, φ) are polar coordinates: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, and g has zero mean over the angle variable φ.

The integration of Eq. (1) is a trivial problem, and the solution can be written out up to an arbitrary function $\mathcal{F}(\rho)$. Averaging procedures involve work with analytic functions of the two variables x, y, and the solution must be estimated by the C-norm of the function g. Such a problem does not arise if the averaging domain is disjoint with a neighborhood of the origin $x = 0$, $y = 0$. However, in physical applications, it is precisely this neighborhood that corresponds to solutions with small amplitudes and it may play the central role[1]. A straightforward application of the Newton–Leibniz formula gives a solution which is analytic in the variables φ and ρ, but it is not analytic in x and y in a neighborhood of the origin. On the other hand, the theory justifying estimates of a solution represented by the Fourier series in the C-norm is rather laborious (1).

In this short communication, we propose a simple integral formula representing the solution of equation (1), which is analytic in the variables x and y and implies simple estimates in the C-norm. In fact, this formula was used earlier in [4], but since it has a general mathematical value and has many potential applications, we believe its separate publication is advisable.

Theorem. *Equation* (1) *has a solution $f(x, y)$ which is analytic in all variables if and only if the mean value of the function g of the variable φ is trivial*:

$$\int_0^{2\pi} g(\rho\cos\varphi, \rho\sin\varphi)\, d\varphi = 0,$$

and each solution has the form

$$f(x, y) = \frac{1}{2}\biggl(\int_0^{\varphi} g(\rho\cos\psi, \rho\sin\psi)\, d\psi + \int_{\pi}^{\varphi} g(\rho\cos\psi, \rho\sin\psi)\, d\psi\biggr) + F(\rho^2), \tag{2}$$

where $F(\mathbb{Z})$ is an arbitrary analytic function.

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[1] An example of such a setup is given by the problem of averaging the motion of a charged particle in the sum of a constant magnetic field and a periodic electric potential $v(x, y)$ (see [4]). The Hamiltonian of this system has the form $H = (p^2 + q^2)/2 + \varepsilon v(p + x, q + y) \equiv I + \varepsilon v(\sqrt{I}\cos\varphi + x, \sqrt{I}\sin\varphi + y)$, $0 < \varepsilon \ll 1$. This function is analytic in the variables (q, p) and is not analytic in the angle-action variables φ, $I = \rho^2/2$ in a neighborhood of the origin $x = 0$, $y = 0$.

Proof. First, let us make the following change of variables:

$$x = u + iv, \qquad y = v + iu.$$

Consider the Taylor expansion of the function g in the variables (u, v):

$$g(u+iv, v+iu) = \sum_{k,l\in\mathbb{N}} g_{kl} u^k v^l \frac{(k+l)!}{k!l!}.$$

The polar coordinates for the variables (u, v) have the form

$$u = \frac{\rho e^{-i\varphi}}{\sqrt{2}}, \qquad v = \frac{\rho e^{i(\varphi-\pi/2)}}{\sqrt{2}}.$$

Then the function g can be represented as follows:

$$g(\rho\cos\varphi, \rho\sin\varphi) = \sum_{k,l\in\mathbb{N}} g_{kl} \frac{(k+l)!}{k!l!} \left(\frac{\rho}{\sqrt{2}}\right)^{k+l} e^{i(l-k)\varphi - il\pi/2}.$$

This expansion may be regarded as the Fourier series for the function $g(\rho\cos\varphi, \rho\sin\varphi)$. Now Eq. (1) can readily be solved:

$$\begin{aligned} f &= \int_0^{\varphi} g(\rho\cos\psi, \rho\sin\psi)\, d\psi + F_0(\rho) = \sum_{k,l\in\mathbb{N}} g_{kl} \frac{(k+l)!}{k!l!} \left(\frac{\rho}{\sqrt{2}}\right)^{k+l} \frac{e^{i(l-k)\psi - il\pi/2}}{i(l-k)}\bigg|_0^{\varphi} + F_0(\rho) \\ &= \sum_{k,l\in\mathbb{N}} g_{kl} \frac{(k+l)!}{k!l!} \left(\frac{\rho}{\sqrt{2}}\right)^{k+l} \frac{e^{i(l-k)\varphi - il\pi/2}}{i(l-k)} - \sum_{k,l\in\mathbb{N}} g_{kl} \frac{(k+l)!}{k!l!} \left(\frac{\rho}{\sqrt{2}}\right)^{k+l} \frac{e^{-il\pi/2}}{i(l-k)} + F_0(\rho), \end{aligned} \tag{3}$$

where $F_0(\rho)$ is a "constant" of integration. The first term in Eq. (3) is an analytic function, because it can be rewritten as a polynomial in the variables $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. The second term is not an analytic function at the origin $x = 0$, $y = 0$ for $(k+l)$ odd; therefore, we must cancel the nonanalytic summand by a proper choice of $F_0(\rho)$. Let us define F_0 as follows:

$$F_0(\rho) = \left(\frac{1}{2}\int_{\pi}^{\varphi} g(\rho\cos\psi, \rho\sin\psi)\, d\psi - \frac{1}{2}\int_0^{\varphi} g(\rho\cos\psi, \rho\sin\psi)\, d\psi\right) + F(\rho^2). \tag{4}$$

Now Eq. (4) cancels the nonanalytic term in Eq. (3), and f is an analytic function of both variables x and y at the origin $x = 0$, $y = 0$. This proves the theorem. $\square$

The corresponding estimates can easily be derived from this formula. Indeed, in the $C(\Omega)$-norm, where Ω is a domain in $\mathbb{R}^2$, we have

$$\begin{aligned} \|f\| &\le \frac{1}{2}\left\| \int_0^{\varphi} g(\rho\cos\psi, \rho\sin\psi)\, d\psi + \int_{\pi}^{\varphi} g(\rho\cos\psi, \rho\sin\psi)\, d\psi \right\|_{C(\Omega)} + \|F(\rho^2)\|_{C(\Omega)} \\ &\le \frac{1}{2}\left(\int_0^{2\pi} \max_{C(\Omega)} |g|\, d\psi + \int_{\pi}^{2\pi} \max_{C(\Omega)} |g|\, d\psi\right) + \max_{C(\Omega)} |F(\rho^2)| = \frac{3}{2}\pi \max_{C(\Omega)} |g| + \max_{C(\Omega)} |F(\rho^2)|. \end{aligned}$$

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