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Hall Conductivity of Minibands Lying at the Wings of Landau Levels

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A semiclassical method is suggested for the description of the energy spectrum of a two-dimensional magnetic Bloch electron in a periodic potential not necessarily smaller than the cyclotron energy. With this method, each Landau band is described as a spectrum of the appropriate one-dimensional Harper-type operator and represents a series of minibands, with the near-edge minibands being flat within the exponential accuracy. It is shown that, irrespective of the potential shape, all these minibands do not contribute to the quantized Hall conductivity.

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In the standard theories of integer quantum Hall effect, each filled Landau level makes a contribution of quantum e^2/h to the Hall conductivity [1], so that, as the Fermi level rises, the Hall conductivity monotonically increases with the e^2/h jumps that were discovered experimentally by K. von Klitzing. In a weak periodic potential V, each Landau level spreads into a band with a width no greater than $2\max|V|$; each band, in turn, is split into magnetic subbands. If the number of magnetic-flux quanta $\Phi_0 = hc/|e|$ of the magnetic flux Φ through the unit cell of the periodic potential V is rational and can be represented as a noncancelable fraction $\Phi/\Phi_0 = N/M$, each of the Landau bands splits into N subbands [2]. As a result, the "flux–energy" diagram for the spectrum of Landau periodic operator assumes a complex fractal structure that was predicted by Azbel and constructed numerically in the approximation of Harper equation [3] (Hofstadter's butterfly). From the well-known gauge arguments of R. Laughlin, it follows that each subband has an integer number of conductivity quanta, which changes in a rather irregular way upon the transition from one subband to the other and obeys a certain Diophantine equation [1] (this number exactly equals the Chern number for the corresponding vector bundle of the Bloch magnetic functions [4]). Therefore, in the presence of a periodic potential in the fields where Φ/Φ_0 is on the order of unity, the dependence of Hall conductivity on the Fermi energy becomes nonmonotonic and, generally, exhibits irregular jumps, again with the magnitudes being multiples of e^2/h. These jumps have recently been observed by K. von Klitzing *et al.* in [5] (in full agreement with the predictions of theory [1]) in measuring the magnetoresistance of a two-dimensional electron gas in a square superlattice with a ~100-nm unit cell. It should be noted that the idea of that experiment was suggested as early as in [3].

Inasmach as V in [5] was ≃0.6 meV, the superposition or partial overlap of the Landau bands did not need to be taken into account in that work. However, this effect cannot be neglected for the larger potential V. As was shown numerically in [6], if V becomes comparable to the cyclotron energy $\hbar\omega_c$, the Landau bands overlap and are even rearranged upon further increase in V. It is significant that, after the crossover, the Chern number for several lower-lying bands is zero; i.e., these bands do not contribute to the Hall conductivity [6]. A more detailed numerical analysis of the influence of the overlap between the Landau bands on the Hall resistivity was carried out in [7]. It should be taken into account that the flux–energy diagram for a periodic Landau operator is different from the ideal self-similar Hofstadter butterfly [8]. The subband Hall conductivity is also affected by the form of the potential curve, in particular, by the presence or absence of the center of inversion [9].

In this work, we propose a semiclassical approach to the Landau bands that is independent of the potential shape and the band overlap. Only two parameters are assumed to be small: $\varepsilon_B = (l_M/L)^2$, where l_M is the magnetic length and L is the characteristic size of the lattice period of potential V, and the value of the parameter $\varepsilon_V = \varepsilon_B\max|V|/\hbar\omega_c$. The ratio $\max|V|/\hbar\omega_c$ should not necessarily be small, so that our approach also applies to the regime of Landau band rearrangement [6]. In the typical situations, the estimates for the parameters ε_B and ε_V are as follows: if $B \simeq 10$ T then $l_M \simeq 10$ nm; for the periodical quantum-dot or quantum-antidot arrays, $L \simeq 100$–500 nm. Hence, $\varepsilon_B \sim 10^{-3}$; for the electron effective mass in GaAs $m = 0.067m_e$, one gets $\hbar\omega_c \simeq$ 15 meV. For this reason, one has $\varepsilon_V \lesssim \varepsilon_B$ for $V \lesssim 15$ meV. Within this approach, we demonstrate that, irrespective of the form of the potential curve, all minibands lying at the wings of Landau levels (and not only the lowest lying subbands) make no contribution to the Hall conductivity. Thus, when describing the influence of the overlap of Landau bands on the quantization of Hall conductivity, one should take into account only the overlap between rather narrow central regions of the smeared Landau levels.

The Hamiltonian of a Bloch magnetic electron in the Landau gauge has the form

$$\hat{H} = \frac{\hbar^2}{2m}((-i\partial_1 + (eB/c\hbar)x_2)^2 - \partial_2^2) + V(x_1, x_2),$$

where the potential V has a lattice with periods and the basis $\mathbf{a}_1 = (L, 0)$ and $\mathbf{a}_2$. In the dimensionless coordinates $\mathbf{X} = \mathbf{x}/L$ and potential $v = V/\max|V|$, $\hat{H}$ is written as $\hat{H} = mL^2\omega_c^2\hat{H}^0$, where

$$\hat{H}^0 = \frac{1}{2}[(\hat{P}_1 + X_{2s})^2 + \hat{P}_2^2] + \varepsilon_V v(X_1, X_2).$$

Here, $\hat{P}_j = -i\varepsilon_B\partial/\partial X_j$.

The perturbation theory with respect to the small parameter ε_V can provide only crude information; moreover, it requires additional assumptions about the relationship between ε_B and ε_V, because the parameter ε_B is small. Nevertheless, the smallness of ε_B allows the fine structure of Landau levels to be described semiclassically.

Since the classical trajectories in the (X_1, X_2) plane for the unperturbed Hamiltonian $\frac{1}{2}((P_1 + X_2)^2 + P_2^2)$ are cyclotron orbits with radii $\sqrt{2I}$ centered at (y_1, y_2), one can pass on to the new canonical variables, namely, to the generalized momenta I, y_1 (or p, y_1) and generalized positions φ, y_2 (or q, y_2), according to the formulas

$$X_1 = q + y_1, \quad P_1 = -y_2, \quad X_2 = p + y_2,$$
$$P_2 = -q, \quad p = \sqrt{2I}\cos\varphi, \quad q = \sqrt{2I}\sin\varphi$$

(φ is the orbital angular coordinate). In these variables, the corresponding classical Hamiltonian H^0 is

$$H^0 = I + \varepsilon_V v(\sqrt{2I}\sin\varphi + y_1, \sqrt{2I}\cos\varphi + y_2).$$

After angular averaging of H_0, the drift of the center of the cyclotron orbit is described by the averaged Hamiltonian

$$\mathcal{H}^{\mathrm{av}}(I, y_1, y_2; \varepsilon_V) = \frac{1}{2\pi}\int_0^{2\pi} H^0 d\varphi$$
$$= I + \varepsilon_V J_0(\sqrt{-2I\Delta})v(y_1, y_2),$$
$$\Delta = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$$

(J_0 is the zero-order Bessel function). One can also show that there is a canonical variable change $(p', q', \mathbf{y}') = (p, q, \mathbf{y}) + O(\varepsilon_V)$ such that

$$H^0(p, q, \mathbf{y}; \varepsilon_V) = \mathcal{H}^{\mathrm{av}}\left(\frac{1}{2}(p'^2 + q'^2), \mathbf{y}'; \varepsilon_V\right) + O(\varepsilon_V^2),$$

but the estimate of the residue $O(\varepsilon_V^2)$ in this formula does not suffice to describe the H^0 fine structure. However, it was shown in [10], that this procedure can be iterated up to the canonical change of the variables $(P, Q, \mathbf{Y}) = (p, q, \mathbf{y}) + O(\varepsilon_V)$, to bring the H^0 Hamiltonian to the form $H^0(p, q, \mathbf{y}; \varepsilon_V) = \mathcal{H}^0\left(\frac{1}{2}(P^2 + Q^2), Y_1, Y_2; \varepsilon_V\right) + O(e^{-C/\varepsilon_V})$, with the right-hand side periodic in the variable Y_j and with a positive constant C.

The quantization of $\mathcal{H}^0$ brings about the $\hat{\mathcal{H}}^0$ operator, whose semiclassical spectrum coincides with that of $\hat{H}^0$ and $\hat{\mathcal{H}}^0$ to an accuracy of $O((\varepsilon_B + \varepsilon_V)^\nu)$ for an arbitrary ν. Since the operators $\hat{P} = -i\varepsilon_B\partial/\partial Q$ and $\hat{Q} = Q$ commute with $\hat{Y}_1 = -i\varepsilon_B\partial/\partial Y_2$ and $\hat{Y}_2 = Y_2$, $\hat{\mathcal{H}}^0$ commutes with the Hamiltonian of harmonic oscillator $\frac{1}{2}(\hat{P}^2 + \hat{Q}^2)$. Therefore, the eigenfunctions Ψ of the operator $\hat{\mathcal{H}}^0$ can be sought in the form $\Psi(Q, Y_2) = \psi_n(Q)\varphi_n(Y_2)$, where ψ_n are the oscillator functions for the $E_n = (n + 1/2)\varepsilon_B$ level ($n = 0, 1, 2, \ldots$) and φ_n satisfy the equation

$$\hat{\mathcal{H}}_n\varphi_n = E\varphi_n. \qquad (1)$$

Here, $\hat{\mathcal{H}}_n$ are found from the classical Hamiltonian $\mathcal{H}_n(Y_1, Y_2) = \mathcal{H}^0(E_n, Y_1, Y_2, \varepsilon_V)$ by the quantization $\hat{Y}_1 = -i\varepsilon_B\partial/\partial Y_2$ (these operators are conventionally called the Harper-type operators). Since $mL^2\omega_c^2\varepsilon_B = \hbar\omega_c$, E_n is exactly the nth Landau level. Hence, the spectrum of operator $\hat{\mathcal{H}}_n$ describes the spreading of the nth Landau level into band under the action of the periodic potential V.

Therefore, each Landau band in our approach is described by Eq. (1), and the initial spectral problem reduces to the family of one-dimensional spectral problems, allowing the problem to be integrated.

Let us now use the analysis of Harper operators [11]. At the edges of the spectrum of operator $\hat{\mathcal{H}}_n$, there are minibands with widths exponentially small in the parameter ε_B. The corresponding Bloch magnetic eigenfunctions of the operator $\hat{H}^0$ for the rational flux Φ/Φ_0 are constructed in [10]. Namely, by denoting $\Phi/\Phi_0 = N/M$ and enlarging the lattice Λ (i.e., going to the lattice with basis $M\mathbf{a}_1$ and $\mathbf{a}_2$ [4]), one has the following semiclassical eigenfunctions in the $\mathbf{X}$ coordinates satisfying the magneto-Bloch periodic conditions with the quasimomentum $\mathbf{k}$:

$$\Psi(\mathbf{X}, \mathbf{k}) = \sum_{l_1, l_2 \in \mathbb{Z}} e^{2\pi i(k_1 l_1 - k_2 l_2) - iNl_2 L_{21}/2} \times \psi(\mathbf{X} - Ml_1\mathbf{L}_1 - l_2\mathbf{L}_2) e^{-iNl_2 X_1}, \tag{2}$$

where $\mathbf{L}_1 = (1, 0)$ and $\mathbf{L}_2 = (L_{21}, L_{22})$ are the periods of the normalized potential v and ψ is a certain localized function quasimode of the $\hat{H}$ operator [12]. From Eq. (2) it directly follows that the vector bundle of the Bloch magnetic functions for the exponentially narrow miniband is trivial and, therefore, has the zero Chern class. According to the standard theory of the Hall quantum effect [1], this means that the Hall conductivity for this miniband is zero. Thus, after the Fermi level crosses the minibands, the quantized Hall conductivity at the Landau level wings does not change. As to the minibands in the middle of Landau bands, the corresponding contribution to the Hall conductivity requires additional calculations, which can conveniently be performed using the Usov formulas [13]. The calculations of this type depend on the particular form of the potential V and have been carried out, e.g., in [9]. The simplest examples indicate that the dependence on the Fermi level is nonmonotonic, in accordance with [1].

In summary, a semiclassical approach is proposed to reduce the description of the Bloch magnetic electron spectrum to a series of one-dimensional problems. With this approach, each Landau band (smeared Landau level) coincides with the spectrum of some one-dimensional Harper-type operator obtained by the quantization of a classical Hamiltonian on the torus for a given level. At the wings of Landau bands, there are exponentially narrow minibands, and the vector bundle of the corresponding semiclassical Block magnetic functions has the zero Chern class. Therefore, these minibands do not contribute to the quantized Hall conductivity, and they can be neglected when considering the influence of overlap of Landau bands on the Hall quantization pattern. It is significant that the method described in this work applies when the lattice potential V is comparable to the cyclotron energy; if $|V| \ll \hbar\omega_c$, our results agree with [14]. Interestingly, the structure of layering of the Bloch magnetic functions for the exponentially narrow (i.e., flat, to the exponentially small field corrections) minibands is the same as for the layering of the fermion eigenfunctions for a lattice in the presence of a magnetic field [15].

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