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The Lyapunov function for Schrödinger operators with a periodic 2×2 matrix potential

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Abstract

We consider the Schrödinger operator on the real line with a 2×2 matrix-valued 1-periodic potential. The spectrum of this operator is absolutely continuous and consists of intervals separated by gaps. We define a Lyapunov function which is analytic on a two-sheeted Riemann surface. On each sheet, the Lyapunov function has the same properties as in the scalar case, but it has branch points, which we call resonances. We prove the existence of real as well as non-real resonances for specific potentials. We determine the asymptotics of the periodic and the anti-periodic spectrum and of the resonances at high energy. We show that there exist two type of gaps: (1) stable gaps, where the endpoints are the periodic and the anti-periodic eigenvalues, (2) unstable (resonance) gaps, where the endpoints are resonances (i.e., real branch points of the Lyapunov function). We also show that periodic and anti-periodic spectrum together determine the spectrum of the matrix Hill operator.

Keywords: Schrödinger operator; Periodic matrix potentials; Spectral bands; Spectral gaps

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1. Introduction and main results

We consider the self-adjoint operator $Ty = -y'' + V(x)y$, acting in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, where V is a symmetric 1-periodic 2×2 matrix potential which belongs to the real space $\mathcal{H}^p$, $p = 1, 2$, given by

$$\mathcal{H}^p = \left\{ V = V^* = V^T = \begin{pmatrix} V_1 & V_3 \\ V_3 & V_2 \end{pmatrix} : \int_0^1 V_3(x)\,dx = 0 \right\},$$

equipped with the norm $\|V\|_p^p = \int_0^1 (|V_1(x)|^p + |V_2(x)|^p + 2|V_3(x)|^p)\,dx < \infty$. Without loss of generality *we assume*

$$V_{(0)} = \int_0^1 V(t)\,dt, \qquad V_{10} \leqslant V_{20}, \qquad V_{30} = 0, \qquad V_{m0} = \int_0^1 V_m(x)\,dx, \quad m = 1, 2, 3.$$

Let us introduce the self-adjoint operator $T^0 = -d^2/dx^2$, with the domain $\mathrm{Dom}(T^0) = W_2^2(\mathbb{R}) \oplus W_2^2(\mathbb{R})$. In order to get self-adjointness of T we use symmetric quadratic forms. We briefly recall a well-known argument (see [19]). We define the form $(V\psi, \psi_1) = -\int_{\mathbb{R}} V\psi\overline{\psi}_1\,dx$, $\psi, \psi_1 \in \mathrm{Dom}(T^0)$. Using the estimate (see [13])

$$\big|(q'f, f)\big| < \varepsilon(f', f') + b_\varepsilon(f, f) \quad \text{for any small } \varepsilon > 0 \text{ and some } b_\varepsilon > 0$$
$$\text{and for any } f \in W_2^2(\mathbb{R}),\ q \in L^2(\mathbb{R}/\mathbb{Z}),$$

we deduce that

$$\big|(V\psi, \psi)\big| < (1/2)(\psi', \psi') + b(\psi, \psi), \quad \psi \in W_2^2(\mathbb{R}) \oplus W_2^2(\mathbb{R}).$$

Thus we can apply the KLMN theorem (see [19]) to define $T = -d^2/dx^2 + V$. There exists a unique self-adjoint operator T with form domain $\mathcal{Q}(T) = W_1^2(\mathbb{R}) \oplus W_1^2(\mathbb{R})$ and

$$(T\psi, \psi_1) = (-\psi'', \psi_1) + (V\psi, \psi_1) \quad \text{all } \psi, \psi_1 \in \mathcal{Q}\big(T^0\big) = W_1^2(\mathbb{R}) \oplus W_1^2(\mathbb{R}).$$

Any domain of essential self-adjointness for T^0 is a form core for T.

It is well known (see [6, pp. 1486–1494], [8]) that the spectrum $\sigma(T)$ of T is absolutely continuous and consists of non-degenerate intervals S_n, $n = 1, 2, \ldots$. These intervals are separated by gaps G_n with lengths $|G_n| > 0$, $n = 1, 2, \ldots, N_G \leqslant \infty$. Introduce the fundamental 2×2 matrix solutions $\varphi(x, \lambda)$, $\vartheta(x, \lambda)$ of the equation

$$-y'' + V(x)y = \lambda y, \quad \lambda \in \mathbb{C}, \tag{1.1}$$

with initial conditions $\varphi(0,\lambda) = \vartheta'(0,\lambda) = 0$, $\varphi'(0,\lambda) = \vartheta(0,\lambda) = I_2$, where I_m, $m \geqslant 1$, is the identity $m \times m$ matrix. Here and below we use the notation $(') = \partial/\partial x$. We define the 4×4 monodromy matrix by

$$M(\lambda) = \mathcal{M}(1,\lambda), \qquad \mathcal{M}(x,\lambda) = \begin{pmatrix} \vartheta(x,\lambda) & \varphi(x,\lambda) \\ \vartheta'(x,\lambda) & \varphi'(x,\lambda) \end{pmatrix}. \tag{1.2}$$

The matrix-valued function M is entire. An eigenvalue of $M(\lambda)$ is called a *multiplier*, it is a root of the algebraic equation $D(\tau,\lambda) = 0$, where

$$D(\tau,\lambda) := \det\big(M(\lambda) - \tau I_4\big), \quad \tau,\lambda \in \mathbb{C}. \tag{1.3}$$

There is an enormous literature on the scalar Hill operator including the inverse spectral theory [7,10,17]. In the recent paper [13] one of the authors solved the inverse problem (including characterization) for the operator $-y'' + v'y$, with a function $v \in L^2(\mathbb{T})$ and v' is its distributional derivative and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. However, in spite of the importance of extending these studies to vector differential equations, apart from the information given by Lyapunov and Krein (see [21]), until recently nothing essential has been done. The matrix potential poses interesting new problems: (1) to construct the Lyapunov function, (2) to define the quasimomentum as a conformal mapping, (3) to derive appropriate trace formulae (e.g. analogous to the trace formulas in [12]), (4) to obtain a priori estimates of potentials in terms of gap lengths, (5) to define and to study the integrated density of states. In fact this is the motivation of our paper.

The basic results in the direct spectral theory for the matrix case were obtained by Lyapunov [16] and Krein [15] (see also Gel'fand and Lidskii [9]). Below we need the following well-known results of Lyapunov [21], which we formulate only for the case of 2×2 matrices.

Theorem *(Lyapunov). Let $V \in \mathcal{H}^1$.*

(i) *If $\tau(\lambda)$ is a multiplier for some $\lambda \in \mathbb{C}$ (or $\lambda \in \mathbb{R}$), then $\tau^{-1}(\lambda)$ (or $\bar{\tau}(\lambda)$) is a multiplier too.*
(ii) *$M(\lambda)$ has exactly four multipliers $\tau_1(\lambda), \tau_1^{-1}(\lambda), \tau_2(\lambda), \tau_2^{-1}(\lambda)$ for all $\lambda \in \mathbb{C}$. Moreover, $\lambda \in \sigma(T)$ iff $|\tau_m(\lambda)| = 1$ for some m.*
(iii) *If $\tau(\lambda)$ is a simple multiplier and $|\tau(\lambda)| = 1$, then $\tau'(\lambda) \neq 0$.*

We mention some papers relevant for our context. For variety problems for periodic systems we refer the reader to [5]. In the papers [2,3] Carlson obtained trace formulas. In [4] he proved the compactness of Floquet isospectral sets for the matrix Hill operator.

By the Lyapunov Theorem, each $M(\lambda)$, $\lambda \in \mathbb{C}$, has exactly four multipliers $\tau_m(\lambda), \tau_m^{-1}(\lambda)$, $m = 1,2$, which are the roots of the characteristic polynomial $\det(M(\lambda) - \tau I_4) = 0$. Define the very important constant in our paper by

$$c_0 = \frac{V_{20} - V_{10}}{2}.$$

Then the multipliers have the following asymptotics:

$$\tau_m(\lambda) = e^{i(\sqrt{\lambda} - V_{m0}/(2\sqrt{\lambda}) + O(1/\lambda))}, \quad |\lambda| \to \infty, \quad |\sqrt{\lambda} - \pi n| > \frac{\pi}{4}, \quad m = 1,2, \tag{1.4}$$

as we prove in Lemma 3.2. Next we define the functions

$$\mu_m(\lambda) = \frac{1}{4}\operatorname{Tr} M^m(\lambda), \quad m = 1, 2, \qquad \rho = \frac{\mu_2 + 1}{2} - \mu_1^2, \qquad \rho_0(\lambda) = c_0 \frac{\sin\sqrt{\lambda}}{2\sqrt{\lambda}}. \tag{1.5}$$

Note that $\varphi(x,\lambda)$, $\vartheta(x,\lambda)$, $\mu_1(\lambda)$, $\mu_2(\lambda)$, $\rho(\lambda)$ are real for all $x, \lambda \in \mathbb{R}$ and entire. If $c_0 > 0$, then Lemma 3.1 yields $\rho(\lambda) = \rho_0^2(\lambda)(1 + o(1))$, as $|\lambda| \to \infty$ in $\mathcal{D}_1$, where

$$\mathcal{D}_r := \left\{ \lambda \in \mathbb{C}\colon |\lambda| > r^2,\ |\sqrt{\lambda} - \pi n| > \frac{\pi}{4},\ n \in \mathbb{N} \right\}, \quad r > 0. \tag{1.6}$$

Thus we may define the analytic function $\sqrt{\rho(\lambda)}$ for $\lambda \in \mathcal{D}_r$ and large $r > 0$ by the requirement that $\sqrt{\rho(\lambda)} = \rho_0(\lambda)(1+o(1))$ as $|\lambda| \to \infty$ in $\mathcal{D}_r$. Then there exists a unique analytic continuation of $\sqrt{\rho}$ from $\mathcal{D}_r$ into the two-sheeted Riemann surface Λ (in general, of infinite genus) defined by $\sqrt{\rho}$. We now introduce our Lyapunov function $\Delta(\lambda)$ by

$$\Delta(\lambda) = \mu_1(\lambda) + \sqrt{\rho(\lambda)}, \quad \lambda \in \Lambda. \tag{1.7}$$

Let $\Delta_1(\lambda) = \mu_1(\lambda) + \sqrt{\rho(\lambda)}$ on the first sheet Λ_1 and let $\Delta_2(\lambda) = \mu_1(\lambda) - \sqrt{\rho(\lambda)}$ on the second sheet Λ_2 of Λ. Now we formulate our main result concerning the function Δ.

Theorem 1.1. *Let $V \in \mathcal{H}^1$ with $c_0 > 0$. Then the function $\Delta = \mu_1 + \sqrt{\rho}$ is analytic on the two-sheeted Riemann surface Λ and has the following properties*:

(i)
$$\Delta_m(\lambda) = \frac{\tau_m(\lambda) + \tau_m^{-1}(\lambda)}{2}, \quad \lambda \in \Lambda_m,\ m = 1, 2, \tag{1.8}$$

$$\Delta_m(\lambda) = \cos\sqrt{\lambda} + V_{m0}\frac{\sin\sqrt{\lambda}}{2\sqrt{\lambda}} + O\left(\frac{e^{|\operatorname{Im}\sqrt{\lambda}|}}{\lambda}\right), \quad m = 1, 2,\ \lambda \in \mathcal{D}_1. \tag{1.9}$$

(ii) $\lambda \in \mathbb{C}$ *belongs to* $\sigma(T)$ *iff* $\Delta_m(\lambda) \in [-1, 1]$ *for some* $m = 1, 2$.
(iii) *If* $\lambda \in \sigma(T)$*, then* $\rho(\lambda) \geqslant 0$.
(iv) (*The monotonicity property*). *Let* Δ_m *be real analytic on some interval* $I = (\alpha_1, \alpha_2) \subset \mathbb{R}$ *and* $-1 < \Delta_m(\lambda) < 1$, *for any* $\lambda \in I$ *for some* $m \in \{1, 2\}$. *Then* $\Delta_m'(\lambda) \neq 0$ *for each* $\lambda \in I$.

Remark. (i) For the scalar Hill operator the monodromy matrix has exactly two eigenvalues τ and τ^{-1}. The Lyapunov function $\frac{1}{2}(\tau + \tau^{-1})$ is an entire function of the spectral parameter and it defines the band-gap structure of the spectrum. By Theorem 1.1, our Lyapunov function for the matrix Hill operator also defines the band-gap structure of the spectrum, but it is the sheeted analytic function.

(ii) Consider the case of a diagonal potential, i.e., $V_3 = 0$. Then the Riemann surface degenerates into two components, and we get

$$\mu_1 = \frac{1}{2}(\Delta_{(1)} + \Delta_{(2)}), \qquad \rho = \frac{1}{4}(\Delta_{(1)} - \Delta_{(2)})^2, \qquad \sqrt{\rho} = \frac{1}{2}(\Delta_{(1)} - \Delta_{(2)}), \tag{1.10}$$

where $\Delta_{(m)}$ is the Lyapunov function for the scalar Hill operator $-y'' + V_m y$, $m = 1, 2$. Thus $\Delta_1 = \mu_1 + \sqrt{\rho} = \Delta_{(1)}$ and $\Delta_2 = \mu_1 - \sqrt{\rho} = \Delta_{(2)}$.

(iii) Many papers are devoted to resonances for the Schrödinger operator with compactly supported potentials on the real line, see [1,14,20,22]. Assume that we have the coupling constant before the potential. If this constant changes then roughly speaking some resonances create the eigenvalues. In our case, if the coupling constant (before the periodic matrix potential) changes, then roughly speaking some resonances create the gaps, see Proposition 1.3.

Let $D_\pm(\lambda) = \frac{1}{4}D(\pm 1, \lambda)$. The zeros of $D_+(\lambda)$ and $D_-(\lambda)$ are the eigenvalues of the periodic and anti-periodic problems associated with the equation $-y'' + Vy = \lambda y$. Denote by $\lambda_{2n,k}$, $n = 0, 1, \ldots,$ and $k \in \{1, 2, 3, 4\}$ the sequence of zeros of D_+ (counted with multiplicity) such that $\lambda_{0,1} \leqslant \lambda_{0,2} \leqslant \lambda_{2,1} \leqslant \lambda_{2,2} \leqslant \lambda_{2,3} \leqslant \lambda_{2,4} \leqslant \lambda_{4,1} \leqslant \lambda_{4,2} \leqslant \lambda_{4,3} \leqslant \lambda_{4,4} \leqslant \cdots$. Denote by $\lambda_{2n-1,k}$, $(n, k) \in \mathbb{N} \times \{1, 2, 3, 4\}$ the sequence of zeros of D_- (counted with multiplicity) such that $\lambda_{1,1} \leqslant \lambda_{1,2} \leqslant \lambda_{1,3} \leqslant \lambda_{1,4} \leqslant \lambda_{3,1} \leqslant \lambda_{3,2} \leqslant \lambda_{3,3} \leqslant \lambda_{3,4} \leqslant \cdots$. Note that $\lambda_{n,k}$, $n = 0, 1, \ldots,$ and $k \in \{1, 2, 3, 4\}$ are the eigenvalues of problems with period 2 for the equation $-y'' + Vy = \lambda y$.

Denote by $\{r_n\}_1^\infty$ the sequence of zeros of ρ in $\mathbb{C}$ (counted with multiplicity) such that $0 \leqslant |r_1| \leqslant |r_2| \leqslant |r_3| \leqslant \cdots$. We call these zeros of ρ *the resonances* of T. We formulate the theorem about the recovering the spectrum of T and the asymptotics of the periodic and anti-periodic eigenvalues and resonances at high energy. Furthermore, we write $a_n = b_n + \ell^2(n)$ iff the sequence $\{a_n - b_n\}_{n\geqslant 1} \in \ell^2$. Recall that $V_{(0)} = \int_0^1 V(t)\,dt = \operatorname{diag}\{V_{10}, V_{20}\}$.

Theorem 1.2.

(i) *Let $V \in \mathcal{H}^2$ with $c_0 > 0$. Then the following asymptotics hold*:

$$\lambda_{n,m+k} = (\pi n)^2 + V_{m0} + \ell^2(n), \quad m, k = 1, 2,\ n \to \infty, \tag{1.11}$$

$$r_{2n-m} = (\pi n)^2 + \ell^2(n), \quad m = 0, 1,\ n \to \infty. \tag{1.12}$$

(ii) *Let $V \in \mathcal{H}^2$, $c_0 > 0$. Then the following statements hold*:
  (a) *The periodic spectrum and the anti-periodic spectrum determine the resonances and the spectrum of the operator T.*
  (b) *The periodic* (*anti-periodic*) *spectrum is determined by the anti-periodic* (*periodic*) *spectrum and the resonances.*

Less precise asymptotics for the case $V \in C^2$ were obtained in [3].

Example. Consider the operator $T_{\gamma,\nu} = -d^2/dx^2 + q_{\gamma,\nu}$, where $q_{\gamma,\nu} = aJ + \gamma v_\nu(x) J_1$ is a potential for some constants $a, \gamma \in \mathbb{R}$,

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $q_{\gamma,\nu}$ satisfies

Condition A. *$a/(2\pi^2) - n_a \in (0,1)$ for some integer $n_a \geqslant 0$. Each function $v_\nu \in L^1(\mathbb{T})$, $\nu = 1, 1/2, 1/3, \ldots$, is such that $v_\nu(x) = v_\nu(1-x)$, $x \in (0,1)$, and $\int_0^1 v_\nu(x)\,dx = 1$ and for any $f \in C(0,k)$, $k \in \mathbb{N}$, the following limit relation holds*:

$$\int_0^k v_\nu(x) f(x)\,dx \to \int_0^k \delta_{\mathrm{per}}(x) f(x)\,dx \quad \text{as } \nu \to 0, \qquad \delta_{\mathrm{per}} = \sum_{-\infty}^{\infty} \delta\left(x - n - \frac{1}{2}\right). \tag{1.13}$$

If $\gamma = 0$, then we have the operator $T^0 = -y'' + q^0$ with a constant potential $q^0 = aJ$. In this case there are no gaps in the spectrum. The fundamental solutions of Eq. (1.1) with $q^0 = aJ$ have the form $\vartheta^0 = \operatorname{diag}(c_+, c_-)$, $\varphi^0 = \operatorname{diag}(s_+, s_-)$, where

$$c_\pm(x,\lambda) = \cos \eta_\pm x, \qquad s_\pm(x,\lambda) = \frac{\sin \eta_\pm x}{\eta_\pm}, \qquad \eta_\pm = \sqrt{\lambda \mp a}, \qquad \operatorname{diag}(b,c) = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}$$

and the branch of $\sqrt{\lambda}$ is given by $\sqrt{1} = 1$. Below we will sometimes write $\rho(\lambda, V)$, $M(\lambda, V), \ldots,$ instead of $\rho(\lambda)$, $M(\lambda), \ldots,$ when several potentials are being dealt with. Then the functions $\rho^0(\cdot) = \rho(\cdot, q^0)$, $\mu_m^0(\cdot) = \mu_m(\cdot, q^0), \ldots$ corresponding to q^0 become the forms

$$\mu_m^0(\lambda) = \frac{c_+(m,\lambda) + c_-(m,\lambda)}{2}, \quad m = 1,2, \qquad \rho^0(\lambda) = \frac{(c_+(1,\lambda) - c_-(1,\lambda))^2}{4}, \tag{1.14}$$

$$D_\pm^0(\lambda) = \bigl(1 \mp c_+(1,\lambda)\bigr)\bigl(1 \mp c_-(1,\lambda)\bigr), \qquad \Delta_{(1)}^0(\lambda) = c_+(1,\lambda), \qquad \Delta_{(2)}^0(\lambda) = c_-(1,\lambda), \tag{1.15}$$

where $\Delta_{(m)}^0$ is the Lyapunov function for the equation

$$-y'' - (-1)^m a y = \lambda y, \quad m = 1,2.$$

The entire function $c_+(1,\lambda) - c_-(1,\lambda)$ has only the simple roots z_n^0 given by

$$z_n^0 = (\pi n)^2 + \frac{a^2}{(2\pi n)^2} = \mp a + \left(\pi n \pm \frac{a}{2\pi n}\right)^2, \quad n \geqslant 1. \tag{1.16}$$

Note that $z_1^0 > z_2^0 > \cdots > z_{n_a}^0$ and $z_{n_a+1}^0 < z_{n_a+2}^0 < \cdots$. Thus all roots of $\rho^0 = \rho(\cdot, q^0)$ have multiplicity 2 and are given by (1.16). If $2a/\pi^2 \notin \mathbb{N}$, then the zeros of $\Delta_{(m)}^0(\lambda)^2 - 1$ have multiplicity 2 and are given by

$$\lambda_{m,n}^0 = a - (-1)^m (\pi n)^2, \quad \lambda_{1,n}^0 \neq \lambda_{2,s}^0, \quad m = 1,2, \ n, s \geqslant 0. \tag{1.17}$$

Note that $\lambda_{m,n}^0$, $m = 1,2$, $n \geqslant 0$ are the roots of $D_\pm^0(\lambda)$.

If $\gamma, \nu \neq 0$ are small enough, then there exist gaps in the spectrum of $T_{\gamma,\nu}$. Define the disk $\mathbb{D}_r = \{\lambda\colon |\lambda| < \pi^2 r^2\}$, $r > 0$. We show that there exist non-degenerate resonance gaps for some potential V. In this example, some resonances are real and some are not.

Proposition 1.3. *Let a potential $q_{\gamma,\nu} = aJ + \gamma v_\nu(x) J_1$, $a > 0$, $\gamma \in \mathbb{R}$, satisfy Condition* A, *and let $4a/\pi^2 \notin \mathbb{N}$. Then for any $N \geqslant 1 + a$ there exist small $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ such that for any $(\gamma, \nu) \in (0, \varepsilon_1) \times (0, \varepsilon_2)$ all zeros $z_n^\pm(q_{\gamma,\nu})$ of $\rho(\lambda, q_{\gamma,\nu})$ and the zeros $\lambda_{m,n}^\pm(q_{\gamma,\nu})$ of $\Delta_m^2(\lambda, q_{\gamma,\nu}) - 1$ in the disk $\mathbb{D}_{N+1/2}$ are simple and have the properties*:

$$z_n^\pm(q_{\gamma,\nu}) \in \mathbb{C}_\pm, \quad 1 \leqslant n \leqslant n_a, \qquad z_n^-(q_{\gamma,\nu}) < z_n^+(q_{\gamma,\nu}), \quad n_a < n \leqslant N,$$
$$\left|z_n^\pm(q_{\gamma,\nu}) - z_n^0\right| < \varepsilon_2, \tag{1.18}$$

$$\lambda_{m,n}^-(q_{\gamma,\nu}) < \lambda_{m,n}^+(q_{\gamma,\nu}), \quad \left|\lambda_{m,n}^\pm(q_{\gamma,\nu}) - \lambda_{m,n}^0\right| < \varepsilon_2, \quad n = 0, \ldots, N, \ m = 1, 2. \tag{1.19}$$

There are no other roots of $\rho(\lambda, q_{\gamma,\nu})$ and $\Delta_m^2(\lambda, q_{\gamma,\nu}) - 1$ in the disk $\mathbb{D}_{N+1/2}$.

Remark. (i) If $0 < a < 2\pi^2$, then $\rho(\lambda, q_{\gamma,\nu})$ has only real roots $z_1^\pm(q_{\gamma,\nu})$ in the disk $\mathbb{D}_{N+1/2}$ for small γ, ν. If $a > 2\pi^2$, then $\rho(\lambda, q_{\gamma,\nu})$ has at least two non-real roots $z_1^\pm(q_{\gamma,\nu})$ for small γ, ν.

(ii) We show that the operator $T_{\gamma,\nu}$ has gaps associated with the periodic or anti-periodic spectrum. Moreover, we show the existence of new type gaps (so-called resonance gaps). The endpoints of the resonance gap are branch points of the Lyapunov function, and, in general, they are neither the periodic n or the anti-periodic eigenvalues. These endpoints are not stable. If they are real ($n = n_a + 1, \ldots, N$), then we have a gap. If they are complex ($1 \leqslant n \leqslant n_a$), then we have no a gap, we have only the branch points of the Lyapunov function in the complex plane.

2. Fundamental solutions

In this section we study ϑ, φ. We begin with some notational conventions. A vector $h = \{h_n\}_1^N \in \mathbb{C}^N$ has the Euclidean norm $|h|^2 = \sum_1^N |h_n|^2$, while a $N \times N$ matrix A has the operator norm given by $|A| = \sup_{|h|=1} |Ah|$. The function φ satisfies the following integral equation:

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} V(t)\varphi(t, \lambda)\, dt, \quad \varphi_0(x, \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} I_2, \tag{2.1}$$

where $(x, \lambda) \in \mathbb{R} \times \mathbb{C}$. The standard iterations in (2.1) yield

$$\varphi(x, \lambda) = \sum\nolimits_{n \geqslant 0} \varphi_n(x, \lambda), \qquad \varphi_{n+1}(x, \lambda) = \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} V(t)\varphi_n(t, \lambda)\, dt. \tag{2.2}$$

A similar expansion gives $\vartheta = \sum_{n \geqslant 0} \vartheta_n$ with $\vartheta_0(x, \lambda) = (\cos\sqrt{\lambda}x) I_2$.

We introduce the functions

$$I_m^0(\lambda) = \int_0^m dt \int_0^t \cos\sqrt{\lambda}(m - 2t + 2s) F(t, s)\, ds, \quad F(t, s) = \operatorname{Tr} V(t)V(s), \tag{2.3}$$

$m = 1, 2$. In Lemma 2.1 we shall show the simple identity

$$I_2^0(\lambda) = 4I_1^0(\lambda)\cos\sqrt{\lambda}. \tag{2.4}$$

We define $|\lambda|_1 \equiv \max\{1, |\lambda|\}$ and

$$V_{(0)} = \int_0^1 V(x)\,dx, \qquad V_{(1)} = \operatorname{Tr} V_{(0)}, \qquad V_{(2)} = \operatorname{Tr} V_{(0)}^2,$$

$$A = e^{|\operatorname{Im}\sqrt{\lambda}|+\varkappa}, \quad \varkappa = \frac{\|V\|_1}{\sqrt{|\lambda|_1}}. \tag{2.5}$$

We prove

Lemma 2.1. *For each $(x, V) \in \mathbb{R}_+ \times \mathcal{H}^1$ the functions $\varphi(x,\cdot), \vartheta(x,\cdot)$ are entire and for any $N \geqslant -1$ the following estimates hold*:

$$\max\left\{\left|\vartheta(x,\lambda) - \sum_0^N \vartheta_n(x,\lambda)\right|, \left|\sqrt{\lambda}\left(\varphi(x,\lambda) - \sum_0^N \varphi_n(x,\lambda)\right)\right|,\right.$$
$$\left.\left|\frac{1}{\sqrt{\lambda}}\left(\vartheta'(x,\lambda) - \sum_0^N \vartheta'_n(x,\lambda)\right)\right|, \left|\varphi'(x,\lambda) - \sum_0^N \varphi'_n(x,\lambda)\right|\right\} \leqslant \frac{(x\varkappa)^{N+1}}{(N+1)!} A^x. \tag{2.6}$$

Moreover, each $\mu_m(\lambda)$, $m = 1, 2$ is real for $\lambda \in \mathbb{R}$ and entire and the following estimates hold:

$$\left|\mu_m(\lambda)\right| \leqslant A^m, \qquad \left|\mu_m(\lambda) - \cos mz\right| \leqslant m\varkappa A^m, \tag{2.7}$$

$$\left|\mu_m(\lambda) - \cos mz - \frac{\sin mz}{4z} m V_{(1)}\right| \leqslant \frac{(m\varkappa)^2}{2} A^m, \tag{2.8}$$

$$\left|\mu_m(\lambda) - \cos mz - \frac{\sin mz}{4z} m V_{(1)} - \frac{1}{8z^2}\left(I_m^0(\lambda) - \frac{m^2 \cos mz}{2} V_{(2)}\right)\right| \leqslant \frac{(m\varkappa)^3}{3!} A^m, \tag{2.9}$$

where I_1^0, I_2^0 satisfy (2.4) *and $z = \sqrt{\lambda}$.*

Proof. We prove the estimates of φ, the proof for φ', ϑ, ϑ' is similar. (2.2) gives

$$\varphi_n(x,\lambda) = \int\limits_{0<x_1<x_2<\cdots<x_{n+1}=x} \varphi_0(x_1,\lambda)\left(\overset{\curvearrowleft}{\prod_{1\leqslant k\leqslant n}} \frac{\sin(\sqrt{\lambda}(x_{k+1}-x_k))}{\sqrt{\lambda}} V(x_k)\right) dx_1\,dx_2\ldots dx_n, \tag{2.10}$$

where for matrices $a_1, a_2, \ldots, a_n$ we denote

$$\overset{\curvearrowleft}{\prod_{1\leqslant k\leqslant n}} a_k = a_n a_{n-1}\ldots a_1.$$

Substituting the estimate $|\sqrt{\lambda}\varphi_0(x,\lambda)| \leqslant e^{|\operatorname{Im}\sqrt{\lambda}|x}$ into (2.10) we obtain $|\sqrt{\lambda}\varphi_n(x,\lambda)| \leqslant (x\varkappa)^n/n! e^{|\operatorname{Im} z|x}$, which shows that for each $x \geqslant 0$ the series (2.2) converges uniformly on bounded subsets of $\mathbb{C}$. Each term of this series is an entire function. Hence the sum is an entire function. Summing the majorants we obtain estimates (2.6).

We have

$$4\mu_m = \operatorname{Tr} M^m(\lambda) = \operatorname{Tr} M(m,\lambda) = \operatorname{Tr} \sum_{n\geqslant 0} M_n(m,\lambda),$$

where $m = 1, 2$ and

$$\operatorname{Tr} M_0(m,\lambda) = 4\cos mz, \qquad \operatorname{Tr} M_n(m,\lambda) = \operatorname{Tr} \vartheta_n(m,\lambda) + \operatorname{Tr} \varphi_n'(m,\lambda), \quad n \geqslant 1. \tag{2.11}$$

The estimates

$$\left|\varphi_n'(m,\lambda)\right| \leqslant \frac{(m\varkappa)^n}{n!} e^{|\operatorname{Im} z| m} \quad \text{and} \quad \left|\vartheta_n(m,\lambda)\right| \leqslant \frac{(m\varkappa)^n}{n!} e^{|\operatorname{Im} z| m}$$

yield

$$\left|\operatorname{Tr} M_n(m,\lambda)\right| \leqslant 4\frac{(m\varkappa)^n}{n!} e^{m|\operatorname{Im}\sqrt{\lambda}|}, \quad n \geqslant 0. \tag{2.12}$$

Using (2.11) we obtain

$$\begin{aligned} \operatorname{Tr} M_1(m,\lambda) &= \frac{1}{z}\int_0^m \bigl(\sin z(m-t)\cos zt + \cos z(m-t)\sin zt\bigr) \operatorname{Tr} V(t)\,dt \\ &= \frac{\sin mz}{z} m V_{(1)}, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \operatorname{Tr} M_2(m,\lambda) &= \frac{1}{z^2}\int_0^m \int_0^t \sin z(t-s) z(m-t+s) F(t,s)\,dt\,ds \\ &= \frac{1}{2z^2}\int_0^m \int_0^t \bigl(\cos z(m-2t+2s) - \cos zm\bigr) F(t,s)\,dt\,ds \\ &= \frac{1}{2z^2}\left(I_m^0(\lambda) - \frac{m^2}{2}\cos mz V_{(2)}\right), \end{aligned}$$

since

$$\int_0^m \int_0^t F(t,s)\,dt\,ds = \frac{1}{2}\operatorname{Tr}\left(\int_0^m V(t)\,dt\right)^2 = \frac{m^2}{2} V_{(2)}.$$

We know that μ_1, μ_2 are entire. Moreover, the trace of the monodromy matrix is the sum of its eigenvalues. By the Lyapunov Theorem (see Section 1), the set of these eigenvalues is symmetric with respect to the real axis, as $\lambda \in \mathbb{R}$. Thus, μ_1, μ_2 are real on $\mathbb{R}$.

We show (2.4). Let $I_m^0 = I_m^0(\lambda)$. We have $2\cos z I_1^0 = Y_0 + Y_1$, where

$$Y_0 = \int_0^1 \int_0^t \cos 2z(t-s) F(t,s)\, dt\, ds, \qquad Y_1 = \int_0^1 \int_0^t \cos 2z(1-t+s) F(t,s)\, dt\, ds.$$

We get $I_2^0 = Y_1 + Y_2 + Y_3$, where

$$Y_2 = \int_1^2 \int_0^1 \cos 2z(1-t+s) F(t,s)\, dt\, ds, \qquad Y_3 = \int_1^2 \int_1^t \cos 2z(1-t+s) F(t,s)\, dt\, ds$$

and using the new variable $\tau = t - 1$ we get

$$Y_2 = \int_0^1 \int_0^\tau \cos 2z(\tau - s) F(\tau, s)\, d\tau\, ds = 2Y_0.$$

We use the new variables $\tau = t - 1$, $\sigma = s - 1$ and obtain

$$Y_3 = \int_0^1 \int_0^\tau \cos 2z(1-\tau+\sigma) F(\tau,\sigma)\, d\tau\, d\sigma = Y_1,$$

which yields $I_2^0 = 2Y_1 + 2Y_0$. Thus we have (2.4). □

We need the basic properties of the monodromy matrix.

Lemma 2.2. *Let $V \in \mathcal{H}^1$. Then the function $D(\tau,\lambda) = \det(M(\lambda) - \tau I_4)$, $\lambda, \tau \in \mathbb{C}^2$ is entire on $\mathbb{C}^2$ and the following identities are fulfilled*:

$$D'_\tau(\tau,\lambda) = -D(\tau,\lambda)\,\mathrm{Tr}\big(M(\lambda) - \tau I_4\big)^{-1}, \tag{2.14}$$

$$\begin{aligned} D(\tau,\cdot) &= \tau^4 - 4\mu_1\tau^3 + 2\big(4\mu_1^2 - \mu_2\big)\tau^2 - 4\mu_1\tau + 1 \\ &= \big(\tau^2 - 2(\mu_1 - \sqrt{\rho})\tau + 1\big)\big(\tau^2 - 2(\mu_1 + \sqrt{\rho})\tau + 1\big). \end{aligned} \tag{2.15}$$

Proof. Let $D(\tau) \equiv D(\tau,\lambda)$. The standard identity

$$D'(\tau) = D(\tau)\,\mathrm{Tr}\bigg((M-\tau)^{-1}\frac{d(M-\tau)}{d\tau}\bigg)$$

yields (2.14). We prove (2.15). Since $\det M = D(0) = 1$, we have: $D(\tau) = 1 + a\tau + b\tau^2 + c\tau^3 + d\tau^4$, $a, b, c, d \in \mathbb{C}$. Then $D(\tau) = (\tau - \tau_1)(\tau - \tau_1^{-1})(\tau - \tau_2)(\tau - \tau_2^{-1})$, where $\tau_1, \tau_1^{-1}, \tau_2, \tau_2^{-1}$ are the multipliers. Therefore $d = 1$, $a = c$. Then we have

$$D(\tau) = 1 + D'(0)\tau + \frac{1}{2}D''(0)\tau^2 + D'(0)\tau^3 + \tau^4. \tag{2.16}$$

Identity (2.14) yields $D'(0) = -\operatorname{Tr} M^{-1} = -\operatorname{Tr} M = -4\mu_1$. Differentiating (2.14) we obtain

$$D''(0) = -D'(\tau)\operatorname{Tr}(M-\tau)^{-1} - D(\tau)\operatorname{Tr}(M-\tau)^{-2}|_{\tau=0} = 4\big(4\mu_1^2 - \mu_2\big).$$

Substituting these identities into (2.16) we obtain the first identity in (2.15). The second identity is proved by direct calculation. □

3. The Lyapunov function

We need some results about the functions ρ, $\rho_0 = c_0 \sin\sqrt{\lambda}/(2\sqrt{\lambda})$, where $c_0 = \frac{1}{2}(V_{20} - V_{10})$.

Lemma 3.1.

(i) *For each $V \in \mathcal{H}^1$ the function $\rho = \frac{1}{2}(\mu_2 + 1) - \mu_1^2$ is entire and real on the real line. Moreover, the following estimate holds*:

$$\big|\rho(\lambda) - \rho_0^2(\lambda)\big| \leqslant 2\varkappa^3 e^{2|\operatorname{Im}\sqrt{\lambda}| + 2\varkappa}, \quad \lambda \in \mathbb{C},\ \varkappa = \|V\|_1/\sqrt{|\lambda|_1}. \tag{3.1}$$

(ii) *Let $V \in \mathcal{H}^1$ with $c_0 = \frac{1}{2}(V_{20} - V_{10}) > 0$. Then for each integer $N > 2^7\|V\|_1^3/c_0^2$ the function ρ has exactly $2N$ roots, counted with multiplicity, in the disk $\{\lambda\colon |\lambda| < \pi^2(N + \frac{1}{2})^2\}$ and for each $n > N$, exactly two roots, counted with multiplicity, in the domain $\{\lambda\colon |\sqrt{\lambda} - \pi n| < 1\}$. There are no other roots.*

(iii) *Let $V \in \mathcal{H}^1$, $c_0 > 0$. Then the function $\sqrt{\rho}$ is an analytic function in the domain $\mathcal{D}_r$, $r = 2^9\|V\|_1^3/c_0^2$ given by* (1.6) *and the following estimate holds*:

$$\big|\sqrt{\rho(\lambda)} - \rho_0(\lambda)\big| \leqslant \frac{3C_0}{5}\frac{|\rho_0(\lambda)|}{\sqrt{|\lambda|}}, \quad C_0 \equiv 4^4\frac{\|V\|_1^3}{c_0^2} < \frac{\sqrt{|\lambda|}}{2},\ \lambda \in \mathcal{D}_r. \tag{3.2}$$

Proof. (i) By Lemma 2.2, ρ is entire and real on the real line. Let $\mu_m \equiv \mu_{m0} + \mu_{m1} + \mu_{m2} + \tilde{\mu}_{m3}$, $m = 1, 2$, where

$$\mu_{m0} = \cos mz, \qquad \mu_{m1} = \frac{\sin mz}{4z} m V_{(1)}, \qquad \mu_{m2} = \frac{1}{8z^2}\left(I_m^0(\lambda) - \frac{m^2}{2}\cos mz V_{(2)}\right), \tag{3.3}$$

and $z = \sqrt{\lambda}$, where $V_{(m)}$ is defined by (2.5). We obtain $\mu_1^2 = B_1 + B_2$, where

$$B_1 = (\mu_{10} + \mu_{11})^2 + 2\mu_{10}\mu_{12}, \qquad B_2 = 2\mu_{10}\tilde{\mu}_{13} + (\mu_1 - \mu_{10} + \mu_{11})(\mu_1 - \mu_{10} - \mu_{11}).$$

Then (3.3) yields

$$B_1 = \cos^2 z + \frac{\sin 2z}{4z} V_{(1)} + \frac{4\cos z I_1^0(\lambda) + \sin^2 z V_{(1)}^2 - 2\cos^2 z V_{(2)}}{16z^2}.$$

Thus we get

$$\rho = \frac{\mu_2 + 1}{2} - \mu_1^2 = G_1 + G_2, \quad G_1 = \frac{1 + \mu_{20} + \mu_{21} + \mu_{22}}{2} - B_1, \ G_2 = \frac{\tilde{\mu}_{23}}{2} - B_2,$$

$$G_1 = \frac{I_2^0(\lambda) - 2\cos 2z V_{(2)} - 4\cos z I_1^0(\lambda) + 2\cos^2 z V_{(2)} - \sin^2 z V_{(1)}^2}{16z^2} = \frac{\sin^2 z}{16z^2}\big(2V_{(2)} - V_{(1)}^2\big),$$

which yields $G_1 = \rho_0^2$. Using (2.7)–(2.9) we obtain

$$|B_2| \leqslant 2\frac{\varkappa^3}{3!}A^2 + (\varkappa A + \varkappa A)\frac{\varkappa^2}{2!}A^2 = \frac{4\varkappa^3}{3}A^2,$$

$$|G_2| \leqslant \frac{|\tilde{\mu}_{m3}|}{2} + |B_2| \leqslant \frac{4(\varkappa)^3}{3!}A^2 + \frac{4\varkappa^3}{3}A^2 = 2\varkappa^3 A^2, \tag{3.4}$$

which yields (3.1).

(ii) Let $N' > N$ be another integer and $r = \pi(N + \frac{1}{2})$. Note that

$$\varkappa \leqslant \frac{c_0^2}{2^7\pi\|V\|_1^2} \leqslant \frac{1}{2^7\pi} \quad \text{for } \lambda \in \mathcal{D}_r.$$

Using the estimate $e^{|\mathrm{Im}\sqrt{\lambda}|} < 4|\sin\sqrt{\lambda}|$, $\lambda \in \mathcal{D}_r$, and (3.1) we obtain (on all contours)

$$\big|\rho(\lambda) - \rho_0^2(\lambda)\big| \leqslant 2\frac{\|V\|_1^3}{|z|^3}e^{2(|\mathrm{Im}\, z|+\varkappa)} \leqslant \frac{C_0}{\sqrt{|\lambda|}}\big|\rho_0^2(\lambda)\big|, \quad C_0 \equiv 4^4\frac{\|V\|_1^3}{c_0^2} < \frac{\sqrt{|\lambda|}}{2}, \ \lambda \in \mathcal{D}_r. \tag{3.5}$$

Hence, by the Rouché theorem, $\rho(\lambda)$ has as many roots, counted with multiplicity, as $\sin^2\sqrt{\lambda}/\lambda$ in the bounded domain $\mathcal{D}_r \cap \{\lambda\colon |\lambda| \leqslant \pi(N' + \frac{1}{2})\}$. Since $\sin^2\sqrt{\lambda}/\lambda$ has exactly one double root at $(\pi n)^2$, $n \geqslant 1$, and since $N' > N$ can be chosen arbitrarily large, (ii) follows.

(iii) Let $\rho \equiv \rho_0^2 + \rho_1$, $\rho_0 = c_0 \sin\sqrt{\lambda}/(2\sqrt{\lambda})$. Estimates (3.5) imply

$$\sqrt{\rho(\lambda)} = \rho_0(\lambda)\sqrt{1 + b(\lambda)}, \quad b = \frac{\rho_1}{\rho_0^2}, \quad |b(\lambda)| \leqslant \frac{C_0}{|z|} < \frac{1}{2}, \ \lambda \in \mathcal{D}_r. \tag{3.6}$$

Using the estimate

$$\big|\sqrt{1 + b(\lambda)} - 1\big| \leqslant \frac{3}{5}\big|b(\lambda)\big| \quad \text{for } \lambda \in \mathcal{D}_r,$$

we obtain

$$\big|\sqrt{\rho(\lambda)} - \rho_0(\lambda)\big| = \big|\rho_0(\lambda)\big(\sqrt{1 + b(\lambda)} - 1\big)\big| \leqslant \frac{3C_0}{5}\frac{|\rho_0(\lambda)|}{\sqrt{|\lambda|}}. \qquad \square$$

We need asymptotics of the eigenvalues of the monodromy matrix.

Lemma 3.2. *Let $V \in \mathcal{H}^1$, $c_0 = (V_{20} - V_{10})/2 > 0$. Then the monodromy matrix $M(\lambda)$ has two multipliers $\tau_m(\lambda)$, $m = 1, 2$ with asymptotics* (1.4). *Moreover, asymptotics* (1.9) *holds.*

Proof. Using the estimates (2.8), (3.2), we obtain asymptotics in (1.9), which yields

$$\begin{aligned}\Delta_m^2(\lambda) - 1 &= \left(\cos\sqrt{\lambda} + V_{m0}\frac{\sin\sqrt{\lambda}}{2\sqrt{\lambda}}\right)^2 - 1 + O\big(E(\lambda)\big)\\ &= -\sin^2\sqrt{\lambda} + V_{m0}\frac{\sin\sqrt{\lambda}\cos\sqrt{\lambda}}{\sqrt{\lambda}} + O\big(E(\lambda)\big)\\ &= -\sin^2\sqrt{\lambda}\left(1 - V_{m0}\frac{\cos\sqrt{\lambda}}{\sqrt{\lambda}\sin\sqrt{\lambda}} + O\big(E(\lambda)\big)\right)\end{aligned}$$

as $\lambda \in \mathcal{D}_1$, $|\lambda| \to \infty$, where $E(\lambda) = e^{2|\operatorname{Im}\sqrt{\lambda}|}/\lambda$, which implies

$$\sqrt{\Delta_m^2(\lambda) - 1} = i\sin\sqrt{\lambda} - iV_{m0}\frac{\cos\sqrt{\lambda}}{2\sqrt{\lambda}} + O\big(E(\lambda)\big). \tag{3.7}$$

By (2.15), the matrix $M(\lambda)$, $\lambda \in \mathcal{D}_r$, has the eigenvalues $\tau_m(\lambda)$ satisfying the identities $\tau_m(\lambda) + \tau_m(\lambda)^{-1} = 2\Delta_m(\lambda)$. Then $\tau_m(\lambda)$ has the form

$$\tau_m(\lambda) = \Delta_m(\lambda) + \sqrt{\Delta_m^2(\lambda) - 1}$$

and the asymptotics give

$$\tau_m(\lambda) = e^{i\sqrt{\lambda}} + V_{m0}\frac{\sin\sqrt{\lambda}}{2\sqrt{\lambda}} - iV_{m0}\frac{\cos\sqrt{\lambda}}{\sqrt{\lambda}} + O\big(E(\lambda)\big) = e^{i\sqrt{\lambda}}\left(1 - \frac{iV_{m,0}}{2\sqrt{\lambda}}\right) + O\big(E(\lambda)\big)$$

which yields (1.4). □

Now we prove our first result about the Lyapunov function $\Delta = \mu_1 + \sqrt{\rho}$.

Proof of Theorem 1.1. By Lemma 3.1 we have Δ is the analytic function on the Riemann surface of the function $\sqrt{\rho}$.

(i) Identity (2.15) shows that $\Delta_m = (\tau + \tau^{-1})/2$ for some multiplier τ. Lemma 3.2 gives the asymptotics of Δ_m and τ_m, $m = 1, 2$.

(ii) The result follows from the statement (i) and the Lyapunov Theorem (see Section 1).

(iii) If $\lambda \in \sigma(T)$, then $\mu_1(\lambda)$ is real. By (ii), $\Delta(\lambda)$ is also real. Hence by (1.7), $\sqrt{\rho(\lambda)}$ is real and $\rho(\lambda) \geqslant 0$.

(iv) Assume that $\Delta_m'(\lambda_0) = 0$ for some $\lambda_0 \in I \subset \sigma(T)$ and $m \in \{1, 2\}$. Then we have

$$\Delta_m(\lambda) = \Delta_m(\lambda_0) + (\lambda - \lambda_0)^p\frac{\Delta_m^{(p)}(\lambda_0)}{p!} + O\big(|\lambda - \lambda_0|^{p+1}\big), \quad \text{as } \lambda - \lambda_0 \to 0, \tag{3.8}$$

where $\Delta_m^{(p)}(\lambda_0) \neq 0$ for some $p \geqslant 1$. By the implicit function theorem, there exists some curve $\Gamma \subset \{\lambda\colon |\lambda - \lambda_0| < \varepsilon\} \cap \mathbb{C}_+$, $\Gamma \neq \emptyset$, for some $\varepsilon > 0$ such that $\Delta_m(\lambda) \in (-1, 1)$ for any $\lambda \in \Gamma$. This contradicts the Lyapunov Theorem in Section 1. □

Recall that $D_\pm(\lambda) = \frac{1}{4}\det(M(\lambda) \mp I_4)$, the set $\{\lambda\colon D_+(\lambda) = 0\}$ is a periodic spectrum and the set $\{\lambda\colon D_-(\lambda) = 0\}$ is an anti-periodic spectrum. Now we prove a lemma about the number of periodic and anti-periodic eigenvalues in a large disc.

Lemma 3.3. *For each $V \in \mathcal{H}^1$ the functions $\Delta_1 + \Delta_2$, $\Delta_1\Delta_2$, $D_\pm$ are entire and satisfy the following identities*:

$$\Delta_1^2 + \Delta_2^2 = 1 + \mu_2, \qquad \Delta_1\Delta_2 = 2\mu_1^2 - \frac{\mu_2 + 1}{2}, \tag{3.9}$$

$$D_\pm = (\mu_1 \mp 1)^2 - \rho = \frac{(2\mu_1 \mp 1)^2 - \mu_2}{2}, \qquad D_+ - D_- = -4\mu_1. \tag{3.10}$$

Let, in addition, $\operatorname{Tr}\int_0^1 V(t)\,dt = 0$. *Then the following estimates and properties are fulfilled*:

$$\max\left\{\left|D_+(\lambda) - 4\sin^4\frac{\sqrt{\lambda}}{2}\right|, \left|D_-(\lambda) - 4\cos^4\frac{\sqrt{\lambda}}{2}\right|\right\} \leqslant \varkappa^2(2+\varkappa)^2 e^{2|\operatorname{Im}\sqrt{\lambda}| + 2\varkappa}, \quad \lambda \in \mathbb{C}. \tag{3.11}$$

(i) *For each integer $N > 8\|V\|_1$ the function D_+ has exactly $4N + 2$ roots, counted with multiplicity, in the disc $\{|\lambda| < 4\pi^2(N + \frac{1}{2})^2\}$ and for each $n > N$, exactly four roots, counted with multiplicity, in the domain $\{|\sqrt{\lambda} - 2\pi n| < \frac{\pi}{2}\}$. There are no other roots.*

(ii) *For each integer $N > 8\|V\|_1$ the function D_- has exactly $4N$ roots, counted with multiplicity, in the disc $\{|\lambda| < 4\pi^2 N^2\}$ and for each $n > N$, exactly four roots, counted with multiplicity, in the domain $\{|\sqrt{\lambda} - \pi(2n + 1)| < \frac{\pi}{2}\}$. There are no other roots.*

Proof. By Lemmas 2.1, 2.2 the functions $\Delta_1 + \Delta_2$, $\Delta_1\Delta_2$, $D_\pm$ are entire and the identities (3.9), (3.10) are fulfilled. Using Lemmas 2.1 and 3.1 we obtain

$$\mu_1(\lambda) = \cos z + \tilde{\mu}_{12}(\lambda), \qquad \left|\tilde{\mu}_{12}(\lambda)\right| \leqslant \frac{\varkappa^2}{2}A^2, \qquad \left|\rho(\lambda)\right| \leqslant (1 + 2\varkappa)\frac{\varkappa^2}{2}A^2.$$

Substituting these relations into $D_\pm = (\mu_1 \mp 1)^2 - \rho$ we get (3.11).

(i) Let $N' > N$ be another integer. Let λ belong to the contours

$$C_0(2N + 1), \qquad C_0(2N' + 1), \qquad C_{2n}\left(\frac{1}{2}\right), \quad |n| > N,$$

where $C_n(r) = \{\sqrt{\lambda}\colon |\sqrt{\lambda} - \pi n| = \pi r\}$, $r > 0$. Note that $\varkappa \leqslant \frac{1}{16\pi}$ on all contours. Then (3.11) and the estimate $e^{\frac{1}{2}\operatorname{Im}\sqrt{\lambda}} < 4|\sin\sqrt{\lambda}/2|$ on all contours yield

$$\left|D_+(\lambda) - 4\sin^4\frac{\sqrt{\lambda}}{2}\right| \leqslant \frac{e^{2\varkappa}(2+\varkappa)^2}{4^4\pi^2}e^{2|\operatorname{Im}\sqrt{\lambda}|} < \frac{1}{4}\left|4\sin^4\frac{\sqrt{\lambda}}{2}\right|.$$

Hence, by Rouché's theorem, $D_+(\lambda)$ has as many roots, counted with multiplicities, as $\sin^4\sqrt{\lambda}/2$ in each of the bounded domains and the remaining unbounded domain. Since $\sin^4\sqrt{\lambda}/2$ has exactly one root of the multiplicity four at $(2\pi n)^2$, and since $N' > N$ can be chosen arbitrarily large, (i) follows.

The proof for D_- is similar. □

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. (i) It is enough to consider the case $V_{10} = c_0 = -V_{20}$. We prove the asymptotics (1.11) for $\lambda_{2n,m}$, $1 \leqslant m \leqslant 4$. The proof for $\lambda_{2n+1,m}$ is similar. Firstly, we prove rough asymptotics of $\lambda_{2n,m}$, r_n. Lemma 3.3(i) yields $\sqrt{\lambda_{2n,m}} = 2\pi n + \varepsilon_n$, $|\varepsilon_n| < 1$, as $n \to \infty$. By Lemma 3.3,

$$D_+(\lambda) = 4\sin^4\frac{\sqrt{\lambda}}{2} + O\big(n^{-2}\big) \quad \text{as } |\sqrt{\lambda} - \pi n| \leqslant 1,\ n \to \infty.$$

Then the identity $D_+(\lambda_{2n,m}) = 0$ implies

$$\sqrt{\lambda_{2n,m}} = 2\pi n + \varepsilon_n, \quad \varepsilon_n = O\big(n^{-1/2}\big),\ 1 \leqslant m \leqslant 4. \tag{3.12}$$

Lemma 3.1(ii) implies $\sqrt{r_{2n-m}} = \pi n + \delta_n$, $|\delta_n| < 1$ for $n \to \infty$, $m = 0, 1$. Moreover, Lemma 3.1(i) gives

$$\rho(\lambda) = c_0^2\frac{\sin^2\sqrt{\lambda}}{4\lambda} + O\big(n^{-3}\big), \quad |\sqrt{\lambda} - \pi n| \leqslant 1 \text{ as } n \to \infty.$$

Since $\rho(r_n) = 0$, we have

$$\sqrt{r_{2n-m}} = \pi n + \delta_n, \quad \delta_n = O\big(n^{-1/2}\big),\ m = 0, 1. \tag{3.13}$$

Secondly, in order to improve these asymptotics of $\lambda_{2n,m}, r_n$ we need asymptotics of the multipliers in a neighborhood of the points πn. We introduce the matrix $\widetilde{M} = U^{-1}MU$ with the same eigenvalues, where

$$U = \begin{pmatrix} I_2 & 0 \\ 0 & \sqrt{\lambda} I_2 \end{pmatrix}.$$

We shall show the asymptotics

$$\widetilde{M}(\lambda) = \widetilde{M}_0(\lambda) + \ell_1^2(n), \quad \widetilde{M}_0(\lambda) = \begin{pmatrix} C(\lambda) & S(\lambda) \\ -S(\lambda) & C(\lambda) \end{pmatrix}, \quad \sqrt{\lambda} = \pi n + O\big(n^{-1/2}\big), \tag{3.14}$$

where

$$\begin{aligned} &C(\lambda) = \operatorname{diag}(\cos\eta_+, \cos\eta_-), \qquad S(\lambda) = \operatorname{diag}(\sin\eta_+, \sin\eta_-), \\ &\eta_\pm = \sqrt{\lambda \mp c_0}. \end{aligned} \tag{3.15}$$

Estimate (2.6) gives

$$\vartheta(1,\lambda) = \cos\sqrt{\lambda} I_2 + \frac{1}{\sqrt{\lambda}} \int_0^1 \sin\sqrt{\lambda}(1-t)\cos\sqrt{\lambda}t V(t)\,dt + O\big(n^{-2}\big)$$

$$= \cos\sqrt{\lambda} I_2 + \frac{1}{2\sqrt{\lambda}} \int_0^1 \big(\sin\sqrt{\lambda} + \sin\sqrt{\lambda}(1-2t)\big) V(t)\,dt + O\big(n^{-2}\big), \qquad (3.16)$$

as $n \to +\infty$, $|\sqrt{\lambda} - \pi n| \leqslant 1$. Let $\sqrt{\lambda} = \pi n + u_n$, $u_n = O(n^{-1/2})$. The Taylor formula gives $\sin 2t(\pi n + u_n) = \sin 2\pi nt + 2tu_n \cos 2\pi nt + O(n^{-1})$ and the similar formula for the case of $\cos 2t(\pi n + u_n)$. Substituting these asymptotics into (3.16) we obtain

$$\vartheta(1,\lambda) = \cos\sqrt{\lambda} I_2 + \frac{\sin\sqrt{\lambda}}{2\sqrt{\lambda}} c_0 J + \ell_1^2(n) = C(\lambda) + \ell_1^2(n), \qquad \sqrt{\lambda} = \pi n + O\big(n^{-1/2}\big).$$

Similar arguments for $\varphi(1,\lambda), \vartheta'(1,\lambda), \varphi'(1,\lambda)$ yield

$$\varphi(1,\lambda) = \frac{1}{\sqrt{\lambda}} S(\lambda) + \ell_2^2(n), \qquad \vartheta'(1,\lambda) = -\sqrt{\lambda} S(\lambda) + \ell^2(n), \qquad \varphi'(1,\lambda) = C(\lambda) + \ell_1^2(n),$$

as $\sqrt{\lambda} = \pi n + O(n^{-1/2})$. Substituting the obtained asymptotics into the definition (1.2) of M and using the identity $\widetilde{M} = U^{-1} M U$ we get (3.14).

We will use the next standard arguments from perturbation theory (see [11, p. 291]). Let A, B be bounded operators, A be normal and denote by $\sigma(A)$, $\sigma(B)$ spectra of A, B. Then $\operatorname{dist}\{\sigma(A), \sigma(B)\} \leqslant \|A - B\|$. Note that $\widetilde{M}_0(\lambda)$ is a normal operator having eigenvalues $e^{i\eta_\pm}$. Hence $\widetilde{M}$ has eigenvalues $\tau_\pm$ satisfying the estimates $|\tau_\pm - e^{i\eta_\pm}| < |\widetilde{M} - \widetilde{M}_0|$. Then (3.14) implies

$$\tau_\pm(\lambda) = e^{i\eta_\pm} + \ell_1^2(n), \quad \text{as } \sqrt{\lambda} = \pi n + O\big(n^{-1/2}\big). \qquad (3.17)$$

Now we improve the asymptotics (3.12) for $\lambda_{2n,m}$. We note that $\lambda = \lambda_{2n,m}$ iff $\tau_+(\lambda) = 1$ or $\tau_-(\lambda) = 1$. Then (3.17) and (3.12) yield

$$e^{i\eta_\pm(\lambda_{2n,m})} = 1 + \ell_1^2(n), \quad \lambda = \lambda_{2n,m}. \qquad (3.18)$$

Substituting the asymptotics (3.12) into (3.18) we have

$$\sqrt{(2\pi n + \varepsilon_n)^2 \pm c_0} = 2\pi n + \ell_1^2(n), \quad n \to +\infty.$$

and therefore $\varepsilon_n = \mp c_0/(4\pi n) + \ell_1^2(n)$. Substituting this asymptotics into (3.12) we obtain (1.11) for $\lambda_{2n,m}$.

Finally, we improve the asymptotics (3.13) of r_n. Note that $\lambda = r_n$, iff the following condition is fulfilled: $\tau_+(\lambda) = \tau_-^{-1}(\lambda)$ or $\tau_+(\lambda) = \tau_-(\lambda)$. The asymptotics (3.17) imply that the second equation has no solutions for large n. We rewrite the first equation in the form:

$$e^{i\eta_+} = e^{-i\eta_-} + \ell_1^2(n), \quad \text{as } \sqrt{\lambda} = \pi n + O\big(n^{-1/2}\big). \qquad (3.19)$$

Substituting asymptotics (3.13) into (3.19) we obtain

$$\sqrt{(\pi n+\delta_n)^2-c_0}+\sqrt{(\pi n+\delta_n)^2+c_0}=2\pi n+\ell_1^2(n),$$

which gives $\delta_n=\ell_1^2(n)$. Substituting (3.13) into we obtain (1.12).

(ii) (a) Assume that we have the periodic spectrum $\lambda_{0,m}$, $m=1,2$, $\lambda_{2n,m}$, $m=1,2,3,4$, $n\geqslant 1$. Using the asymptotics (1.11) and repeating the standard arguments (see [18, pp. 39–40]) we obtain the Hadamard factorization for the function D_+:

$$D_+(\lambda)=\frac{(\lambda-\lambda_{0,1})(\lambda-\lambda_{0,2})}{4}\prod_{m=1,2,3,4,n\geqslant 1}\frac{\lambda_{2n,m}-\lambda}{(2\pi n)^2}.$$

In a similar way, we determine D_- by the anti-periodic spectrum. Using (3.10) we obtain ρ. Thus, we recover the resonances.

(b) Suppose that we know the periodic spectrum and the set of resonances. Then we determine the functions D_+ by the periodic spectrum and ρ by the resonances. Using (3.10) we get μ_1, μ_2 and then D_-. Thus, we recover the anti-periodic spectrum. The proof of another case is similar. □

4. Example

1. Periodic δ-potentials. Consider the operator

$$T^\gamma=-\frac{d^2}{dx^2}+q^\gamma,\quad \gamma\in\mathbb{C},$$

where $q^\gamma=aJ+\gamma\delta_{\mathrm{per}}J_1$ with the potential $\delta_{\mathrm{per}}=\sum_{-\infty}^{\infty}\delta(x-n-\frac{1}{2})$. Let $\mu_1^\gamma=\mu_1(\cdot,q^\gamma)$, $\rho^\gamma=\rho(\cdot,q^\gamma),\ldots.$

Lemma 4.1. *For the operator $T^\gamma=-d^2/dx^2+aJ+\gamma\delta_{\mathrm{per}}J_1$ the following identities are fulfilled*:

$$\mu_1^\gamma=\mu_1^0,\quad \mu_2^\gamma=\mu_2^0+2h,\quad \rho^\gamma=\rho^0+h,\quad D_\pm^\gamma=D_\pm^0-h,\quad h=\frac{\gamma^2}{4}s_+s_-,\quad \gamma\in\mathbb{C}. \tag{4.1}$$

Proof. The solution $y(x), x\in\mathbb{R}$ of the system $-y''+q^\gamma y=\lambda y$ is continuous and $y'(x_n+0)-y'(x_n-0)=\gamma J_1 y(x_n)$ at the points $x=x_n=n+\frac{1}{2}$. Then the fundamental solutions have the form:

$$\vartheta^\gamma(x)=\vartheta^0(x),\quad \varphi^\gamma(x)=\varphi^0(x),\quad 0\leqslant x<\frac{1}{2},$$

and

$$\vartheta^\gamma(x)=\begin{pmatrix} c_+(x) & \gamma c_-(\frac{1}{2})s_+(x-\frac{1}{2})\\ \gamma c_+(\frac{1}{2})s_-(x-\frac{1}{2}) & c_-(x)\end{pmatrix},\quad \frac{1}{2}\leqslant x<\frac{3}{2},$$

$$\varphi^\gamma(x) = \begin{pmatrix} s_+(x) & \gamma s_-(\frac{1}{2})s_+(x-\frac{1}{2}) \\ \gamma s_+(\frac{1}{2})s_-(x-\frac{1}{2}) & s_-(x) \end{pmatrix}, \quad \frac{1}{2} \leqslant x < \frac{3}{2},$$

and

$$\vartheta^\gamma(x) = \begin{pmatrix} c_+(x) + \gamma^2 c_+(\frac{1}{2})s_-(1)s_+(x-\frac{3}{2}) & * \\ * & c_-(x) + \gamma^2 c_-(\frac{1}{2})s_+(1)s_-(x-\frac{3}{2}) \end{pmatrix},$$

$$\varphi^\gamma(x) = \begin{pmatrix} s_+(x) + \gamma^2 s_+(\frac{1}{2})s_-(1)s_+(x-\frac{3}{2}) & * \\ * & s_-(x) + \gamma^2 s_-(\frac{1}{2})s_+(1)s_-(x-\frac{3}{2}) \end{pmatrix},$$

for $\frac{3}{2} \leqslant x < \frac{5}{2}$, where $*$ is some term. These relations yield

$$\vartheta(1) = \begin{pmatrix} c_+(1) & * \\ * & c_-(1) \end{pmatrix}, \qquad \varphi'(1) = \begin{pmatrix} c_+(1) & * \\ * & c_-(1) \end{pmatrix},$$

$$\vartheta(2) = \begin{pmatrix} c_+(2) + 2h & * \\ * & c_-(2) + 2h \end{pmatrix}, \qquad \varphi'(2) = \begin{pmatrix} c_+(2) + 2h & * \\ * & c_-(2) + 2h \end{pmatrix}.$$

The last identities and (1.14) imply

$$\mu_1^\gamma = \frac{\operatorname{Tr}(\vartheta^\gamma(1) + (\varphi^\gamma)'(1))}{4} = \mu_1^0, \qquad \mu_2^\gamma = \frac{\operatorname{Tr}(\vartheta^\gamma(2) + (\varphi^\gamma)'(2))}{4} = \mu_2^0 + 2h,$$

which give $\rho^\gamma = \frac{1}{2}(1 + \mu_2^\gamma) - (\mu_1^\gamma)^2 = \rho^0 + h$ and $D_\pm^\gamma = (\mu_1^\gamma \mp 1)^2 - \mu_2^\gamma = D_\pm^0 - h$. □

We describe the spectrum of the operator T^γ.

Lemma 4.2. *Let the operator $T^\gamma = -d^2/dx^2 + q^\gamma$, where $q^\gamma = aJ + \gamma\delta_{\mathrm{per}}J_1$, $a > 0$ and $\gamma \in \mathbb{R}$.*

(i) *Let $0 < a/(2\pi^2) - n_a < 1$, for some integer $n_a \geqslant 0$. Then for each $n \in \mathbb{N}$ there exist analytic functions $z_n^\pm(\gamma)$, $|\gamma| < \gamma_n$ for some $\gamma_n > 0$ such that $z_n^\pm(\gamma)$ are the zeros of $\rho^\gamma(\lambda)$ and*

$$z_n^\pm(\gamma) = z_n^0 \pm (-1)^n \gamma i^{k_n}\big(c_n + O(\gamma)\big),$$
$$c_n > 0, \quad k_n = \begin{cases} 1 & \text{if } 1 \leqslant n \leqslant n_a, \\ 0 & \text{if } n > n_a, \end{cases} \quad \text{as } \gamma \to 0. \tag{4.2}$$

Moreover, each spectral interval $(z_n^-(\gamma), z_n^+(\gamma)) \subset \mathbb{R}$, $n > n_a$ is a gap in the spectrum of T^γ.

(ii) *If $2a/\pi^2 \notin \mathbb{N}$, then for each $n \geqslant 0, m = 1, 2$ there exist real analytic functions $\lambda_{m,n}^\pm(\gamma)$, $\gamma \in (-\gamma_n, \gamma_n)$ for some $\gamma_n > 0$, such that $\lambda_{n,m}^\pm(\gamma)$ is the zero of the function $\Delta_m^2(\lambda) - 1$ and*

$$\lambda_{n,m}^-(\gamma) < \lambda_{n,m}^+(\gamma), \qquad \lambda_{n,m}^\pm(0) = \lambda_{n,m}^0. \tag{4.3}$$

Moreover, each spectral interval $(\lambda_{n,m}^-(\gamma), \lambda_{n,m}^+(\gamma)) \neq \emptyset$, $n \geqslant 1$, has multiplicity 2.

Proof. (i) Recall that $\eta_\pm = \sqrt{\lambda \mp a}$. The zero of $\rho^\gamma(\lambda) = 0$ satisfies the equation

$$0 = \rho^\gamma(\lambda) = f^2(\lambda) + \frac{\gamma^2}{4} s_+(\lambda) s_-(\lambda), \quad f \equiv 2\sin\frac{\eta_+ - \eta_-}{2}\sin\frac{\eta_+ + \eta_-}{2}. \tag{4.4}$$

The zeros z_n^0 of $\rho^0 = \rho(\cdot, q^0)$ have the form (1.16) and satisfy the following identities:

$$\sqrt{z_n^0 + a} + \sqrt{z_n^0 - a} = \frac{a}{\pi n}, \quad \sqrt{z_n^0 + a} - \sqrt{z_n^0 - a} = 2\pi n, \quad \text{if } 1 \leqslant n \leqslant n_a, \tag{4.5}$$

$$\sqrt{z_n^0 + a} + \sqrt{z_n^0 - a} = 2\pi n, \quad \sqrt{z_n^0 + a} - \sqrt{z_n^0 - a} = \frac{a}{\pi n}, \quad \text{if } n > n_a \geqslant 0. \tag{4.6}$$

Using (4.5), (4.6) we have the identity

$$\sin\eta_+(\lambda)\sin\eta_-(\lambda)|_{\lambda = z_n^0} = \begin{cases} 1 - \cos\frac{a}{\pi n} > 0 & \text{if } 1 \leqslant n \leqslant n_a, \\ \cos\frac{a}{\pi n} - 1 < 0 & \text{if } n > n_a. \end{cases} \tag{4.7}$$

Recall that the function f has only simple zeros $\lambda = z_n^0, n \geqslant 1$. Consider the case $n > n_a$, the proof for $1 \leqslant n \leqslant n_a$ is similar. We rewrite $\rho^\gamma(\lambda) = 0$ in the form

$$\big(f(\lambda) - \gamma F(\lambda)\big)\big(f(\lambda) + \gamma F(\lambda)\big) = 0, \qquad F(\lambda) \equiv \sqrt{-s_+(\lambda)s_-(\lambda)/2},\ F\big(z_n^0\big) > 0.$$

Applying the implicit function theorem to $\Phi_\pm(\lambda, \gamma) = 0$, where $\Phi_\pm(\lambda, \gamma) = f(\lambda) \pm \gamma F(\lambda)$ and $\partial\Phi_\pm(z_n^0, 0)/\partial\lambda \neq 0$ we obtain a unique solution $z_n^{\gamma,\pm} = u = u(\gamma)$, $\gamma \in (-\gamma_0, \gamma_0)$, $u(0) = z_n^0$ of the equation $\Phi_\pm(\lambda, \gamma) = 0$, such that $\Phi_\pm(u(\gamma), \gamma) = 0$, $\gamma \in (-\gamma_0, \gamma_0)$ for some $\gamma_0 > 0$.

(ii) Consider the equation $0 = D_+^\gamma(\lambda) = D_+^0(\lambda) - \frac{\gamma^2}{4} s_+(\lambda)s_-(\lambda)$. Using

$$D_+^0(\lambda) = 4\sin^2\frac{\eta_+}{2}\sin^2\frac{\eta_-}{2}, \qquad s_\pm(\lambda) = \frac{2}{\eta_\pm}\sin\frac{\eta_\pm}{2}\cos\frac{\eta_\pm}{2}, \quad \cos\eta_-(\lambda_{2n,1}) \neq 1, \tag{4.8}$$

we obtain two equations for the zeros of $D_+^\gamma(\lambda)$

$$\Phi(\lambda, \gamma) \equiv \sin\frac{\eta_+}{2} - \frac{\gamma^2}{4\eta_-\eta_+}\cos\frac{\eta_+}{2}\cot\frac{\eta_-}{2} = 0, \qquad \sin\frac{\eta_+}{2} = 0. \tag{4.9}$$

The equation $\sin(\eta_+/2)/\eta_+ = 0$ has the zeros $\lambda_{1,2n}^0$. Consider the first equation in (4.9). Applying the implicit function theorem to $\Phi(\lambda, \gamma) = 0$, where $\partial\Phi(\lambda_{2n,1}^0, 0)/\partial\lambda \neq 0$ we obtain a unique solution $u = u_n(\gamma)$, $\gamma \in (-\gamma_n, \gamma_n)$, $u_n(0) = \lambda_{2n,1}^0$ of the equation $\Phi(\lambda, \gamma) = 0$, such that $\Phi(u_n(\gamma), \gamma) = 0$, $\gamma \in (-\gamma_n, \gamma_n)$ for some $\gamma_n > 0$. The proof for $D_-^\gamma(\lambda) = 0$ is similar. □

2. The perturbed operator. We consider the operator $T_{\gamma,\nu} = -d^2/dx^2 + q_{\gamma,\nu}$ where the potential $q_{\gamma,\nu} = aJ + \gamma v_\nu J_1$ satisfies Condition A, $\gamma \in \mathbb{R}$ is small and $a > 0$. We determine the asymptotics of the function $\rho(\lambda, q_{\gamma,\nu})$, $\mu_m(\lambda, q_{\gamma,\nu})$, $m = 1, 2$.

Lemma 4.3. *Each function* $\rho(\lambda, q_{\gamma,\nu})$, $\mu_m(\lambda, q_{\gamma,\nu})$, $m = 1, 2$, $\nu = 1, \frac{1}{2}, \frac{1}{3}, \dots$, *is analytic in* $\mathbb{C}^2$. *Moreover, uniformly on any compact subset in* $\mathbb{C}^2$ *the following asymptotics are fulfilled*:

$$\rho(\lambda, q_{\gamma,\nu}) = \rho^\gamma(\lambda) + o(1), \quad \mu_m(\lambda, q_{\gamma,\nu}) = \mu_m^\gamma(\lambda) + o(1), \quad m = 1, 2, \ \nu \to 0. \tag{4.10}$$

Proof. The fundamental solutions $\vartheta_{\nu,\gamma}, \varphi_{\nu,\gamma}$ of the equation $-y'' + q_{\gamma,\nu} y = \lambda y$, satisfying the conditions $\vartheta_{\nu,\gamma}(0, \lambda) = (\varphi_{\nu,\gamma})'(0, \lambda) = I_2$, $(\vartheta_{\nu,\gamma})'(0, \lambda) = \varphi_{\nu,\gamma}(0, \lambda) = 0$ are the solutions of the integral equation

$$\varphi_{\nu,\gamma}(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x \varphi_0(x - t, \lambda) q_{\gamma,\nu}(t) \varphi_{\nu,\gamma}(t, \lambda)\, dt. \tag{4.11}$$

The standard iterations in (4.11) yield

$$\varphi_{\nu,\gamma} = \sum\nolimits_{n \geqslant 0} \varphi_{n,\nu,\gamma}, \quad \varphi_{n,\nu,\gamma}(x, \lambda) = \int_0^x \varphi_0(x - t, \lambda) q_{\gamma,\nu}(t) \varphi_{n-1,\nu,\gamma}(t, \lambda)\, dt. \tag{4.12}$$

The last identity gives

$$\varphi_{n,\nu,\gamma}(x, \lambda) = \int\limits_{0 < x_1 < x_2 < \dots < x_{n+1} = x} \varphi_0(x_1, \lambda) \left(\overset{\curvearrowleft}{\prod_{1 \leqslant k \leqslant n}} \varphi_0(x_{k+1} - x_k, \lambda) q_{\gamma,\nu}(x_k) \right) dx_1\, dx_2 \dots dx_n, \tag{4.13}$$

where for matrices $a_1, a_2, \dots, a_n$ we define $\overset{\curvearrowleft}{\prod}_{1 \leqslant k \leqslant n} a_k = a_n a_{n-1} \dots a_1$. Substituting the estimate $\|\sqrt{\lambda} \varphi_0(x, \lambda)\| \leqslant e^{|\mathrm{Im}\sqrt{\lambda}| x}$ into (4.13) we obtain

$$\left\| \sqrt{\lambda} \varphi_{n,\nu,\gamma}(x, \lambda) \right\| \leqslant \frac{(2x(a + |\gamma|))^n}{n!} e^{|\mathrm{Im}\, z| x},$$

which shows that for each $x \geqslant 0$ the formal series (4.12) converges uniformly on bounded subsets of $\mathbb{C}$. Each term of this series is an entire function. Hence the sum is an entire function. Since $v_\nu \to p$ in the sense of distributions we obtain $\varphi_{n,\nu,\gamma}(x, \lambda) \to \varphi_{n,0,\gamma}(x, \lambda)$ as $\nu \to 0$ uniformly on bounded subset of $\mathbb{R} \times \mathbb{C}^2$, which yields (4.10). □

We give

Proof of Proposition 1.3. Lemma 4.3 yields $\rho(\lambda, q_{\gamma,\nu}) \to \rho(\lambda, q^\gamma,)$ and $\mu_m(\lambda, q_{\gamma,\nu}) \to \mu_m(\lambda, q^\gamma)$, $m = 1, 2$, uniformly on any compact subset in $\mathbb{C}^2$ as $\nu \to 0$. Then their zeros converge to the corresponding zeros at $\nu = 0$, uniformly on any compact subset in $\mathbb{C}^2$. Due to Lemma 3.3 we have convergence of $D_\pm(\lambda, q_{\gamma,\nu})$ and of the Lyapunov function $\Delta(\lambda, q_{\gamma,\nu})$. Thus using Lemmas 4.1, 4.2 we obtain the proof of Proposition 1.3. □

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