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*Dedicated to the memory of Vladimir Andreevich Borovikov,
who often chose a problem as an engineer
and solved it by creating new and surprising mathematics*

Calculation of the Kirchhoff Coefficients for the Helmholtz Resonator

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Abstract. A Helmholtz resonator is a shell $\Omega_{\rm shell}$ separating a compact cavity $\Omega_{\rm int}$ from a noncompact outer domain $\Omega_{\rm out}$. A small opening Ω^δ in the shell connects the cavity with the outer domain, causing the transformation of real eigenfrequencies of the Neumann Laplacian in the cavity into the complex scattering frequencies of the full spectral problem for the Neumann Laplacian on $\Omega = \mathbb{R}^3 \backslash \Omega_{\rm shell}$.

The Kirchhoff model of 1882, see [21], gives a convenient ansatz

$$\Psi_{\rm out}(x,\nu,\lambda) = \Psi^N_{\rm out}(x,\nu,\lambda) + A_{\rm out}\, G^N_{\rm out}(x,a,\lambda), \qquad x \in \Omega_{\rm out}, \tag{1}$$

for the approximate calculation of the outer component of the scattered wave of the full spectral problem on Ω in terms of the scattered wave $\Psi^N_{\rm out}(x,\nu,\lambda)$ and the Green function $G^N_{\rm out}(x,a,\lambda)$ of the Neumann Laplacian on the outer domain, with a pole at the pointwise opening $\Omega^\delta \approx a$.

In this paper, we suggest an explicit formula for the Kirchhoff coefficient $A_{\rm out}$, based on the construction of a fitted solvable model for the Helmholtz resonator with a narrow short channel Ω^δ connecting the cavity with the outer domain. The correcting term of the scattering matrix of the model serves as a rational approximation, on a certain spectral interval, for the correcting term of the full scattering matrix of the Helmholtz resonator.

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1. PRELIMINARIES

A Helmholtz resonator is a compact shell $\Omega_{\rm shell}$ in $\mathbb{R}^3$ with a piecewise smooth boundary. The shell separates the outer domain $\Omega_{\rm out}$ from the inner domain $\Omega_{\rm int}$, the cavity. In this paper, we assume that the outer domain and the cavity are connected by a cylindrical channel Ω^δ, of length H, of radius δ, and with imaginable "upper" and "lower" lids Γ_H and Γ (disks) separating the channel from $\Omega_{\rm out}$ and $\Omega_{\rm int}$, respectively.

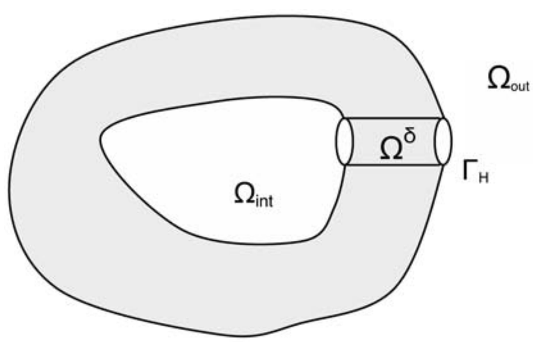

Fig. 1. Helmholtz Resonator.

On the domain $\Omega = \Omega_{\text{out}} \cup \Omega_{\text{int}} \cup \Omega^{\delta}$, we consider the Neumann Laplacian $L^N =: L$ with the Meixner conditions imposed on the inner surfaces (roughly speaking, the assumption that the energy of the scattered field remains finite in every finite region). This condition provides automatic convergence of the integrals arising in our formulas.

We discuss the full stationary scattering problem for the Neumann Laplacian L in Ω with the wave-number $k = \sqrt{\lambda}$ from an "essential spectral interval" Δ situated on the positive semi-axis, in the range of relatively small wave-numbers $kH < \pi/2$ or, equivalently, of large wavelengths, compared with the length H of the connecting channel.

Typically, one considers the spectral problem for the Helmholtz resonator with a small opening as a perturbation of the corresponding spectral problem for the Laplacian on $\Omega_{\text{int}} \cup \Omega_{\text{out}}$ without a connecting channel. This is then a standard perturbation problem with embedded eigenvalues. Lord Rayleigh noticed around 1916, see [39], that a small opening in the shell causes the transformation of the eigenvalues of the inner problem for the Laplacian in Ω_{int} into complex "resonances" and posed the problem of calculating these resonances. The spectral nature of resonances was understood only fifty years later, see [28]. It has become clear since then that the problem of calculating the resonances cannot be solved by methods of the spectral theory of self-adjoint operators, see [2, 3, 33, 34]. Researchers have used advanced asymptotic methods to approximately calculate the resonances for a small opening (see, e.g., [8, 9, 10, 11, 12]), but the spectral nature of resonances was hidden in technical details of the asymptotic approach.

Kirchhoff believed, see [21], that the resonator with a small opening can be replaced by a resonator with a "pointwise" opening localized at the center a of the upper lid, and supplied with an asymptotic boundary condition. He suggested an ansatz for the Green function of the above spectral problem, in the form of a linear combination of the unperturbed Green functions. The corresponding scattering ansatz (1), with an undefined coefficient A_{out}, satisfies the Helmholtz equation $-\Delta\Psi = \lambda\Psi$ in Ω_{out} and the Neumann homogeneous boundary condition on $\partial\Omega_{\text{out}}$. A similar ansatz appears in the earlier work of Faddeev and Pavlov [7] as an eigenfunction of a "fake" Hamiltonian for the Helmholtz resonator in the form of a self-adjoint extension, see [19], of the Neumann Laplacian defined by some asymptotic boundary conditions at the point a. This fake Hamiltonian considered in [7] contains several extension parameters which do not seem to have any naïve physical meaning, so fitting them turned out to be a problem. The general question of fitting all parameters of the model [7] has remained unsolved.

In this paper, we modify the model in such a way that it is fitted, in a sense, on a spectral interval Δ, namely, the scattering matrix of the modified model serves as an approximation on Δ of the full scattering matrix. The fitted solvable model can be applied in different situations. First, it can give an explicit approximate formula for the scattered waves in a remote zone, and it also enables us to calculate the Kirchhoff coefficients approximately. Using this solvable model, one can construct approximate solutions of the scattering problem with several resonators and, eventually, the methods developed here can serve as a jump-start in the two-step analytic perturbation procedure for embedded eigenvalues, as was suggested in [30, 26, 31] by using ideas of [35, 37, 38]. The procedure of fitting of the solvable model proposed in this paper is based on a rational approximation of the Neumann-to-Dirichlet mapping (ND-mapping) of the Laplacian in Ω_{int} and on transport properties of a short thin channel. In particular, we suggest an explicit formula for the Kirchhoff coefficient A_{out}. Most of the analytic results of this paper are derived under the assumption that the channel is short and thin, $kH \ll \pi/2$ and $\delta/H \ll 1$.

2. SOLVABLE MODELS FOR HELMHOLTZ RESONATORS

In this section, we recall the basic setup from [7] for the scattering problem in question. We assume that the upper and lower disks Γ_H and Γ of the channel Ω^{δ} are parts of the common boundary of the shell, of the outer domain Ω_{out} and the inner domain Ω_{int}, respectively, $\Gamma \subset \partial\Omega_{\text{int}}$, $\Gamma_H \subset \partial\Omega_{\text{out}}$. We restrict the inner and the outer Neumann Laplacian $L^{\text{int,out}} \to L_0^{\text{int,out}}$ to the smooth functions which vanish near the centers $a_{\Gamma_H} = a_H \in \partial\Omega_{\text{out}}$ and $a_\Gamma \in \partial\Omega_{\text{int}}$ of the upper and lower disks, respectively.

The deficiency indices of the restricted operators L_0^{int} and L_0^{out} are $(1, 1)$, and the deficiency elements at any complex point of the spectral parameter $\bar{\lambda}$ are the Green functions $G_{\text{int}}(x, a_\Gamma, \lambda)$ and $G_{\text{out}}(x, a_H, \lambda)$, see [7].

We next discuss the asymptotic formulas for the Green functions $G_{\mathrm{int,out}}$ with poles at points $x_H \in \Gamma_H$ and $x_\Gamma \in \Gamma$ on the upper and lower disks. We have

$$G_{\mathrm{int}}(x, x_\Gamma, \lambda) = G_{\mathrm{int}}(x, x_\Gamma, M) + C_\Gamma(x, x_\Gamma, M) + (\lambda - M)^3 \sum_{s=1}^{\infty} \frac{\psi_s(x)\,\psi_s(x_\Gamma)}{(\lambda_s - \lambda)(\lambda_s - M)^3}$$
$$= G_{\mathrm{int}}(x, x_\Gamma, M) + \mathcal{M}_{\mathrm{int}}(x_\Gamma, \lambda) + \cdots, \tag{2}$$

for $x \to x_\Gamma$ and $\lambda \neq \lambda_s$. Here the dots stand for terms vanishing at x_Γ and, for $x \in \Gamma$, M is an arbitrary spectral point (but will typically be large and negative in applications), $C_\Gamma(x, x_\Gamma, M)$ is a generalized kernel of a compact operator acting on Γ which does not depend on λ, whereas $\mathcal{M}_{\mathrm{int}}(x, x_\Gamma, \lambda)$ is a convergent spectral series

$$\mathcal{M}_{\mathrm{int}}(x, x_\Gamma, \lambda) = (\lambda - M)^3 \sum_{s=1}^{\infty} \frac{\psi_s(x)\,\psi_s(x_\Gamma)}{(\lambda_s - \lambda)(\lambda_s - M)^3}$$

with the sum over the discrete spectrum $\{\lambda_s\}$ and with associated eigenfunctions ψ_s. Note that, since the disks Γ, Γ_H are situated on flat pieces of the boundary of the shell, the inner and the outer logarithmic terms, e.g., $\alpha_{\mathrm{int}} \log|x - x_\Gamma| := (1/8\pi)\big[1/R_1 + 1/R_2\big] \log|x - x_\Gamma|$, ordinarily found on the right-hand side of (2) and depending on the mean curvature, see [36], can be omitted.

Similar asymptotic formulas for the outer Green function when $x \to x_H \in \Gamma_H$ contain a spectral characteristic of the outer problem which is represented in the form of the spectral integral over the scattered waves of L_{out} at the point x_H on the upper section. The eigenfunctions of continuous spectrum are $\psi_\omega(x, k)$ for $k \geqslant 0$, the eigenvalue, and $\omega \in \Sigma$, the unit sphere, $\psi_\omega(k^2, x) \approx e^{-ik\langle x,\omega\rangle} + e^{ik|x|}/(4\pi|x|)\; f_k(\omega, \nu)$ as $x \to \infty\cdot\nu$ with the scattering amplitude $f_k(\omega, \nu)$. If $x \to x_H$ and $\operatorname{Im}\lambda \neq 0$, then

$$G_{\mathrm{out}}(x, x_H, \lambda) = G_{\mathrm{out}}(x, x_H, M) + C_H(x, x_H, M) + (\lambda - M)^3 \frac{1}{8\pi^3} \int_0^\infty k^2\, dk \int_\Sigma d\omega \frac{\bar\psi_\omega(x, k)\,\psi_\omega(x_H, k)}{(k^2 - \lambda)(k^2 - M)^3}$$
$$= \frac{1}{2\pi|x - x_H|} + \mathcal{M}_{\mathrm{out}}(x_H,\, x). \tag{3}$$

Here $C_H(x, x_H, M)$ can be found by iterating the Hilbert identity which reads, suppressing the other variables, $G(\lambda) = G(M) + (\lambda - M)G(\lambda)G(M) = G(M) + (\lambda - M)[G(M) + (\lambda - M)G(\lambda)G(M)]G(M) = G(M) + (\lambda - M)G(M)G(M) + (\lambda - M)^2 G(M)G(\lambda)G(M) = \cdots$. The limit of the spectral integral on the positive semi-axis, as $\lambda \to \lambda_0 = k_0^2 \in \mathbb{R}_+$, on the right-hand side of (3), is defined by the Plejel (or Cauchy principal value) formula

$$\lim_{\lambda \to k_0^2} \frac{1}{8\pi^3} \int_0^\infty k^2\, dk \int_\Sigma d\omega \frac{\bar\psi_\omega(x, k)\,\psi_\omega(x_H, k)}{(k^2 - \lambda)(k^2 - M)^3}$$
$$= \frac{i\pi\, k_0^2}{8\pi^3} \int_\Sigma d\omega \frac{\bar\psi_\omega(x, k_0)\,\psi_\omega(x_H, k_0)}{2k_0(k_0^2 - M)^3} + \frac{PV}{8\pi^3} \int_0^\infty k^2\, dk \int_\Sigma d\omega\, \frac{\bar\psi_\omega(x, k)\,\psi_\omega(x_H, k)}{(k^2 - k_0^2)(k^2 - M)^3}$$
$$= \frac{i\pi}{(k_0^2 - M)^3} \frac{\partial \mathcal{E}}{\partial \lambda}(x, x_H, \lambda) + \frac{I}{(k_0^2 - M)^3} \mathcal{M}_{PV}(x, x_H, \lambda).$$

Here PV stands for the principal value of the integral and $\mathcal{E}$ for the spectral density of scattered waves.

By the uniqueness theorem for matrix R-functions [27], for any x_H, there are only finitely many degenerate points λ_0 (with $\frac{\partial \mathcal{E}}{\partial \lambda}(x_H, x_H, \lambda_0) = 0$) on the chosen essential spectral interval Δ.

Assumption 1. Assume that, for a given small flat upper disk centered at a_H, there are no zeros of the imaginary part $\operatorname{Im} G_{\mathrm{out}}(x, y, \lambda + i0)$ on the chosen essential interval Δ,

$$\operatorname{Im} G_{\mathrm{out}}(x, y, \lambda + i0) \approx \pi\, (\partial \mathcal{E}/\partial \lambda)(a_H, a_H, \lambda) > 0, \tag{4}$$

for $\lambda = k^2 \in \Delta$, $x, y \in \Gamma_H$.

Now, combining the asymptotic formula (3) for the Green function at the pole $x_H \in \Gamma_H$ with the asymptotic formula for the imaginary part of $G_{\mathrm{out}}(x, x_H, \lambda + i0)$ as $x \to x_H$, we obtain the asymptotic formula for the outer Green function as $x \to x_H$.

Lemma 2.1. *An asymptotic formula for the outer Green function $G(x, x_H, \lambda + i0)$ on the upper shore of the continuous spectrum $[0, \infty)$, as $x \to x_H$, is given by*

$$G_{\text{out}}(x, x_H, \lambda + i0) = \frac{1}{2\pi|x - x_H|} + C(x, x_H, M) + i\pi \frac{\partial \mathcal{E}}{\partial \lambda}(x_H, x_H) + \mathcal{M}_{PV}(x_H, x_H, \lambda) + \cdots$$
$$= \frac{1}{2\pi|x - x_H|} + \mathcal{M}_{\text{out}}(x, x_H) + \cdots , \tag{5}$$

where the dots stand for terms vanishing at x_H.

Remark 1. Due to Assumption 1, the imaginary part of $\mathcal{M}_{\text{out}}(x, x_H)$ for $\lambda \in \mathbb{R}$ is strictly positive on the essential spectral interval Δ. We do not provide an explicit formula for $\mathcal{M}_{\text{out}}(x, x_H)$ here and assume that this quantity is known, as well as the necessary information on the inner and outer Neumann Laplacians.

In [7], a solvable model of the Helmholtz resonator was suggested in the form of a self-adjoint extension of the orthogonal sum $L_0^{\text{int}} \oplus L_0^{\text{out}}$ of the restricted Neumann Laplacian on $\Omega_{\text{int,out}}$. The domain of the extension is obtained by imposing a special boundary condition on the asymptotic boundary values A, B as $x \to a_\Gamma$, a_H. As was assumed, $H = 0$, and hence $a_\Gamma = a_H$. Thus, the asymptotic boundary values $A_{\text{int,out}}$ and $B_{\text{int,out}}$ are defined as the coefficients at the leading terms at $a_\Gamma = a_H = a$,

$$u_{\text{int}} = \frac{A_{\text{int}}}{2\pi|x - a|} + B_{\text{int}} + \cdots , \qquad u_{\text{out}} = \frac{A_{\text{out}}}{2\pi|x - a|} + B_{\text{out}} + \cdots , \tag{6}$$

of elements in the domain of the corresponding adjoint operators $(L^{\text{int}})_0^+ \oplus (L^{\text{out}})_0^+$. The boundary forms of the adjoint operators are calculated in terms of the boundary values A and B. For instance,

$$\mathcal{J}_{\text{out}}^{u,v} := \lim_{\varepsilon \to 0} \int_{\Omega_{\text{out}} \setminus B(a,\varepsilon)} [-\Delta \bar{u}\, v + \bar{u}\, \Delta v]\, dx^3 = \bar{B}_{\text{out}}^u A_{\text{out}}^v - \bar{A}_{\text{out}}^u B_{\text{out}}^v. \tag{7}$$

Here $B(a, \varepsilon) = \{x \in \mathbb{R}^3 : |x - a| \leqslant \varepsilon\}$. A similar formula holds for the inner boundary form. Therefore, the sum of the inner and the outer boundary forms vanishes if the asymptotic boundary values $\vec{A} := (A_{\text{int}}, A_{\text{out}})$ and $\vec{B} := (B_{\text{int}}, B_{\text{out}})$ are subjected to a self-adjoint boundary condition with a Hermitian 2×2 matrix β (for instance, if $\beta \vec{B} = \vec{A}$).

Unfortunately, this naïve model cannot be fitted to a resonator with nonzero channel length $H > 0$. For a thin short channel ($\delta \approx 0$), another (*modified*) model can be constructed by using the same outer operator $L_{\text{out}}^N =: L_{\text{out}}$ and an "inner structure" $\mathbf{A}, E$ attached to the lower end Γ of the one-dimensional "link." This linking is an interval $[a_\Gamma, a_H]$, of length H, and the inner structure is formed by a finite-dimensional Hilbert space $E_\Gamma =: E$ and a Hermitian matrix $\mathbf{A}\colon E \to E$. The modified model is constructed as a "zero-range model with an inner structure," see, e.g., [29, 4]. We shall show (see Section 5) that this model can be fitted on a chosen essential spectral interval.

Here we use the standard notation for the zero-range models with inner structure that can be found in, say, [31] and which we briefly recall. The elements in the domain of the formal adjoint operator $U = U_0 + \frac{\mathbf{A}}{\mathbf{A} - iI}\xi_+ - \frac{I}{\mathbf{A} - iI}\,\xi_-$, where U_0 is chosen from the restricted domain D_0, and the elements $\xi_\pm \in N_i$ of the one-dimensional deficiency subspace N_i play the role of symplectic coordinates for every element U in the domain $D_0 + N_i + N_{-i} = D_0^+$ of the formal adjoint operator $\mathbf{A}_0^+$ of the restricted operator A_0. The corresponding boundary form is represented as

$$\mathcal{J}_\Gamma^{\xi_\pm} = \langle \xi_+^u, \xi_-^v \rangle_E - \langle \xi_-^u, \xi_+^v \rangle_E, \tag{8}$$

see [29, 31]. The symplectic coordinates $\xi_\pm \in N_i$ of the solution of the adjoint homogeneous equation $(\mathbf{A}_0^+ - \lambda I)U = 0$ are connected (see [19]) by the Krein matrix function $\mathcal{M}$,

$$\xi_- = -P \frac{I + \lambda \mathbf{A}}{\mathbf{A} - \lambda I} P\, \xi_+ =: -\mathcal{M}\, \xi_+, \tag{9}$$

with $\mathcal{M} = \text{Trace}\, P\frac{I+\lambda\mathbf{A}}{\mathbf{A}-\lambda I}P$. Here P is a 1D orthogonal projection onto N_i. Generally, the Krein function and its inverse $-\mathcal{M}^{-1}$ are R-functions which admit the standard Herglotz representation

$$\mathcal{M}(\lambda) := \text{Trace}\, P\frac{I+\lambda\mathbf{A}}{\mathbf{A}-\lambda I}P = \mathcal{M}_0 + \mathcal{M}_1\lambda + \sum_{l=1}^{N}\frac{1+\lambda A_l}{A_l-\lambda I}q_l, \tag{10}$$

where $\mathcal{M}_0$ is real, $\mathcal{M}_1$ is positive, A_l are the eigenvalues of $\mathbf{A}$, and $q_l = \text{Trace}\, Q_l P$ are the spectral projections Q_l of $\mathbf{A}$ framed as $\text{Trace}\, PQ_lP$ by the projections P onto the chosen 1D deficiency subspace N_i.

To construct a fittable model of the Helmholtz resonator, we attach the inner structure $\{E, \mathbf{A}\}$ to the pointwise opening $a = a_H$ via a one-dimensional link, of length H, connecting the points (a_Γ, a_H), see Fig. 2, and set $\mathcal{M} =: \mathcal{M}_\Gamma$.

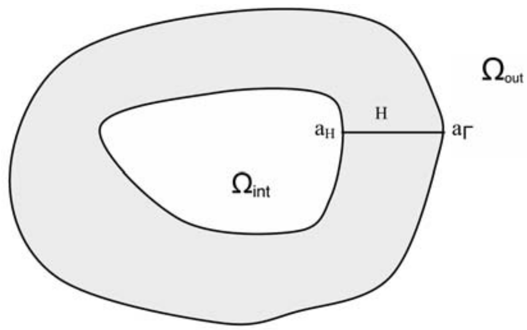

Fig. 2. Inner link structure for the cavity of a resonator.

We assume that the link (a_Γ, a_H) is directed (inner to outer) and one-dimensional, with the operator $-d^2$ on it. Then the transmission of the Neumann/Dirichlet data $u'_\Gamma, u_\Gamma \longrightarrow u'_H, u_H$ from the lower end a_Γ of the channel a_γ to the upper end a_H is defined by the 1-dimensional Neumann-to-Dirichlet mapping of the link, in particular:

$$\mathcal{ND}_\Gamma \longrightarrow \mathcal{ND}_H, \tag{11}$$

where $\mathcal{ND}_\Gamma = \frac{u_\Gamma}{u'_\Gamma} =: -\frac{1}{\mathcal{M}_\Gamma}$ and $\mathcal{ND}_H = \frac{-1+\lambda^{-1/2}\tan\sqrt{\lambda}H\mathcal{M}_\Gamma}{\lambda^{1/2}\tan\sqrt{\lambda}H+\mathcal{M}_\Gamma} =: -\frac{1}{\mathcal{M}_H}$. The direction of differentiation is defined by the vector $(a_\Gamma,\, a_H)$. Considering the boundary forms $J_\Gamma^{u,v} = \bar{u}'v - \bar{u}v'\big|_\Gamma$, together with the boundary form of the inner structure $\mathcal{J}_\Gamma^{\xi_\pm} = \bar{\xi}_+^u\xi_-^v - \bar{\xi}_-^u\xi_+^v$, we connect the symplectic coordinates $\xi_\pm^u$ with u'_Γ, u'_Γ at a_Γ by the symmetric boundary condition,

$$\begin{pmatrix}\xi_+\\ u'_\Gamma\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\xi_-\\ u_\Gamma\end{pmatrix}, \tag{12}$$

and thus the sum of the forms vanishes, $\mathcal{J}_\Gamma^{\xi_\pm} + J_\Gamma^{u,v} = 0$, if the symplectic variables $\xi_\pm^u$, u_Γ, u'_Γ belong to the same element (U, u) of the domain of the adjoint $\mathbf{A}_0^+ \oplus -(d_0^2)^+$ of the properly restricted operator. In this way, we also connect $\mathcal{M} =: \mathcal{M}_\Gamma$ with the Krein function of the inner structure,

$$\xi_- = -P\frac{I+\lambda\mathbf{A}}{\mathbf{A}-\lambda I}\,\xi_+ \; = -\mathcal{M}\xi_+ =: -\mathcal{M}_\Gamma\,\xi_+. \tag{13}$$

The boundary form of the Laplacian in Ω_{out} and the boundary form on the link at the upper lid are $\mathcal{J}_H^{B,A} = \bar{B}^u\,A^v - \bar{A}^u\,B^v$ and $\mathcal{J}_H^{u,v} = -\bar{u}'_H v_H + \bar{u}_H\,v'_H$. The sum of these forms vanishes if a symmetric boundary condition is imposed at the upper end a_H of the link,

$$\begin{pmatrix}0 & \beta_{01}\\ \beta_{10} & \beta_{11}\end{pmatrix}\begin{pmatrix}B\\ -u'_H\end{pmatrix} = \begin{pmatrix}A\\ u_H\end{pmatrix}. \tag{14}$$

We now have the following elementary lemma.

Lemma 2.2. *The Weyl function of the link at the upper lid,* $\frac{u_H}{u'_H} = -\frac{1}{\mathcal{M}_H} = \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}} - \frac{1}{\cos^2\sqrt{\lambda}H}\frac{1}{\sqrt{\lambda}\tan\sqrt{\lambda}H + \mathcal{M}_\Gamma}$, *for any relatively small* $\sqrt{\lambda}H \ll 1$ *is approximately equal to the expression* $-\frac{1}{\mathcal{M}_H} \approx H - \frac{1}{\mathcal{M}_\Gamma(\lambda)+\lambda H}$, *with a small error order* $\lambda H^2 \ll 1$, *on the complement of a small neighborhood of zeros of the denominator.*

This lemma will be used when fitting the solvable model of the Helmholtz resonator in Section 5.

Theorem 2.1. *The hybrid operator constructed of the inner structure, the double differentiation on the link and the Neumann Laplacian in the outer domain, and supplied with the boundary conditions* (12), (14) *at the lower and upper lids of the link, respectively, is self-adjoint and defines a solvable model of the Helmholtz resonator parametrized by the boundary conditions and by the inner structure* $\mathbf{A}$. *The outer Kirchhoff constant of the model thus constructed is*

$$-\frac{\psi_{\mathrm{out}}(\omega, a_H)}{\mathcal{M}_{\mathrm{out}} + |\beta_{01}|^{-2}\left[-\frac{1}{\mathcal{M}_H} + \beta_{11}\right]} =: A_{\mathrm{mod}}, \tag{15}$$

and the additional term of the amplitude of the solvable model, for the above boundary condition and the inner structure, is represented as

$$\frac{1}{8\pi^3}\;\frac{\psi_{\mathrm{out}}(\omega, a_H)\psi_{\mathrm{out}}(\nu, a_H)}{\mathcal{M}_{\mathrm{out}} + |\beta_{01}|^{-2}\left[-\frac{1}{\mathcal{M}_H} + \beta_{11}\right]} =: \delta a_{\mathrm{mod}}(\omega, \nu, \lambda), \tag{16}$$

and thus the full model amplitude is equal to $a_\beta(\omega, \nu, k) = a_{\mathrm{out}}(\omega, \nu, k) + \delta a_{\mathrm{mod}}(\omega, \nu, \lambda)$.

Here a_{out} *is the scattering amplitude of the outer operator* L_{out} *and* $\mathcal{M}_{\mathrm{out}}$ *is the kernel defined in* (3). *On the smooth part of the boundary, it is continuous and can be calculated for* $x = x_H$.

Note that, below, the choice of inner structure defined by (65) or by Theorem 5.2 connects the characteristics $\mathcal{D}$ of the inner problem, see formula (51), with the inner structure.

Proof. Denote by $-\triangle_0$ the restriction of the outer Neumann Laplacian to the functions vanishing near a_H. Then the restricted operator $\mathbf{A}_0 \oplus -d^2 \oplus -\triangle_0$ (defined on functions vanishing at the ends of the link near the point a_H on the upper lead and on the vectors $e \in E_0$ of the inner space such that $\mathbf{A}e - ie\perp N_i$) is symmetric and has equal finite deficiency indices $(3, 3)$. Then, after switching to the original Neumann version of the extension procedure, which is based on isometries, we conclude that this is also self-adjoint, because every isometry between two finite-dimensional spaces is injective.

Then our formulas for the scattering amplitude and for the Kirchhoff constant are obtained as the joint solution of the linear system (12), (14) with the help of (11).

The model constructed in this way is parametrized by the inner structure $\mathbf{A}$ and by the matrix $\{\beta_{ik}\}$. One may guess that the eigenvalues of the inner structure $\mathbf{A}$ should simulate properly renormalized eigenvalues of the Neumann Laplacian on the cavity. The boundary parameters β_{ik} have no naïve physical meaning. We shall fit these parameters by the comparison of the model scattering amplitude with the approximate full scattering amplitude.

The referee points out that constants similar to that of A_{out}, which was defined at (6), can be obtained from the asymptotic results of Gadyl'shin [13–15]. These depend on the asymptotic behavior at infinity of special harmonic functions in an unbounded domain. We shall discuss these relationships elsewhere.

3. SCATTERING MATRIX OF THE RESONATOR VIA THE NEUMANN-TO-DIRICHLET MAPPING

We proceed in this section with studying the extended inner domain $\Omega_{\mathrm{int}} \cup \Omega^\delta =: \Omega^*_{\mathrm{int}}$ and considering two boundary problems, namely, the "inner" and the "extended inner" problem for the Neumann Laplacian L_{int}, L^*_{int} in Ω_{int}, Ω^*_{int}, respectively, with the normal on Γ, Γ_H, directed towards

$\Omega_{\rm out}$, $-\Delta u = \lambda u$, $\partial u/\partial n\big|_\Gamma = \rho_\Gamma$, $x \in \Omega_{\rm int}$, $-\Delta u = \lambda u$, $\partial u/\partial n\big|_{\Gamma_H} = \rho_H$, and $x \in \Omega^*_{\rm int}$. Solutions to these problems are given by integral transforms with kernels defined by the corresponding Neumann Green functions $G_{\rm int}, G^*_{\rm int}$ of the inner and the extended inner problems, respectively. For instance,

$$[\mathbf{Q}^*_{\rm int}(\lambda)\,\rho_H\,](x) = \int_{\Gamma_H} G^*_{\rm int}(x,\, y,\, \lambda)\rho_H(y)d\Gamma, \quad x \in \Omega^*_{\rm int}. \tag{17}$$

The trace of $\mathbf{Q}^*_{\rm int}\rho_{\Gamma_H}$ on Γ_H defines the restrictions of the standard Neumann-to-Dirichlet mapping in $\Omega^*_{\rm int}$ to Γ_H, $\partial \mathbf{Q}^*_{\rm int}\rho_H/\partial n\big|_{\Gamma_H} = \rho_H$, $\partial \mathbf{Q}^*_{\rm int}\rho_H/\partial n\big|_{\partial\Omega^*_{\rm int}\setminus\Gamma_H} = 0$. Hence, the trace of $\mathbf{Q}^*_{\rm int}\rho_H$ on Γ_H coincides with the ND-mapping $\mathcal{ND}^*_{\rm int}\rho_H$. The inverse mapping $[\mathcal{ND}^*_{\rm int}]^{-1}$ exists if λ is not an eigenvalue of the corresponding "mixed" boundary problem $-\Delta u = \lambda u$, $\partial u/\partial n\big|_{\partial\Omega^*_{\rm int}\setminus\Gamma_H} = 0$, $u\big|_{\Gamma_H} = 0$, and then this inverse is the associated *relative* Dirichlet-to-Neumann mapping obtained as the restriction to Γ_H of the boundary current for the solution of the *relative* Dirichlet boundary problem $-\Delta u = \lambda u$, $\partial u/\partial n\big|_{\partial\Omega^*_{\rm int}\setminus\Gamma_H} = 0$, $u\big|_{\Gamma_H} = u_H$, $\partial u/\partial n\big|_{\Gamma_H} =: \mathcal{DN}^*_{\rm int}u_H$, and

$$\mathcal{DN}^*_{\rm int}\ \mathcal{ND}^*_{\rm int}\ \rho_H = \rho_H. \tag{18}$$

A similar statement holds for Γ.

One can consider a similar boundary problem in $\Omega_{\rm out}$ with the normal on Γ_H directed towards $\Omega_{\rm out}$, $-\Delta u = \lambda u$, $\partial u/\partial n\big|_{\Gamma_H} = -\rho_H$, and $\partial u/\partial n\big|_{\partial\Omega_{\rm out}\setminus\Gamma_H} = 0$. The solution of this problem is given by the integral transform $u = [\mathbf{Q}\rho_H](x) = \int_{\Gamma_H} G^N_{\rm out}(x,s)\rho_H(s)d\Gamma_H$, because, with the normal defined as above, $\partial\mathbf{Q}\rho_H/\partial n\big|_H = -\rho_H$, the corresponding *standard* Neumann-to-Dirichlet mapping $\mathcal{ND}_{\rm out}$ (associated with the normal on Γ_H directed outside $\Omega_{\rm out}$) is defined by the trace of $\mathbf{Q}\rho_H$ on Γ_H, $\mathcal{Q}_{\rm out}\rho_H = \text{Trace}_{\Gamma_H}\mathbf{Q}_{\rm out}\rho_H$. It is again convenient to note that the inverse $[\mathcal{ND}^*_{\Gamma_H}]^{-1}$ of the ND-mapping associated with Γ_H is obtained as the *relative* Dirichlet-to-Neumann mapping for the boundary problem on $\Omega_{\rm out}$ with relative Dirichlet boundary data, for instance, $-\Delta u = \lambda u$, $\partial u/\partial n\big|_{\partial\Omega_{\rm out}\setminus\Gamma_H}=0$, $u\big|_{\Gamma_H} = u_{\Gamma_H}$, and $\partial u/\partial n\big|_{\Gamma_H} =: \mathcal{DN}^H_{\rm out}\ u_{\Gamma_H}$. Matching outer normals on Γ gives

$$\mathcal{DN}^H_{\rm out}\ \mathcal{ND}^H_{\rm out}\big|_{\Gamma_H}\ \rho_{\Gamma_H} = \rho_{\Gamma_H}. \tag{19}$$

As was proved in [32], the singularities of $\mathcal{DN}_{\rm int}(\lambda)$ (regarded as an unbounded operator on $W_2^{3/2}(\Gamma)$) and the poles of $\mathcal{DN}_{\rm int}(\lambda)$ at the eigenvalues of the inner Dirichlet problem can be separated, see Theorems 3.1 and 3.2 below. These statements are valid both in case of classical DN-mappings and in case of the relative DN-mapping, by the above formulas (18) and (19). In the following theorems, Theorems 3.1 and 3.2 quoted from [32], we mean, respectively, both the standard and relative DN-mappings associated with Dirichlet or relative Dirichlet boundary problems,

Theorem 3.1. *Consider the Dirichlet Laplacian $L^D_{\rm int}$ or the relative Dirichlet Laplacian in $L_2(\Omega_{\rm int})$ on a compact domain $\Omega_{\rm int} \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega = \Gamma$ or $\partial\Omega \supset \Gamma$, respectively. The DN-mapping (the relative DN-mapping) of $L^D_{\rm int}$ has the following representation on the complement of the corresponding spectrum $\sigma^D_{\rm int}$ in the complex λ plane, for $M > 0$:*

$$\mathcal{DN}^\Gamma(\lambda) = \mathcal{DN}^\Gamma(M) - (\lambda - M)\mathcal{P}^+(M)\mathcal{P}(M) - (\lambda - M)^2\mathcal{P}^+(M)R_\lambda\mathcal{P}(M), \tag{20}$$

where R_λ stands for the resolvent of $L^D_{\rm int}$ and $\mathcal{P}(-M)$ for the corresponding Poisson kernel. A similar formula is true for the ND mapping and, after two iterations of the resolvent equation, we obtain

$$\mathcal{ND}_\Gamma(\lambda) = \mathcal{ND}_\Gamma(M) + (\lambda - M)\mathbf{Q}^+_{\rm int}(M)\mathbf{Q}_{\rm int}(M) + (\lambda - M)^2\mathbf{Q}^+_{\rm int}(M)R_\lambda\mathbf{Q}_{\rm int}(M). \tag{21}$$

Here $\mathcal{ND}_\Gamma(\lambda)$ is the trace of $\mathbf{Q}(\lambda)\rho$ on Γ. The operators $\mathcal{DN}^\Gamma(M)$ and $\mathcal{P}^+(M)\mathcal{P}(M)$ are bounded from $W_2^{3/2}(\Gamma)$ to $W_2^{1/2}(\Gamma)$ (onto) and bounded on $W_2^{3/2}(\Gamma)$, respectively, and the operator function

$$\left[\mathcal{P}^+(M)R_\lambda\mathcal{P}(M)\right](x_\Gamma, y_\Gamma) = \sum_{\lambda_s\in\Sigma_L} \partial\varphi_s/\partial n(x_\Gamma)\,\partial\varphi_s/\partial n(y_\Gamma)\,(\lambda_s - M)^{-2}(\lambda_s - \lambda)^{-1} \tag{22}$$

is compact in $W_2^{3/2}(\Gamma)$.

On the continuous spectrum, $\lambda \geqslant 0$, the Dirichlet-to-Neumann mapping is defined as the boundary current of the outgoing solution of the corresponding boundary problem, which is obtained as a limit of the square-integrable solution $\lim_{\varepsilon\to 0} u_{\lambda+i\varepsilon}$ of the boundary problem with data on Γ.

Let us calculate the boundary currents for the inner and outer domain by differentiating the outgoing solution of the corresponding boundary problem (in the outward direction on Γ_H). A statement similar to Theorem 3.1 quoted above is also true for the DN-mapping of the exterior domain, see [32]. In particular,

$$\mathcal{DN}_{\rm out}(\lambda) = \mathcal{DN}_{\rm out}(M) - (\lambda - M)\mathcal{P}^+(M)\mathcal{P}(M) - (\lambda - M)^2\mathcal{P}^+(M)R_\lambda\mathcal{P}(M), \tag{23}$$

with the only difference that the first terms of the decomposition contain the DN-mapping and the Poisson mapping for the exterior domain and the generalized kernel in the last term is represented via the integral over the absolutely continuous spectrum $\sigma_L^a = [0, \infty)$, and the integrand is combined of normal derivatives of the scattered waves $\psi(x, |k|, \nu)$, $k = |k|\nu$, $|\nu| = 1$, $\mathrm{Im}\,\lambda \neq 0$, namely,

$$\mathcal{P}^+(M)R_\lambda\mathcal{P}(M)(x_\Gamma,\, y_\Gamma) = (2\pi)^{-3}\int_{|k|^2\,\in\,\Sigma_L^a} \frac{\partial\bar{\psi}/\partial n(x_\Gamma,\, |k|, \nu)\,\partial\psi_s/\partial n(y_\Gamma\, |k|, \nu)}{(|k|^2 - M)^2(|k|^2 - \lambda)} d^3k.$$

The absolutely continuous spectra $\sigma_{\rm out}^{D,N}$ of both the Dirichlet and Neumann Laplacians $L_{\rm out}^{D,N}$ fill the positive semi-axis $0 \leqslant \lambda < \infty$ with infinite multiplicity, and the scattered waves $\psi(x, k)$ are parametrized by the energy $\lambda = k^2 > 0$ and the direction $\nu, |\nu| = 1$, or just by the *momentum* $k\nu \in \mathbb{R}^3$.

The normal limit values of $\mathcal{ND}_{\rm out}$ can be calculated as above, due to absolute continuity of the spectrum of $L_{\rm out}$, via the principal values $\lim_{\lambda\to\lambda_0+i0}\mathcal{ND}_{\rm out} = i\pi d\mathcal{E}/d\lambda + PV\,\mathcal{Q}_{\rm out}$, where $PV\,\mathcal{Q}_{\rm out}$ is found as the principal value of the sum of spectral integrals. For instance, if $M > 0$, then

$$PV\,\mathcal{Q}_{\rm out}(M)\,R_\lambda\,\mathcal{Q}_{\rm out}(M) = \frac{1}{(2\pi)^3}PV\int_0^\infty k^2dk\,\int_\Sigma d\Sigma\frac{\psi(x_\Gamma, k, \nu)\psi(y_\Gamma, k, \nu)}{(k^2 - M)^2(k^2 - \lambda)},$$

where the PV limit of the integrals is taken over the complements of a sequence of small nesting intervals centered at $\lambda \in \mathbb{R}_+$.

In [32], the Hilbert identity is transformed into the corresponding Krein formula for the resolvent, see (24) below. The transformed expression has a form of Schur complement, see [25], and contains the denominator $\mathcal{Q}_{\rm int}^*(\lambda) + \mathcal{Q}_{\rm out}(\lambda)$ constructed of Neumann-to-Dirichlet mappings of the extended inner and the outer boundary problems. The Krein formula connects the resolvent of the self-adjoint operator $\mathcal{L}$ on the composite domain $\Omega_{\rm int}^* \cup \Omega_{\rm out}$ with the characteristics of the orthogonal sum of the self-adjoint operators $L_{\rm int}^* \oplus L_{\rm out}$ defined in $L_2(\Omega_{\rm int}^*) \oplus L_2(\Omega_{\rm out})$ by the homogeneous Neumann boundary conditions.

Theorem 3.2. *The resolvent kernel $G(x, y, \lambda)$ of the operator $\mathcal{L}$ for regular λ and x, y in $\Omega_{\rm out}$ is represented by the Krein formula*

$$G(x, y, \lambda) = G_{\rm out}(x, y, \lambda) - G_{\rm out}(x, *, \lambda)\left[\mathcal{Q}_{\rm int}^*(\lambda) + \mathcal{Q}_{\rm out}(\lambda)\right]^{-1} G_{\rm out}(*, y, \lambda), \tag{24}$$

where the asterisk stands for the variable in Γ_H. In particular, the above formula implies a relationship between the scattered waves $\psi(x, \nu, \lambda)$ of the perturbed problem (with opening) and the corresponding scattering amplitude with the corresponding characteristics of the outer Neumann problem,

$$\psi(x, \nu, \lambda) = \psi_{\rm out}(x,\, \nu, \lambda) - G_{\rm out}(x, *, \lambda)\left[\mathcal{Q}_{\rm int}^*(\lambda) + \mathcal{Q}_{\rm out}(\lambda)\right]^{-1} \psi_{\rm out}(*, \nu, \lambda), \tag{25}$$

$$a(\omega, \nu, \lambda) = a_{\rm out}(\omega, \nu, \lambda) + \frac{1}{8\pi^3}\psi_{\rm out}(*,\, \omega, \lambda)\left[\mathcal{Q}_{\rm int}^*(\lambda) + \mathcal{Q}_{\rm out}(\lambda)\right]^{-1} \psi_{\rm out}(*,\, \nu, \lambda). \tag{26}$$

Formulas (25), (26) *admit an analytic continuation on the spectral sheet of the variable $\lambda = k^2$* ($\mathrm{Im}\,k > 0$), *and all operator functions on the upper and lower shores of the continuous spectrum are calculated as weak limits of the values in the corresponding half-planes.*

Formula (26) is a multi-dimensional analog of the popular 1D formula for the scattering matrix in terms of the Weyl function, see, e.g., [16]. The role of the Weyl function in (26) is played by the

Dirichlet-to-Neumann mapping (actually, its inverse, the ND-mapping). For the use of DN-mapping in spectral analysis, we refer the reader to the recent papers [17, 18].

Remark 2. The calculation of the amplitude by using (26) requires the solution of the equation

$$\big[\mathcal{Q}^*_{\rm int}(\lambda) + \mathcal{Q}_{\rm out}(\lambda)\big]u = \psi_{\rm out}(*, \nu, \lambda). \tag{27}$$

Both the operators on the left-hand side of this equation exist and, generically, are mappings $W_2^{1/2}(\Gamma) \xrightarrow{\mathcal{Q}_{\rm out}} W_2^{3/2}(\Gamma)$ if λ is not an eigenvalue of the relative Dirichlet problem $-\Delta u = \lambda u$, $u\big|_\Gamma = 0$, $\partial u/\partial n\big|_{\partial\Omega_{\rm int}\setminus\Gamma} = 0$. Both $\mathcal{Q}_{\rm out}$ and $\mathcal{Q}^*_{\rm int}$ can be extended from $W_2^{1/2}(\Gamma_H)$ to $L_2(\Gamma_H)$. The extended operators are compact in $L_2(\Gamma_H)$. Their sum is also a compact operator. To construct the corresponding inverse, we must regularize the problem.

If λ_0 is an eigenvalue of the relative Dirichlet problem and u_0 is the corresponding eigenfunction, then $\partial u_0/\partial n \xrightarrow{\mathcal{Q}_{\rm int}} 0$, and hence $\mathcal{Q}_{\rm int}$ has a zero eigenvalue. Moreover, $\mathcal{Q}_{\rm int}L_2(\partial\Omega) = L_2(\partial\Omega) \ominus \big\{\partial u_0/\partial n\big|_\Gamma\big\}$. Due to the absence of eigenvalues of the outer problem, the operator $\mathcal{Q}_{\rm out}$ is invertible, and the inverse is an operator of differential order 1, $\mathcal{Q}^{-1}_{\rm out} = \mathcal{DN}^H_{\rm out}\colon W_2^{3/2}(\Gamma_H) \to W_2^{1/2}(\Gamma_H)$ or $\mathcal{Q}^{-1}_{\rm out} = \mathcal{DN}^H_{\rm out}\colon W_2^1(\Gamma_H) \to L_2(\Gamma_H)$. The inverse coincides with the relative DN-mapping which is associated with the boundary problem $-\Delta u = \lambda u$, $\partial u/\partial n\big|_{\partial\Omega_{\rm out}\setminus\Gamma_H} = 0$, $u\big|_{\Gamma_H} = u_H$, with the Meixner condition imposed on the inner domain ($u \in W^2_{2,\rm loc}(\Omega) \cap W_2^1(\Omega)$): $\mathcal{DN}^H_{\rm out} := \mathcal{DN}^{\Gamma_H}_{\rm out}\colon u_H \longrightarrow c\partial u/\partial n\big|_{\Gamma_H}$.

Because of (21), the operator $\mathcal{Q}^*_{\rm int}(\lambda)$ can be represented as

$$\mathcal{Q}^*_{\rm int}(\lambda) = \mathcal{Q}^*_{\rm int}(M) + (\lambda - M)\mathbf{Q}^+_{\rm int}(M)\mathbf{Q}_{\rm int}(M) + (\lambda - M)^2\mathbf{Q}^+_{\rm int}(M)R_\lambda\mathbf{Q}_{\rm int}(M). \tag{28}$$

Here we need no smoothness, because the Meixner conditions ensure the convergence of the integrals.

The third term on the right-hand side of (28) can be represented as a series (of one-dimensional polar terms) convergent with respect to the operator norm on $W_2^1(\Gamma) \times L_2(\Gamma)$,

$$\begin{aligned}(\lambda - M)^2 \sum_{l=1}^N \frac{\varphi_l\big|_\Gamma\rangle\ \langle\varphi_l\big|_\Gamma, g\rangle}{(\lambda_l - M)^2(\lambda_l - \lambda)} + (\lambda - M)^2\ O\Big(\sum_{N+1}^\infty \lambda_l^{-3+\alpha/2+\alpha'/2}\Big)\\ = \mathcal{Q}^N_{\rm int} + (\lambda - M)^2\ O\Big(\sum_{N+1}^\infty \lambda_l^{-3+\alpha/2+\alpha'/2}\Big) =: \mathcal{Q}^N_{\rm int} + \mathcal{K}^N,\end{aligned} \tag{29}$$

with the estimates $\alpha > 1/2$, $\alpha' > 3/2$, and $\alpha/2 + \alpha'/2 < 3/2$ following from embedding results. The details are like those for the similar estimate in [24].

Note that the compact operator $\mathcal{Q}_{\rm out}\colon L_2(\Gamma) \to W_2^1(\Gamma)$ is invertible. Its inverse exists for any λ, $\operatorname{Im}\lambda \geqslant 0$, and acts as the relative DN-mapping $\mathcal{DN}_{\rm out}$ associated with the generalized $W_2^{3/2}(\Omega_{\rm out})$ solution of the boundary problem $-\Delta u = \lambda u$, $\partial u/\partial n\big|_{\partial\Omega_{\rm out}\setminus\Gamma} = 0$, $u\big|_\Gamma = u_\Gamma$, $\operatorname{Im}\lambda > 0$. The generalized $W_2^{3/2}(\Omega_{\rm out})$ solution of this problem is unique, see [22], and the corresponding relative DN mapping is a closed operator $\mathcal{DN}_\Gamma : W_2^1(\Gamma) \to L_2(\Gamma)$. Then the inverse operator is a closed mapping $\mathcal{Q}_{\rm out}$ onto $L_2(\Gamma) \to W_2^1(\Gamma)$.

The operator R-function on the upper half-plane $\operatorname{Im}\lambda \geqslant 0$, $\mathbf{Q}(\lambda) := \mathcal{Q}_{\rm out}(\lambda) + \mathcal{Q}^*_{\rm int}(M) + (\lambda - M)\mathbf{Q}^+_{\rm int}(M)\mathbf{Q}_{\rm int}(M)$, is compact in $L_2(\Gamma)$ and defines a closed operator $L_2(\Gamma) \to W_2^1(\Gamma)$. The mapping is "onto" if $\mathbf{Q}(\lambda)$ does not have the zero eigenvalue. In this case, the corresponding inverse exists and is bounded, due to the closed graph theorem, see, e.g., [6], $\mathbf{Q}^{-1}(\lambda)\colon W_2^1(\Gamma) \to L_2(\Gamma)$, $\|\mathbf{Q}^{-1}(\lambda)\|_{W_2^1(\Gamma)\times L_2(\Gamma)} < \infty$. The operator R-function $\mathbf{Q}(\lambda)$ is smooth on the closed upper half-plane, and its vector zeros μ_s are real, $\mathbf{Q}(\mu_s)e_s = 0$, $e_s \in L_2(\Gamma)$. Further, according to [27], it can have only finitely many vector zeros of this kind on any finite interval of the real axis of spectral parameter. Denote by Δ_μ the finite set of all vector zeros on the essential spectral interval Δ and select, for a given rational approximation (29), a real neighborhood of Δ_μ such that the condition

$$\sup_{\lambda \in \Delta \setminus \Delta_\mu} \|\mathbf{Q}^{-1}(\lambda)\mathcal{K}^N\|_{L_2(\Gamma)} =: q < 1 \tag{30}$$

holds on the complement of Δ_μ in Δ. Then the operator function $\mathbf{Q}+\mathcal{K}^N(\lambda)$ is invertible on $\Delta\setminus\Delta_\mu$, and equation (27) can be rewritten on $\Delta\setminus\Delta_\mu$ in a finite-dimensional form,

$$u + \left[\mathbf{Q} + \mathcal{K}^N(\lambda)\right]^{-1} \mathcal{Q}^N_{\text{int}} u = \left[\mathbf{Q} + \mathcal{K}^N(\lambda)\right]^{-1} \psi_{\text{out}}(*,\, \nu, \lambda). \tag{31}$$

Summarizing the above discussion, we obtain the desired regularization of problem (27). An extended discussion can be found in [24].

Theorem 3.3 (cf. [24]). *Problem* (27) *is reduced on* $\Delta\setminus\Delta_\mu$ *to a finite-dimensional equation and has a unique smooth solution* $u \in W_2^{1/2}(\Gamma)$ *if* $\lambda \in \Delta\setminus\Delta_\mu$ *is not a zero of the corresponding determinant,* $\det\left[I + \left[\mathbf{Q} + \mathcal{K}^N(\lambda)\right]^{-1} \mathcal{Q}^N_{\text{int}}\right] =: \mathbf{D}(\lambda) \neq 0.$

Proof. We use the smoothness of the trace $\psi_{\text{out}}\big|_\Gamma \in W_2^{3/2}(\Gamma)$ on Γ of the generalized eigenfunction ψ_{out} (scattered waves) of L_{out}.

Theorem (3.3) is used below in Section 5 when fitting the solvable model.

Remark 3. The precise smoothness of the scattered waves ψ_{out} is determined by the smoothness of the boundary. In particular, for a sufficiently smooth boundary, $\partial\Omega_{\text{out}} \in C_2$, we have at least $\psi_{\text{out}}\big|_\Gamma \in W_2^{5/2}(\Gamma)$. Then the right-hand side of (31) belongs to $W_2^{3/2}(\Gamma)$, and hence $u \in W_2^{3/2}(\Gamma) \in C(\Gamma)$.

In the next section, we shall develop another regularization method for (27) to obtain an approximate solution of (31). This regularization is based on filtering the signals by a thin channel barred at a certain level of frequencies.

4. TRANSPORT PROPERTIES OF A SHORT THIN CHANNEL

Let us now evaluate the contribution of the channel to the additional term of the amplitude (26) under the assumption that the channel is "relatively short and thin," $kH < \pi/2$, $\delta/H \ll 1$, see Fig. 2.

In fact, each of these conditions can be loosened a little, and we make a few concluding comments about this. In this section, we shall also impose the harder conditions $kH \ll \pi/2$, $\delta/H \ll 1$, which define "short thin" channels.

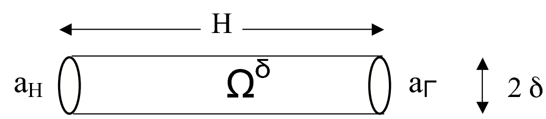

Fig. 3. A thin short channel.

To evaluate the denominator $\mathcal{Q}^*_{\text{int}}(\lambda)+\mathcal{Q}_{\text{out}}(\lambda)$ in formula (26) in terms of spectral characteristics of Ω_{int}, we must transfer the ND mapping of the inner domain from the lower lid Γ to the upper lid Γ_H along the channel Ω^δ. This will be done by using transport properties of the channel. If the channel is relatively short and thin, the final formulas turn out to be sufficient for an explicit asymptotic calculation of the additional term of scattering amplitude. As in Section 1 above, assume that the channel is a relatively short and thin circular cylinder, see Fig. 3, of height H, $0 < x < H$, of radius δ, $0 < r < \delta$, and with the lower lid Γ and the upper lid Γ_H.

Denote by $\lambda_{n,s} = \nu_{n,s}^2(\delta) = \delta^{-2}\,[\nu_{n,s}^1]^2$ the eigenvalues of the Laplacian on the cross-section of the cylinder with homogeneous Neumann boundary conditions at $r = \delta$ and by $P_{n,s}$ the projections to the corresponding normalized eigenfunctions $Y_n J_n(\nu_{n,s}\, r)$, $n = 1, 2, \dots$, with $Y_n(\varphi) = \text{Const}\, e^{\pm\, in\,\varphi}$. The eigenvalues $\nu_{n,s}^2 = \delta^{-2}\,(\nu^1)_{n,s}^2$ are defined by the zeros of the derivative of the Bessel functions, $J_n'(\nu_{n,s}\delta) = 0$. Denote by P_0 the projection to the constant eigenfunction $Y_{0,0} = (\sqrt{2\pi}\,\delta)^{-1}$ corresponding to the eigenvalue $\nu_0^2 = 0$. In this case, $\sum_{(n,s)\neq(0,0)} P_{n,s} = P^\perp$ is the projection to the orthogonal complement of the constants on the cross-sections Γ, Γ_H, $P_0 \oplus P^\perp = I$ in $L_2(\Gamma)$, $L_2(\Gamma_H)$. The complementary projections $P_0, P^\perp$ in $L_2(\Gamma), L_2(\Gamma_H)$ are represented as

$$P_0 = \frac{\chi(x)\rangle\,\langle\chi(y)}{\pi\delta^2}, \qquad P^\perp = I - \frac{\chi(x)\rangle\,\langle\chi(y)}{\pi\delta^2}. \tag{32}$$

Here $\chi(x) = \chi_\Gamma(x), \chi_H(x)$ is an indicator of the corresponding lid Γ, Γ_H, e.g., $\chi_H(x) = 1$ if $x \in \Gamma_H$ and $\chi_H(x) = 0$ on the complement $\partial\Omega_{\text{out}}\backslash\Gamma_H$. Hereafter, we use spectral data for the scaled Neumann Laplacian $-\triangle^\perp$ on an orthogonal cross-section of the channel Ω_δ with respect to the scaled variables $\delta^{-1}\,r =: \xi,\ 0 < \xi < 1$, with the scaled eigenvalues $\lambda_{n,s}^1 = (\nu_{n,s}^1)^2$.

We also consider the boundary problem for the Laplacian on the channel with Neumann boundary condition at $r = \delta$ and nonhomogeneous Dirichlet boundary conditions on the lids, $-\Delta u = \lambda u$, $u\big|_\Gamma = u_\Gamma$, $u\big|_H = u_H$. The relative Dirichlet-to-Neumann mapping Λ^δ of this problem on Γ, Γ_H is defined by the normal outward derivatives with respect to the inner domain, on both the sections Γ, Γ_H, and is obtained via the separation of variables,

$$\Lambda^\delta = \begin{pmatrix} \Lambda^{HH} & \Lambda^{H\Gamma} \\ \Lambda^{\Gamma H} & \Lambda^{\Gamma\Gamma} \end{pmatrix} = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}\,H} \begin{pmatrix} \cos\sqrt{\lambda}H & -1 \\ 1 & -\cos\sqrt{\lambda}H \end{pmatrix} P_0$$
$$+ \sum_{n,s=1}^{\infty} \frac{\sqrt{\nu_{n,s}^2 - \lambda}}{\sinh\sqrt{\nu_{n,s}^2 - \lambda}\,H} \begin{pmatrix} \cosh\sqrt{\nu_{n,s}^2 - \lambda}\;H & -1 \\ 1 & -\cosh\sqrt{\nu_{n,s}^2 - \lambda}\;H \end{pmatrix} P_{n,s} := \Lambda_0^\delta + \Lambda_\perp^\delta. \tag{33}$$

For a thin channel, the nontrivial (off-diagonal) component of the DN-mapping, which is responsible for the transmission of Dirichlet/Neumann data from one lid to another, is essentially defined by the constant eigenfunction of the cross-section,

$$\Lambda^\delta \approx \Lambda_0^\delta + \begin{pmatrix} \delta^{-1}\sqrt{-\triangle^\perp - \delta^2\lambda I}\,P_H^\perp & 0 \\ 0 & -\delta^{-1}\sqrt{-\triangle^\perp - \delta^2\lambda I}\,P_\Gamma^\perp \end{pmatrix}, \tag{34}$$

where $-\triangle^\perp$ is the Neumann Laplacian on the orthogonal complement of constants on the lids Γ, Γ_H represented in terms of the scaled variables (on the corresponding scaled section of radius 1). Let us discuss the approximation suggested above.

The inverse operator, which is the Neumann-to-Dirichlet mapping, can be calculated as

$$\mathcal{Q}^\delta := \begin{pmatrix} \mathcal{Q}_{HH}^\omega & \mathcal{Q}_{H\Gamma}^\omega \\ \mathcal{Q}_{\Gamma H}^\omega & \mathcal{Q}_{\Gamma\Gamma}^\omega \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{\lambda}\tan\sqrt{\lambda}H} & \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} \\ -\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} & \frac{1}{\sqrt{\lambda}\tan\sqrt{\lambda}H} \end{pmatrix} P_0$$
$$+ \sum_{n,s} \begin{pmatrix} \frac{1}{\sqrt{\lambda_{n,s}-\lambda}\tanh\sqrt{\lambda_{n,s}-\lambda}\,H} & -\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\,\sinh\sqrt{\lambda_{n,s}-\lambda}\,H} \\ \frac{1}{\sqrt{\lambda_{n,s}-\lambda}\,\sinh\sqrt{\lambda_{n,s}-\lambda}\,H} & -\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\,\tanh\sqrt{\lambda_{n,s}-\lambda}\,H} \end{pmatrix} P_{n,s} =: \mathcal{Q}_0^\delta + \mathcal{Q}_\perp^\delta.$$

This coincides with the restriction of the full ND-mapping of the channel to the lids. Note that the second sum here also admits the spectral representation

$$\mathcal{Q}_\perp^\delta = \begin{pmatrix} 1 & e^{-\sqrt{-\Delta^\perp - \lambda I^\perp}H} \\ e^{-\sqrt{-\Delta^\perp - \lambda I^\perp}H} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{-\Delta^\perp - \lambda I^\perp}H} & 0 \\ 0 & -\frac{1}{\sqrt{-\Delta^\perp - \lambda I^\perp}H} \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & -e^{-\sqrt{-\Delta^\perp - \lambda I^\perp}H} \\ -e^{-\sqrt{-\Delta^\perp - \lambda I^\perp}H} & 1 \end{pmatrix} \frac{1}{1 - e^{-2\sqrt{-\Delta^\perp - \lambda I^\perp}H}}.$$

Therefore, for a thin channel, $\delta/H \ll 1$, the values of the spectral parameter, below the second threshold $\mathcal{Q}^\delta_\perp$, can be replaced with small error by the diagonal matrix

$$\begin{pmatrix} \frac{1}{\sqrt{-\Delta^\perp-\lambda I^\perp}\tanh\sqrt{-\Delta^\perp-\lambda I^\perp}H} & 0 \\ 0 & -\frac{1}{\sqrt{-\Delta^\perp-\lambda I^\perp}\tanh\sqrt{-\Delta^\perp-\lambda I^\perp}H} \end{pmatrix},$$

or even by the matrix $\begin{pmatrix} \frac{1}{\sqrt{-\Delta^\perp-\lambda I^\perp}H} & 0 \\ 0 & -\frac{1}{\sqrt{-\Delta^\perp-\lambda I^\perp}H} \end{pmatrix}$, because $\tanh\sqrt{-\Delta^\perp-\lambda I^\perp}H \approx 1$ for a thin channel, $\delta/H \ll 1$.

Hereafter, for a short thin channel

$$0 < \sqrt{\lambda}H < \pi/2, \qquad \delta H^{-1} \ll 1, \tag{35}$$

we use the following approximation:

$$\mathcal{Q}^\delta \approx \mathcal{Q}^\delta_0 + P_\perp\left(-\Delta-\lambda I\right)^{-1/2}P_\perp\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{36}$$

where the second term on the right-hand side deviates from the corresponding exact term by an exponentially small error

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda_{n,s}-\lambda}\sinh\sqrt{\lambda_{n,s}-\lambda}\,H} \\ -\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\sinh\sqrt{\lambda_{n,s}-\lambda}\,H} & 0 \end{pmatrix} \approx e^{-\sqrt{\lambda^1_{n,s}-\delta^2\lambda}\;H/\delta}. \tag{37}$$

For the short channel $\sqrt{\lambda}H \ll \pi/2$, a further simplification is possible:

$$\mathcal{Q}^\delta \approx \frac{1}{\lambda H}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}P_0 + P_\perp\left(-\Delta-\lambda I\right)^{-1/2}P_\perp\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

but now the deviation of the first term on the second line of (36) from the corresponding exact term contains powers of $\sqrt{\lambda}\,H$.

In what follows, we neglect the exponentially small terms (37) in the second term on the first line of (36) and *retain the exact expression for the first term* $\mathcal{Q}_0$,

$$\begin{aligned}\mathcal{Q}^\delta_{\text{appr}} &= \begin{pmatrix} -\frac{1}{\sqrt{\lambda}\tan\sqrt{\lambda}H} & \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} \\ -\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} & \frac{1}{\sqrt{\lambda}\tan\sqrt{\lambda}H} \end{pmatrix}P_0 + \sum_{n,s}\begin{pmatrix} \frac{1}{\sqrt{\lambda_{n,s}-\lambda}\tanh\sqrt{\lambda_{n,s}-\lambda}\,H} & 0 \\ 0 & -\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\tanh\sqrt{\lambda_{n,s}-\lambda}\,H} \end{pmatrix}P_{n,s} \\ &= \frac{1}{\lambda H}\begin{pmatrix} -\gamma_t & \gamma_s \\ -\gamma_s & \gamma_t \end{pmatrix}P_0 + \begin{pmatrix} d^\perp & 0 \\ 0 & -d^\perp \end{pmatrix} =: \begin{pmatrix} \mathcal{Q}^\delta_{HH} & \mathcal{Q}^\delta_{H\Gamma} \\ \mathcal{Q}^\delta_{\Gamma H} & \mathcal{Q}^\delta_{\Gamma\Gamma} \end{pmatrix}.\end{aligned} \tag{38}$$

We use the following notation:

$$\sqrt{\lambda}\,H\tan^{-1}\sqrt{\lambda}\,H =: \gamma_t, \qquad \sqrt{\lambda}\,H\sin^{-1}\sqrt{\lambda}\,H =: \gamma_s,$$

$$d^\perp = \sum_{n,s}\frac{1}{\sqrt{\lambda_{n,s}-\lambda}\tanh\sqrt{\lambda_{n,s}-\lambda}\;H}P_{n,s} = \frac{1}{\sqrt{-\Delta_\perp-\lambda P_\perp}\tanh\sqrt{-\Delta_\perp-\lambda P_\perp}\;H}, \tag{39}$$

and the exponentially small off-diagonal elements of $\mathcal{Q}^\delta_\Gamma$ are neglected. Note that the diagonal elements $\mathcal{Q}^\delta_{\Gamma\Gamma}, \mathcal{Q}^\delta_{HH}$ are invertible if the channel Ω^δ is thin. Indeed, for $\Omega^\delta_{\Gamma\Gamma}$, we have

$$\mathcal{Q}^\delta_{\Gamma\Gamma} \approx \frac{1}{\sqrt{\lambda}\tan\sqrt{\lambda}H}\left[P_0 - \frac{\sqrt{\lambda}\tan\sqrt{\lambda}H}{[-\Delta_\perp-\lambda P_\perp]^{1/2}}\right] \approx \frac{P_0}{\sqrt{\lambda}\tan\sqrt{\lambda}H} - d^\perp. \tag{40}$$

Substituting, for thin channel, $\delta/H \ll 1$, the exact ND-mapping of Ω^δ by the above approximation $\mathcal{Q}^\delta_{\rm appr}$, we admit an exponentially small error $O(e^{-H/\delta})$.

We carry out the subsequent calculations with the precision of $o(\delta/H), o(\sqrt{\lambda}H)$ and just replace $\mathcal{Q}^\delta$ by $\mathcal{Q}^\delta_{\rm appr}$. In particular, using the above approximation (38) for $\mathcal{Q}^\delta$, we shall calculate the restriction to Γ_H of the ND-mapping for the extended inner domain $\Omega^*_{\rm int} := \Omega_{\rm int} \cup \Omega^\delta$. Using [5], one can construct an approximate spectral representation for the DN-mapping of the basic domain $\Omega_{\rm int}$,

$$\mathcal{DN}_\Gamma = \sum_{l=1}^{N} \frac{\partial\varphi_l/\partial n\rangle\, \langle\partial\varphi_l/\partial n}{\lambda - \lambda_l} + \mathcal{K}^N_{\rm dn} =: \mathcal{DN}^N_\Gamma + \mathcal{K}^N_{\rm dn},$$

with respect to Γ, on an essential spectral interval Δ and on its complex neighborhood G_Δ. Choose a number N such that $\|\mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N_{\rm dn}\|_{W_2^{3/2}(\Gamma_H)} < 1$. Then the operators $I - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N$, $\mathcal{Q}^\delta_{\Gamma\Gamma} - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N_{\rm dn}\mathcal{Q}^\delta_{\Gamma\Gamma}$ are invertible, and therefore we obtain

$$[\mathcal{Q}^\delta_{\Gamma\Gamma} - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N_{\rm dn}\mathcal{Q}^\delta_{\Gamma\Gamma}]^{-1} =: V = [\mathcal{Q}^\delta_{\Gamma\Gamma}]^{-1}\frac{I}{I - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N_{\rm dn}} \approx [\mathcal{Q}^\delta_{\Gamma\Gamma}]^{-1} \approx \lambda H\, P_0. \tag{41}$$

Denote by $E^N = \bigvee_{l=1}^N \varphi_l$ the invariant subspace of $L_{\rm int}$ which corresponds to the eigenfunctions $\{\varphi_l\}_{l=1}^N$ and introduce the spectral projection $P^N = \sum_{l=1}^N \varphi_l\rangle\,\langle\varphi_l$ and the part $L^N = \sum_{l=1}^N \lambda_l\varphi_l\rangle\,\langle\varphi_l$ of $L_{\rm int}$ in E^N. Below we also use the mappings

$$T := \sum_{l=1}^{N} \varphi_l\Big\rangle\Big\langle\frac{\partial\varphi_l}{\partial n},\ \mathcal{Q}^\delta_{\Gamma\Gamma} *\Big\rangle_\Gamma, \qquad T^+ := \sum_{l=1}^{N} \mathcal{Q}^\delta_{\Gamma\Gamma}\frac{\partial\varphi_l}{\partial n}\Big|_\Gamma\Big\rangle\ \Big\langle\varphi_l, *\Big\rangle. \tag{42}$$

The following statement describes the transmission of the Dirichlet/Neumann data from the lower lid Γ of the channel Ω_δ to the upper lid Γ_H.

Theorem 4.1. *For a thin channel* ($\delta/H \ll 1$), *the approximation* (38) *for* $\mathcal{Q}^\delta$ *implies the following approximate formula for the relative ND-mapping* $\mathcal{ND}^*_H$ *of* $\Omega^*_{\rm int}$ *on* Γ_H:

$$\mathcal{ND}^*_H = d^\perp + \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}}\ P_0 - \frac{I}{\lambda\sin^2\sqrt{\lambda}H}P_0\Big[VT^+\frac{I}{\lambda I^N - L^N - TVT^+}TV + V\Big]P_0. \tag{43}$$

Proof. The DN-mapping of $L_{\rm int}$ with respect to Γ is related to the ND mapping of $L^*_{\rm int}$ on the extended domain $\Omega^*_{\rm int}$ by the linear system

$$\begin{pmatrix} \mathcal{Q}^\delta_{HH} & \mathcal{Q}^\delta_{H\Gamma} \\ \mathcal{Q}^\delta_{\Gamma H} & \mathcal{Q}^\delta_{\Gamma\Gamma}\end{pmatrix}\begin{pmatrix}\rho_H \\ \mathcal{DN}_\Gamma u_\Gamma\end{pmatrix} = \begin{pmatrix}\mathcal{ND}_H\ \rho_H \\ u_\Gamma\end{pmatrix}. \tag{44}$$

This system yields the representation

$$\mathcal{ND}_H = \Big[\mathcal{Q}^\delta_{HH} - \mathcal{Q}^\delta_{H\Gamma}\left[\mathcal{Q}^\delta_{\Gamma\Gamma}\right]^{-1}\mathcal{Q}^\delta_{\Gamma H}\Big] + \mathcal{Q}^\delta_{H\Gamma}\left[\mathcal{Q}^\delta_{\Gamma\Gamma} - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{DN}_\Gamma\mathcal{Q}^\delta_{\Gamma\Gamma}\right]^{-1}\mathcal{Q}^\delta_{\Gamma H}, \tag{45}$$

which we now simplify by using the explicit expressions for $\mathcal{Q}^\delta$ substituted with $\mathcal{Q}^\delta_{\rm appr}$ picking up an exponentially small error. In particular, the first term on the right-hand side of (45) simplifies to $\mathcal{Q}^\delta_{HH} - \mathcal{Q}^\delta_{H\Gamma}\left[\mathcal{Q}^\delta_{\Gamma\Gamma}\right]^{-1}\mathcal{Q}^\delta_{\Gamma H} = d^\perp + \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}}\,P_0$. To calculate the second term in (45), we must solve the equation

$$\left[\mathcal{Q}^\delta_{\Gamma\Gamma} - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{DN}_\Gamma\mathcal{Q}^\delta_{\Gamma\Gamma}\right]u = \mathcal{Q}^\delta_{\Gamma H}\rho_H. \tag{46}$$

If the spectral rational approximation of $\mathcal{DN}$ is chosen in such a way that the mapping

$$[\mathcal{Q}^\delta_{\Gamma\Gamma} - \mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N \mathcal{Q}^\delta_{\Gamma\Gamma}]^{-1} =: V \tag{47}$$

exists, then, because of the domination condition $\|\mathcal{Q}^\delta_{\Gamma\Gamma}\mathcal{K}^N\|_{W_2^{3/2}(\Gamma_H)} < 1$, the solution of the above equation can be found by inverting a finite matrix. Indeed, in terms of the new variable $v = \sum_{l=1}^N \varphi_l \frac{\left\langle \mathcal{Q}^\delta_{\Gamma\Gamma}\frac{\partial\varphi_l}{\partial n}, u\right\rangle}{\lambda-\lambda_l} = (\lambda I^N - L^N)^{-1} T u$, equation (46) can be represented as

$$(\lambda I^N - L^N)v - TVT^+ v = TV\mathcal{Q}^\delta_{\Gamma H}\rho; \tag{48}$$

hence, $v = \frac{I}{\lambda I^N - L^N - TVT^+} TV\mathcal{Q}^\delta_{\Gamma H}\rho$. Then $u = VT^+ v + V\mathcal{Q}^\delta_{\Gamma H}\rho = VT^+ \frac{I}{\lambda I^N - L^N - TVT^+} TV\mathcal{Q}^\delta_{\Gamma H}\rho + V\mathcal{Q}^\delta_{\Gamma H}\rho$, and $\mathcal{ND}_H\rho = d^\perp\rho + \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}} P_0\rho + \mathcal{Q}^\delta_{H\Gamma}\Big[VT^+ \frac{I}{\lambda I^N - L^N - TVT^+} TV + V\Big]\mathcal{Q}^\delta_{\Gamma H}\rho$. Note that $\mathcal{Q}^\delta_{H\Gamma} = \frac{I}{\sqrt{\lambda}\sin\sqrt{\lambda}H} P_0 = -\mathcal{Q}^\delta_{\Gamma H}$. This completes the proof.

Remark 4. When deriving the expression for $\mathcal{ND}^*_H$, we have neglected the exponentially small off-diagonal terms $O(e^{-H/\delta})$ of the component $\mathcal{Q}_\perp$ and obtained an approximation for $\mathcal{ND}^*_H$ in the form of the diagonal matrix

$$\mathcal{ND}^*_H \approx d^\perp + \mathbf{M}P_0 \tag{49}$$

with the scalar function $\mathbf{M} = \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}} - \frac{I}{\lambda\sin^2\sqrt{\lambda}H}\,\mathrm{Trace}\Big[VT^+ \frac{I}{\lambda I^N - L^N - TVT^+} TV + V\Big]P_0$. Denoting $\mathbf{D} = \mathrm{Trace}\Big[VT^+ \frac{I}{\lambda I^N - L^N - TVT^+} TV\Big]P_0$, using (41), and replacing the last term in (43) by $-1/\lambda H$ (on the complement of poles), we also obtain the approximate expression for $\mathbf{M}$, $\mathbf{M} = \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}} - \frac{\mathbf{D}}{\lambda\sin^2\sqrt{\lambda}H} - \frac{\mathrm{Trace}\,P_0V}{\lambda\sin^2\sqrt{\lambda}H} \approx \frac{\tan\sqrt{\lambda}H}{\sqrt{\lambda}} - \frac{\mathbf{D}}{\lambda\sin^2\sqrt{\lambda}H} - \frac{1}{\lambda H}$, for short thin channels. It can readily be seen that the diagonal expression (49) differs from the exact value of $\mathcal{ND}^*_H$ by an exponentially small error estimated as

$$\|\mathcal{ND}^*_H - d^\perp - \mathbf{M}P_0\| \leqslant C\mathbf{D}e^{-H/\delta}, \tag{50}$$

which is small for a thin short channel and for λ outside an open neighborhood of the poles of $\mathbf{D}$.

Remark 5. We might also consider another approximation for $\mathcal{ND}^*_H$ (less accurate but more convenient for fitting) for a short thin channel ($kH \ll \pi/2$, $\delta/H \ll 1$). This approximation is obtained by replacing V by $P_0\lambda H$ and $\mathbf{D}$ by $\lambda^2H^2\mathcal{D}$ for $\mathrm{Im}\,\mathcal{D} < 0$ and set

$$\begin{aligned}\mathcal{D}(\lambda) &:= \mathrm{Trace}\left[P_0 \sum_{l,m=1}^N \frac{\partial\varphi_l}{\partial n}\Big|_\Gamma\Big\rangle \Big\langle \varphi_l, \frac{I}{\lambda I^N - L^N - TVT^+}\varphi_m\Big\rangle \Big\langle \frac{\partial\varphi_m}{\partial n}\Big|_\Gamma\right] \\ &= \sum_{l,m} \pi^{-1}\delta^{-2}\Big[\int_\Gamma \frac{\partial\varphi_l}{\partial n}d\Gamma\Big]\Big\langle \varphi_l, \frac{I}{\lambda I^N - L^N - TVT^+}\varphi_m\Big\rangle\Big[\int_\Gamma \frac{\partial\varphi_m}{\partial n}d\Gamma\Big] = O(\delta^2),\end{aligned} \tag{51}$$

again, outside the poles. Then, using (41), for a short channel $\sqrt{\lambda}H \ll 1$, we can substitute P_0V by $P_0\lambda H$ to obtain $\mathbf{M} \approx H - \mathcal{D} - \frac{1}{\lambda H}$ with a controllable error for small $\mathcal{K}^N_{\mathrm{nd}}$, $[VP_0 - \lambda HP_0] \approx \mathcal{K}^N_{\mathrm{nd}}$. Next, we can estimate the difference $\mathbf{D} - \lambda H\,\mathcal{D}\,\lambda H\,P_0$ on the complement $G_\mathcal{D} := \Delta\backslash U_\mathcal{D}$ of an open neighborhood $U_\mathcal{D}$ of the poles of $\mathcal{D}$,

$$\Big|\frac{1}{\lambda\sin^2\sqrt{\lambda}H}\,\mathrm{Trace}[\mathbf{D} - \lambda H\,\mathcal{D}\,\lambda H]P_0\Big| \leqslant \|P_0\mathcal{K}^N_{\mathrm{nd}}\||\mathcal{D}|. \tag{52}$$

Under the same condition, we estimate

$$\mathrm{Trace}[P_0V - P_0(\mathcal{Q}^\delta_{\Gamma\Gamma})^{-1}] \approx \mathrm{Trace}\,\mathcal{K}^N_{\mathrm{nd}}P_0. \tag{53}$$

Summarizing the above observations (52) and (53), we obtain the following assertion for short thin channels.

Lemma 4.1. *For small* $\|\mathcal{K}^N_{\rm nd}\|$ *and* $\sqrt{\lambda}H \ll 1$, *the function* $\mathbf{M}$ *can be approximated, with a relatively minor error on* $G_{\mathcal{D}}$, *by the expression* $\mathbf{M} \approx H - \mathcal{D} - \frac{I}{\lambda H}$.

Remark 6. Inserting the expression for $\mathbf{M}$ into (49), we obtain, for $\mathcal{Q}_H(\lambda) =: \mathcal{Q}^*_{\rm int}(\lambda) = d^\perp + \mathbf{M}P_0 + \cdots$, a convenient approximate expression for the denominator of the additional term of the scattering amplitude (26), namely,

$$\frac{1}{8\pi^3}\psi_{\rm out}(*,\,\omega,\lambda)\,[\mathcal{Q}^*_{\rm int}(\lambda) + \mathcal{Q}_{\rm out}(\lambda)]^{-1}\,\psi_{\rm out}(*,\,\nu,\lambda)$$
$$\approx \frac{1}{8\pi^3}\psi_{\rm out}(*,\,\omega,\lambda)\,\left[d^\perp + \mathbf{M}\,P_0 + \mathcal{Q}_{\rm out}(\lambda)\right]^{-1}\psi_{\rm out}(*,\,\nu,\lambda) =: \delta a_M. \qquad (54)$$

We also have an exponential estimate for the deviation of $\mathcal{Q}^*_{\rm int}$ from $d^\perp + \mathbf{M}\,P_0$ on the complement of some open neighborhood of the poles of $\mathbf{M}$, or, following the above arguments, a slightly coarser estimate for the deviation on the complement $G_{\mathcal{D}}$ of an open neighborhood $U_{\mathcal{D}}$ of the poles of $\mathcal{D}$.

5. FITTING OF THE SOLVABLE MODEL

A solvable model of the Helmholtz resonator is said to be "fitted" if the corresponding model for the scattering matrix serves as a local approximation for the original scattering matrix of the Helmholtz resonator on the major part $G_{\mathcal{D}} = \Delta\backslash U_{\mathcal{D}}$ of the essential spectral interval Δ, i.e., on the complement of an open neighborhood of the poles of the corresponding intermediate DN-mapping.

We can fit a solvable model using a comparison of the additional term of the model amplitude (16) with the corresponding rational approximation.

Note first of all that the values $\psi_{\rm out}(x,\,\omega,\lambda)$, $\psi_{\rm out}(x,\,\nu,\lambda)$ of the scattered waves in a solvable model are taken at the center a_H of the upper lid. According to Remark 3, the solution u of (27) obtained by the corresponding regularization is unique and smooth, $u \in W_2^{3/2}(\Gamma)$, for nonsingular λ. Now we can evaluate the solution u of (27) by replacing $\mathcal{Q}^*_{\rm int}$ by the approximate expression (49), see Remark 4 above. We also replace $\psi_{\rm out}(*,\,\omega,\lambda)$ and $\psi_{\rm out}(*,\,\nu,\lambda)$ on Γ_H by their values at the center a_H of the upper lid which are multiplied by the indicator function $\chi_{\Gamma_H} = \chi_H$ of the upper lid, $\psi_{\rm out}(\gamma,\,\omega,\lambda) \longrightarrow \psi_{\rm out}(a_H,\,\omega,\lambda)\chi_H(x)$, $\gamma \in \Gamma_H$. Since $\psi_{\rm out} \in W_2^2(\Omega)$, it follows that the trace of $\psi_{\rm out}$ on Γ_H is smooth, $\psi_{\rm out} \in W_2^{3/2}(\Gamma_H)$. By the embedding theorem, the inclusion $W_2^{3/2}(\Gamma_H) \subset {\rm Lip}_{1/3}(\Gamma_H)$ holds, and thus we have the estimate

$$|\psi_{\rm out}(x,\,\omega,\lambda) - \psi_{\rm out}(a_H,\,\omega,\lambda)\chi(x)| \leqslant C\,\delta^{1/3}|\lambda|^{3/4}, \qquad x \in \Gamma_H, \qquad (55)$$

for an absolute constant C, since $\|\psi_{\rm out}\|_{W_2^{3/2}(\Gamma_H)} \leqslant C|\lambda|$. The resulting expression for the approximate correcting term of the amplitude is

$$\frac{1}{8\pi^3}\;\psi_{\rm out}(a_H,\,\omega,\lambda)\;\left\langle \chi_H,\,\frac{1}{\mathcal{Q}^*_{\rm int} + \mathcal{Q}_{\rm out}}\chi_H\right\rangle\;\psi_{\rm out}(a_H,\,\nu,\lambda) =: \delta\hat{a}. \qquad (56)$$

Let us now estimate the difference between $\delta\hat{a}$ and the original correcting term,

$$\frac{1}{8\pi^3}\psi_{\rm out}(*,\,\omega,\lambda)\,[\mathcal{Q}^*_{\rm int}(\lambda) + \mathcal{Q}_{\rm out}(\lambda)]^{-1}\,\psi_{\rm out}(*,\,\nu,\lambda) =: \delta a. \qquad (57)$$

Lemma 5.1. *On the major part of the essential spectral interval* Δ, *i.e., on the complement to* Δ *of a small open neighborhood of the poles of the relative DN-mapping, the deviation of the solutions of equations* (56) *and* (57) *from each other can be estimated as* $\max_{\Gamma_H}|u - \hat{u}| \leqslant {\rm Const}\big(\delta\sqrt{\lambda}\big)^{1/3}$.

Proof. Let us denote by u and $\hat{u}$ the solutions of the equations $[\mathcal{Q}^*_{\rm int} + \mathcal{Q}_{\rm out}]u = \psi_{\rm out}(\gamma)$ and $[\mathcal{Q}^*_{\rm int} + \mathcal{Q}_{\rm out}]\hat{u} = \chi_H(\gamma)\psi_{\rm out}(a_H,\,\nu,\lambda)$, respectively. Applying the procedure described in Remark 4 to the above equations, we reduce them (for a thin channel, $\delta/H \ll 1$) to the finite-dimensional linear system with the determinant $\det\Big[I + \big(I + \mathcal{DN}_{\rm out}\mathcal{K}^N_{\rm nd}\big)^{-1}\mathcal{DN}_{\rm out}\mathcal{Q}^N_{\rm int}\Big] =: \det\mathcal{B}(\lambda)$. We have $\det\mathcal{B}(\lambda)\neq 0$ on the complement to Δ of the discrete spectrum of the intermediate Hamiltonian. In this case, $\max_{\Gamma_H}|u-\hat u| \leqslant C\delta^{1/3}|\lambda|^{3/4}/\det\mathcal{B}(\lambda) = C\big(\delta\sqrt{\lambda}\big)^{1/3}\ [\det\mathcal{B}(\lambda)]^{-1}|\lambda|^{7/12}$ with a constant C independent of λ. This implies the corresponding estimate for the difference $\delta a - \delta\hat a$, again with some constant C independent of λ.

Our next step is to compare the approximate correcting term

$$\frac{1}{8\pi^3}\psi_{\rm out}(a_H,\,\omega,\lambda)\big\langle\chi_H,[d^\perp + \mathbf{M}\,P_0 + \mathcal{Q}_{\rm out}(\lambda)]^{-1}\chi_H\big\rangle\psi_{\rm out}(a_H,\,\nu,\lambda) \tag{58}$$

with the correcting term obtained from the solvable model, see (16). To calculate the approximate correcting term (58), we must solve the equation

$$\big[P_0\mathbf{M} + d^\perp + \mathcal{Q}_{\rm out}\big]\,u = \chi_H. \tag{59}$$

Although the problem is ill-posed, we have already discussed above (in Remark 4) a way to regularize the problem and establish the smoothness of the solution. The difference between the approximate correcting term and the original correcting term (26) is estimated by using the results of the previous section. In particular, the term $\mathcal{Q}^*_{\rm int}$ can be replaced with a minor error by $P_0\mathbf{M} + d^\perp$.

Writing $u = (\rho_0\,\chi_H + u_\perp)$ and applying $P_0 = [\pi\delta^2]^{-1}\chi_H\rangle\langle\chi_H,\ P_\perp = I - P_0$ to (59), we obtain the system of two equations for $\rho_0, u_\perp$, namely,

$$P_\perp d^\perp u_\perp + P_\perp\mathcal{Q}_{\rm out}\,u_\perp + P_\perp\mathcal{Q}_{\rm out}\,\chi_H\rho_0 = 0, \qquad \mathbf{M}\,\chi_H\rho_0 + P_0\mathcal{Q}_{\rm out}\,u_\perp + P_0\mathcal{Q}_{\rm out}\,\chi_H\,\rho_0 = \chi_H. \tag{60}$$

The middle term of the above formula (56) is directly related to the component ρ_0 of the solution of the system

$$\Big\langle\chi_H,\frac{1}{\mathcal{Q}^*_{\rm int} + \mathcal{Q}_{\rm out}}\chi_H\Big\rangle = \langle\rho_0\,\chi_H,\,\chi_H\rangle = \rho_0\,\pi\,\delta^2. \tag{61}$$

The kernel of the integral operator $\mathcal{Q}_{\rm out}$ coincides with Green's function for the Neumann Laplacian in the outer domain; this kernel can be represented by (5). Hence,

$$\mathcal{Q}_{\rm out}u = \int_{\Gamma_H} G_{\rm out}(x,y,\lambda)u(y)dy = \int_{\Gamma_H} G_{\rm out}(x,y,M)u(y)dy + \int_{\Gamma_H}\mathcal{M}_{\rm out}(x,y,\lambda,M)u(y)dy,$$

with a large negative M and a continuous kernel

$$\mathcal{M}_{\rm out}(x,y) = (\lambda - M)\int_{\Omega_{\rm out}} G_{\rm out}(x,z,M)G_{\rm out}(z,y,\lambda)dz.$$

One can show that, near a smooth point $y\in\partial\Omega_{\rm out}$, Green's function $G_{\rm out}(x,y,M)$ has an asymptotic expansion as $x\in\Gamma_H\ \to\ y\in\Gamma_H$ of the form $G_{\rm out}(x,y,M) = \frac{1}{2\pi|x-y|} - \frac{\sqrt{|M|}}{2\pi} + \cdots$ for $|x-y|\,\sqrt{M}\ll 1$. Below we use the notation

$$\gamma_1 = \delta^{-3}\int_\Gamma\int_\Gamma\frac{dxdy}{2\pi|x-y|},\quad\text{and therefore}\quad\int_\Gamma\int_\Gamma\frac{dxdy}{2\pi|x-y|} = \gamma_1\,\delta^3.$$

We also use the notation $\mathcal{M}_{\rm out}(\lambda+i0,a)$ for the limit of $\mathcal{M}_{\rm out}$ at the center of the upper lid $a_H\equiv a$, see (4), $\lim_{x,y\to a}\mathcal{M}_{\rm out}(x,y,\lambda,M) = C(a,M) + i\pi d\mathcal{E}/d\lambda\,(a) + \mathcal{M}_{VP}\equiv\mathcal{M}_{\rm out}(\lambda+i0,a)$.

Theorem 5.1. *For small δ, the component ρ_0 of the solution of* (59) *is approximately*

$$\rho_0 = \frac{1 - o(\gamma_1 \delta^{4/3}\, \lambda^{3/4})}{\mathbf{M} + \gamma_1\, \delta\, \pi^{-1} + \mathcal{M}_{\rm out} \pi\, \delta^2{}_{\rm out}} \approx \frac{1}{\mathbf{M} + \gamma_1\, \delta\, \pi^{-1} + \mathcal{M}_{\rm out} \pi\, \delta^2}$$

on the major part of the essential spectral interval (*on the complement of a small neighborhood of the zeros of the denominator*).

Proof. We have already noted at the end of Section 3 that, due to the local smoothness of eigenfunctions of the Laplacian, see Remark 3, the solution u of the above equation (59) is smooth and, in particular, $u \in W_2^{3/2}(\Gamma) \subset {\rm Lip}_{1/2}(\Gamma)$. More precisely, due to the corresponding embedding theorem on the small upper lid and to the relations $u \in W_2^2(\Omega)$ and $u\big|_{\Gamma_H} \in W_2^{3/2}(\Gamma_H)$, we see that $\sup_{\Gamma_H} |u_\perp| = \sup_\Gamma |u - P_0 u| \leqslant {\rm Const}\ \delta^{1/3} \|u\|_{W_2^{3/2}(\Gamma_H)}$. Therefore, equation (60) yields

$$\begin{aligned} &\mathbf{M}\rho_0 + (\pi\delta^2)^{-1} \int_\Gamma \int_\Gamma \frac{dxdy}{2\pi|x-y|}\ \rho_0 + \mathcal{M}_{\rm out}(a)\ \pi\delta^2\ \rho_0 \\ &\quad = 1 - (\pi\delta^2)^{-1} \int_\Gamma \int_\Gamma \frac{u_\perp}{2\pi|x-y|} dxdy = 1 - {\rm Const}\ \delta^{4/3}\ \lambda^{3/4}. \end{aligned} \tag{62}$$

This gives an approximate expression for ρ_0 on the major part of the essential spectral interval.

Remark 7. Combining the above result with earlier assertions in Lemmas 4.1 and 2.2, Theorem 2.1, and formula (61), we see that, on a major part of the essential spectral interval (on the complement of a small neighborhood of zeros of the denominator),

$$\left\langle \chi_{\Gamma_H}, \frac{1}{\mathcal{Q}^*_{\rm int} + \mathcal{Q}_{\rm out}} \chi_{\Gamma_H} \right\rangle = \frac{\pi\delta^2}{[H - \mathcal{D} - (\lambda H)^{-1} + \gamma_1\, \delta\, \pi^{-1}] + \mathcal{M}_{\rm out}\ \pi\delta^2},$$

for a thin short channel, and the correcting term (56) for the amplitude is

$$\delta a \approx [8\pi^3]^{-1}\ \frac{\psi_{\rm out}(a_H,\, \omega, \lambda)\rangle\, \pi\delta^2\, \langle \psi_{\rm out}(a_H,\, \nu, \lambda)}{[H - \mathcal{D} - (\lambda H)^{-1} + \gamma_1\, \delta\, \pi^{-1}] + \mathcal{M}_{\rm out}\ \pi\delta^2}. \tag{63}$$

Comparing this expression with the corresponding model correcting term at (16) gives

$$\delta a_{\rm mod} = [8\pi^3]^{-1}\ \frac{\psi_{\rm out}(a_H,\, \omega, \lambda)\rangle\ \langle \psi_{\rm out}(a_H,\, \nu, \lambda)}{(\beta_{01})^{-2}\big[H - \frac{1}{\mathcal{M}_\Gamma + \lambda H} + \beta_{11}\big] + \mathcal{M}_{\rm out}}. \tag{64}$$

We see that the solvable model constructed in Theorem 2.1 is fitted on the complement of an open neighborhood of the zeros of the denominators of $\mathcal{D}$ and $\mathcal{M}_H$ (on the major part of the essential spectral interval) if

$$|\beta_{01}|^{-2}\Big[H - \frac{1}{\mathcal{M}_\Gamma + \lambda H} + \beta_{11}\Big] = \frac{1}{\pi\delta^2}[H - \mathcal{D} - (\lambda H)^{-1} + \gamma_1\, \delta\, \pi^{-1}] \tag{65}$$

for $\lambda \in \Delta$. Since the analytic functions on the left- and right-hand sides of the above equation are in the Nevanlinna class, and therefore they can be represented by the Herglotz formula (10), we can compare the polynomial terms of the formula and the polar sum separately and insert the calculated values of the model parameters into (65) to obtain the following main result of the paper.

Theorem 5.2. *If* $|\beta_{01}|^2 = \pi\delta^2$ *and* $\beta_{11} = \gamma_1\delta^{-1}\pi$ *and if* $\mathcal{M}_\Gamma$ *is such that*

$$-\frac{1}{\mathcal{M}_\Gamma + \lambda H} = -\mathcal{D} - \frac{1}{\lambda H}, \tag{66}$$

then, under Assumption 1, *the solvable model constructed in Theorem* 2.1 *is fitted on the major part of the essential spectral interval* Δ, $\delta a_{\text{mod}} = \delta a + o(1)$, $A_{\text{mod}} = A_{\text{out}} + o(1)$, *if the channel* Ω^δ *is thin and short.*

Remark 8. Note that the inner structure **A** is subjected to an essential renormalization (66) compatible with the restriction L^N of the inner Laplacian to the spectral subspace corresponding to the essential spectral interval. Moreover, the chosen values of the model parameters are not uniquely defined, because in (66), we apply the Herglotz formula to a rational function of $\mathcal{M}_\Gamma$ rather than to a linear one. By adding equivalent rational terms to the left- and right-hand sides of (65), we can transfer some part of the sum of polar expressions to the polynomial part of the Herglotz formula. However, this is not used here.

6. HISTORICAL REMARKS AND BEST PROSPECTS

This work is essentially a continuation of the paper [7] published jointly by M. D. Faddeev with one of us (B.P.). In that paper, the connection between the Kirchhoff model and the operator extensions was discovered and developed later into the zero-range model with inner structure, see [29]. The problem with that paper was that the authors could not present a persuading procedure to choose parameters of the operator extension to "fit" the solvable model. This problem proved to be related to the perturbation of embedded eigenvalues. Eventually, these difficulties were overcome, inspired by the idea of H. Poincaré, see [35], about the "elimination of dangerous resonances." We developed in [30, 26] a modified two-step analytic perturbation procedure based on a "jump-start" technique. We also used essentially recent progress in spectral representations of the Dirichlet-to-Neumann mappings and the connection between the DN-mapping and the scattering matrix discovered in 2001, see [32].

The first problem to discuss next is fitting of the original zero-range model suggested in [7], which actually corresponds to an "almost point-like opening" obtained as a limit of a short channel, $\delta^2\,\lambda \to 0,\ H/\delta \to 0$. One can expect that, in contrast to the solvable model of the resonator with thin short channel considered here, this fitting needs just a restriction of the inner Laplacian to the spectral subspace corresponding to the essential spectral interval rather than an essential renormalization of the inner structure **A**. We feel that the fitting of the original solvable model [7] of the Helmholtz resonator with "short channel" could be recognized as a solution to the original 1882 Kirchhoff problem, in the original formulation, and simultaneously as a confirmation of the optimistic 15-year old Gadyl'shin conjecture quoted above [8].

Another tempting problem suggested in [39] is an approximate calculation of the poles of the scattering matrix and, in particular, the poles of the additional term of the amplitude placed near the continuous spectrum. These resonances belong to the series of resonances, see [33, 34], which appear due to the opening. Certainly, it is natural to attempt to calculate them formally as zeros of the denominator of the additional term δa_{mod} of the model amplitude. A simple result which is obtained in this way (and will be published elsewhere) needs a serious verification based on the matrix version of Rouché's theorem, see [20]. This verification requires an accurate estimation in the *complex plane* of the errors we neglected on the real axis, when deriving approximate expressions for $\mathcal{N}\mathcal{D}^*_{\text{int}}, \mathcal{D}, \mathcal{M}_{\text{out}}$. We discuss the related problems elsewhere.

The problem of approximate calculation of the scattering matrix for the Helmholtz resonator with a relatively wide opening is more interesting. Recall that we successively simplified the basic equation (27) for a thin channel due to the fact that all exponential modes in the channel that correspond to positive cross-section eigenvalues $\lambda_{l,m} : \mathcal{J}'_m(\lambda_{m,l}) = 0$, are "filtered out" by the channel. Indeed, they do not contribute to the transfer of the Dirichlet/Neumann data from the lower section to the upper one, because $\lambda^1_{l,m}\,\delta^{-2} \gg \lambda$ beginning with $l = 1$. When using direct computing, we are able to relax the above condition, neglecting only the transfer by the higher modes, namely, $[\sinh \sqrt{\lambda^1_{m,l} - \delta^2\lambda}\; H/\delta]^{-1} \ll 1$ if $\lambda_{m,l} > \Lambda_0$, beginning with some Λ_0, which is to be be chosen large enough. There are only finitely many modes below that level, which can be taken into account in an explicit way, by the direct computation. The procedure thus suggested is actually an extension of the perturbation procedure (suggested recently in [1]) for the junction of a quantum network to the case of Helmholtz resonator. Note that the computational procedure is suggested there for resonances in any resonator, without additional conditions on the diameter of the opening. The procedure described above enables us to extend our analysis to Helmholtz resonators with wide openings, with potentially intriguing applications of this extension, as was discussed in [24].

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