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# Some Solutions of the 3D Laplace Equation in a Layer with Oscillating Boundary Describing an Array of Nanotubes and an Application to Cold Field Emission. I. Regular Array

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**Abstract.** The aim of this paper is to construct solutions of the Dirichlet problem for the 3D Laplace equation in a layer with highly oscillating boundary. The boundary simulates the surface of a nanotube array, and the solutions are applied to compute the cold field electron emission. We suggest a family of exact solutions that solve the problem for a boundary with appropriate geometry. These solutions, along with the Fowler–Nordheim formula, allow one to present explicit asymptotic formulas for the electric field and the emission current. In this part of the paper, we consider the main mathematical aspects, restricting ourselves to the analysis of properties of the potential created by a single tube and a regular array of tubes. In the next part, we shall consider some cases corresponding to nonregular arrays of tubes and concrete physical examples.

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## 1. INTRODUCTION. FORMULATION OF THE PROBLEM

Consider the Dirichlet problem for the Laplace equation,

$$\begin{aligned} \triangle u(x,y,z) = 0, \quad (x,y,z) \in \Omega; \\ u|_{z=g(x,y)} = 0, \quad u|_{z=1} = 1, \end{aligned} \tag{1.1}$$

in a layer

$$\Omega = \{g(x,y) < z < 1\} \subset \mathbb{R}^3$$

with fast oscillating lower boundary

$$\Gamma = \big\{(x,y,z)\big| z = g(x,y)\big\}.$$

We first restrict ourselves to a boundary of the form of a regular array of vertically aligned identical axial-symmetric tubes (see Fig. 1). The geometry of the array is defined by several parameters, namely, the height $h$ and the diameter $d$ ($d \ll h \ll 1$) of the tubes and the distance $s$ between them. Here we consider arrays of medium density, $s \sim h \gg d$.

Problems of this kind come from the study of emission effects on a screen surface organized by an array of nanotubes placed into a stationary external electric field ([2]). In our case, the direction of the external field coincides with the axes of the tubes. The geometrical sizes (the height $\tilde h$ and the diameter $\tilde d$ of the tubes, the distance between the tubes $\tilde s$, and coordinates $\tilde x, \tilde y, \tilde z$, all measured in $\mu m = 10^{-6} m$) are rated to the anode-cathode distance $D$ (in $\mu m$). This procedure gives dimensionless variables $h = \tilde h/D$, $x = \tilde x/D$, etc.

The electrostatic potential $\tilde u$ (in $V$) is rated to anode-cathode voltage $U$ (in $V$), which gives a dimensionless function $u = \tilde u/U$. This normalization leads to the boundary condition $u|_{z=1} = 1$ in problem (1.1). The $z$-derivative of the solution $u$ gives the field enhancement factor $\beta$,

$$\beta \equiv \frac{F}{F_0}, \quad F_0 \equiv \frac{U}{D}: \quad \frac{\partial u}{\partial z} = \frac{1}{F_0}\frac{\partial \tilde u}{\partial \tilde z} = \beta, \tag{1.2}$$

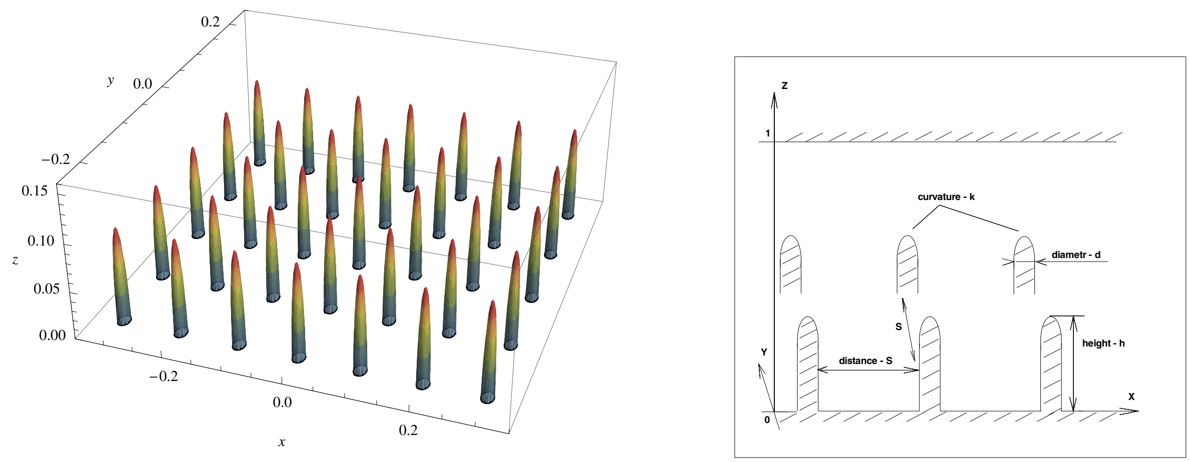


**Fig. 1.** Fast oscillating boundary. The scheme of the regular array of tubes.

where $F$ and $F_0$ (both in $V\ \mu m^{-1}$) is a field in presence of tubes and the external field, respectively.

The emission current density of an array $I$ (in $A\ \mu m^{-2}$) is an object of interest in physical problems. This density is given by the integral under the surface of the array of tubes $\Gamma$:

$$I = \frac{1}{s^2}\int_{-\frac{s}{2}}^{\frac{s}{2}}\int_{-\frac{s}{2}}^{\frac{s}{2}} J(x,y)dxdy. \tag{1.3}$$

Value of $J(x,y)$ (the emission current density of a small area $dxdy$, in $A\ \mu m^{-2}$) can be evaluated by the Fowler–Nordheim law using $z$-component of the electric field $F(x,y,z) \equiv F_0\ \partial u/\partial z$ on the surface $\Gamma$ (see [2]),

$$J = AF^2 \exp\Big\{-\frac{B}{F}\Big\}\Big|_{z=g(x,y)} = AF_0^2\beta^2 \exp\Big\{-\frac{B}{F_0}\frac{1}{\beta}\Big\}\Big|_{z=g(x,y)}. \tag{1.4}$$

Here the constants are given by

$$A = 1.54\times 10^{-6}\times\varphi^{-1} = 0.31\times 10^{-6}\ A\ V^{-2}, \quad B = 6.8\times 10^{3}\times\varphi^{3/2} = 76\times 10^{3}\ V\ \mu m^{-1},$$

and the so-called work function $\varphi$ is taken to be equal to $\varphi = 5eV$ (see, e.g., [1, 3, 4, 5]). It is important to analyze the dependence of the current density $I$ on the geometry of the array, i.e., to find the optimum distance maximizing the emission current.

Assume that we have a solution of problem (1.1). For a high aspect ratio $\lambda \equiv h/d \gg 1$ of the tubes, it seems reasonable to calculate the integral (1.3) approximately by the Laplace method. Let the point $(x,y,z) = (0,0,z_{\max})$ be the top point of the central tube in a regular array (i.e., $(0,0) \in \operatorname{argmax} g(x,y)$). Then

$$I = 2\pi AF_0^2\frac{\beta^2}{s^2}\exp\Big\{-\frac{B}{F_0}\frac{1}{\beta}\Big\}\cdot\frac{F_0}{B}\frac{\beta^2}{\sqrt{\det\Big(-\operatorname{Hess}\beta\Big)}}\Big|_{x=y=0,\ z=z_{\max}}\Big(1+O\big(\frac{F_0}{B}\big)\Big). \tag{1.5}$$

Here Hessian matrix is

$$\operatorname{Hess}\beta \equiv \begin{pmatrix} \partial^2\beta/\partial x^2 & \partial^2\beta/\partial x\partial y \\ \partial^2\beta/\partial x\partial y & \partial^2\beta/\partial y^2 \end{pmatrix}.$$

Estimates for the accuracy of the Laplace method can be found in textbooks (see, e.g., in [6]).

## 2. SUGGESTED SOLUTION

For the solution $u$ of problem (1.1), one can use the well-known analytical formula for a double layer potential [7], where the function $\varphi(P)$ is defined by the integral Fredholm equation derived using the boundary conditions. This formula is based on the calculation of some integrals admitting no evaluation in explicit form, and they are of small use from the point of view of applications.

Our idea is to construct special solutions for (1.1) (in the space $\mathbb{R}^3$) which satisfy the related boundary condition (i.e., the boundary surface $\Gamma$ should simulate the surface of an array of tubes with given geometrical characteristics $h$, $d$, and $s$). The lower surface is not arbitrary in this approach; however, it seems to be sufficient to simulate the emission effects (like field enhancement and field screening). Due to the general theory of the Laplace equation, it is impossible to construct a regular solution in the layer $\{0 < z < 1\} \in \mathbb{R}^3$ with the desired properties (like high aspect ratio of tubes). Thus, the function $u$ must have singularities in this layer, out of the domain $\Omega$. Actually, it can satisfy the Poisson equation with the right-hand side having a special $\delta$-like form. Here we use an idea close to that used in [8, 9].

First, we simulate one tube by taking the right-hand side in the form $4\pi\delta(x)\delta(y)\rho(z)$. This right-hand side corresponds to a vertically aligned infinitely thin "charged stick" with linear charge density $\rho(z) = \nu_1 z + \nu_3 z^3,\ z \in [-h, h]$. The cubic density enables us to simulate a tube with almost vertical walls of given diameter $d$, whereas the linear charge density $\nu_1 z$ gives a tube of the form of a smoothed triangle. We use dipole-like "sticks" (with symmetric charge density with respect to $z$) to obtain fast decaying potentials.

To construct an array of tubes, one can place the set of such "sticks" in different points and then summarize the corresponding potentials. This procedure changes the characteristics of individual tubes, which reflects the screening effect that is important for applications. We shall discuss the details when presenting the formulas. The above ideas could be used in some general (irregular) situations; however, as was mentioned above, we restrict ourselves to an array of tubes with "regular" properties.

The geometrical parameters in real arrays differ from tube to tube, and thus it is important to obtain a simple family of solutions including several parameters. Changing these parameters, one can easily analyze the emission current dependence on the averaged geometrical characteristics. Our approach is based on a selection of such a solution solving the problem for a certain boundary surface $z = g(x, y)$, which is appropriate for our purposes. We suggest a family of solutions with parameters $h, s, \nu_1, \nu_3$ as follows:

$$u(x,y,z) = z - V(x,y,z) + V_{\mathrm{refl}}(x,y,z)\ \Big(V(x,y,z) \equiv \sum_{i,j\in\mathbb{Z}} V_0(R_{ij}, z),\ V_{\mathrm{refl}}(x,y,z) \equiv V(x,y,2-z)\Big),$$

$$V_0(R_{ij}, z) \equiv \int_{-h}^{h} \frac{(\nu_1\zeta + \nu_3\zeta^3)\,d\zeta}{\sqrt{R_{ij}^2 + (z-\zeta)^2}}, \quad R_{ij}^2 \equiv (x - is)^2 + (y - js)^2. \tag{2.1}$$

**Proposition 1.** *The function $u(x, y, z)$ given by (2.1) is a solution of problem (1.1) for the lower boundary condition $u|_\Gamma = 0$. This lower boundary $\Gamma$ is the zero level surface of the potential $u(x, y, z)$ and has the form of an array of tubes (as in Fig. 1). The parameters $\nu_1(h, s, d)$ and $\nu_3(h, s, d)$ depend on the geometrical characteristics $h$, $s$, and $d$ of the array.*

The function $V_0$ gives the potential of a thin "charged stick" with charge density

$$\rho(x,y,z) = (\nu_1\zeta + \nu_3\zeta^3)\delta(x)\delta(y), \qquad |z| \leqslant h,$$

with the Dirac delta functions $\delta(x)$. The functions $V$ and $V_{\mathrm{refl}}$ solve the Poisson equation for an array of thin "charged sticks" (see Fig. 2),

$$\Delta_{ij} = \{x = is,\ y = is,\ -h \leqslant z \leqslant h\}, \quad \Delta_{ij}^{\mathrm{refl}} = \{x = is,\ y = is,\ 2 - h \leqslant z \leqslant 2 + h\},$$

respectively. Thus, the function $u(x, y, z)$ provides a solution for the Laplace equation in the layer

$$\Omega = \{0 < z < 1\} \Big\backslash \bigcup_{i,j} \Delta_{ij}.$$

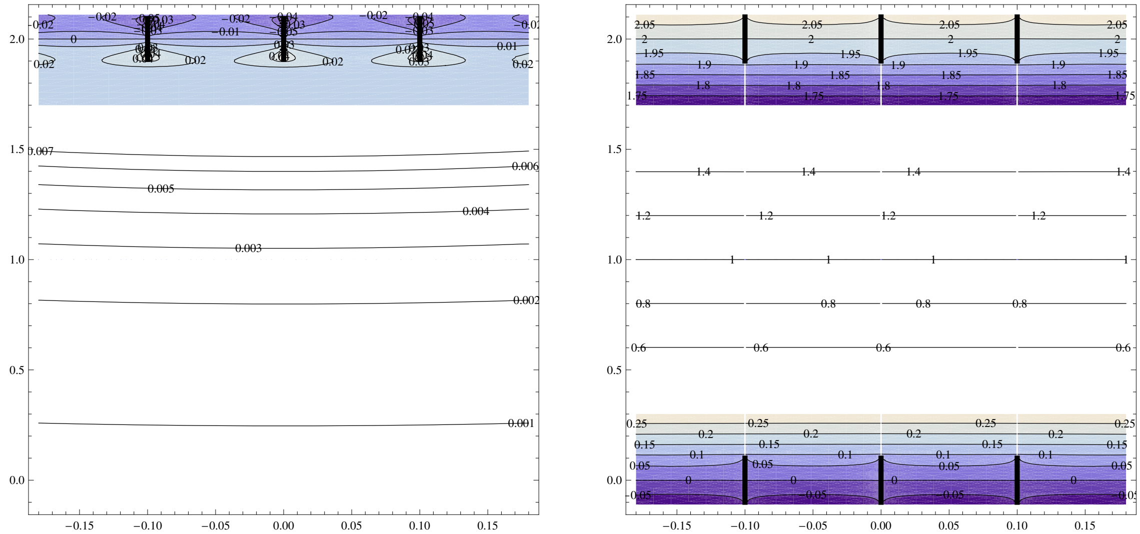


**Fig. 2.** Level curves for the potential $V_{\mathrm{refl}}$ (left) and for the solution $u = z - V + V_{\mathrm{refl}}$. Thick vertical intervals illustrate the localized charges.

**Proposition 2.** *The integral in the definition of $V_0(R, z)$ can be evaluated using elementary functions,*

$$V_0(R,z) = \int_{-h}^{h} \frac{(\nu_1\zeta + \nu_3\zeta^3)\,d\zeta}{\sqrt{R^2 + (z-\zeta)^2}} = \frac{\big(-\nu_1 + \frac{2}{3}\nu_3 R^2 - \frac{1}{3}\nu_3 h^2 - \frac{11}{6}\nu_3 z^2\big)4hz}{\sqrt{R^2+(z+h)^2} + \sqrt{R^2+(z-h)^2}} + \frac{5}{6}\nu_3 hz\big(\sqrt{R^2+(z+h)^2}$$
$$+ \sqrt{R^2+(z-h)^2}\big) + z\Big(\nu_1 - \frac{3}{2}\nu_3 R^2 + \nu_3 z^2\Big)\log\frac{\sqrt{R^2+(z+h)^2}+z+h}{\sqrt{R^2+(z-h)^2}+z-h}. \tag{2.2}$$

*The potential $V_0(R, z)$ has the following asymptotics at infinity*:

$$V_0(R,z) = zh^3\Big(\frac{2}{3}\nu_1 + \frac{2}{5}\nu_3 h^2\Big)\frac{1}{R^3} + O\Big(\frac{1}{R^5}\Big), \quad R \to +\infty,$$
$$V_0(R,z) = h^3\Big(\frac{2}{3}\nu_1 + \frac{2}{5}\nu_3 h^2\Big)\frac{1}{z^2} + O\Big(\frac{1}{z^4}\Big), \quad z \to +\infty.$$

This allows us to write explicit asymptotic formulas for the potential $u$ and for the emission current density $I$ for an array of tubes with given geometrical characteristics $h, d, s$.

The potential $V_{\mathrm{refl}}$ is quite small in the layer $0 < z < 1$, namely, it is of order $O(\frac{h^3}{(2-z)^2})$ (see Fig. 2, left). It gives a very small correction to the zero-level surface of the solution $u(x, y, z)$ and to the derivatives of the function $u(x, y, z)$ for $z \leqslant h$. Thus, when studying the solution for $z \leqslant h$, we can omit the term $V_{\mathrm{refl}}$ in our future considerations and use the following formula for solution $u(x, y, z)$:

$$u(x,y,z) = z - V(x,y,z) + O(h^3/(2-z)^2). \tag{2.3}$$

## 3. ANALYSIS OF THE POTENTIAL DESCRIBING A SINGLE TUBE

We need to calculate the field enhancement factor $\beta$ and the emission current density $I$ for given geometrical characteristics of a given array of tubes. We calculate them in two steps: we express parameters $h, s, \nu_1, \nu_3$ of the model via geometrical characteristics of the array and then calculate $\beta$ and $I$.

First we derive formulas for the case of a single tube. In the case of array of tubes, the algorithm is the same. The presence of neighboring tubes corrects formulas for $\nu_1$ and $\nu_3$. However, the dependence of $\beta$ and $I$ on the parameters of the model $h, s, \nu_1, \nu_3$ is asymptotically the same, because $\beta$ and $I$ depend only on derivatives of the potential $u$, and the major impact to this derivatives at the point near a tube is given by the potential of this tube.

Consider the potential for a single tube,

$$u(x,y,z) = z - V_0(r,z), \quad r^2 = x^2 + y^2.$$

It consists of the plane $z = 0$ and the surface that simulates a tube and has the form of a "hat." These surfaces intersect along a curve $\gamma \in \{z = 0\}$, which is almost a circle. Below, by the zero level surface we mean the surface of the tube with the part of the plane $z = 0$ outside the curve $\gamma$ (and hence outside the tube). The zero level surface of the function $u$ (which has the form of a "hat") obtained by using the Wolfram Mathematica software is represented on Fig. 3 by thin lines. We present the cut lines laying on the plane $y = 0$.

### *3.1. Analysis of the Surface of a Single Tube*

Let us study the equation $u = 0$ with respect to variable $z$ outside the vicinity of the top ($z < h - r$). The interesting surface $r = r_0(z)$ corresponds to $r \sim d \Rightarrow r \ll h$. Thus, we can try to expand the potential in a neighborhood of point $r = 0$. Omitting a small correction $O(\frac{r}{h})$, we obtain

$$r_0^2(z) = 4(h^2 - z^2)\exp\Big\{-\frac{1 + 2\nu_1 - \nu_3 h^2 + \frac{11}{3}\nu_3 z^2}{\nu_1 + \nu_3 z^2}\Big\}\big(1 + O(r/h)\big). \tag{3.1}$$

The radius $r_0(0)$ of the bottom $z = 0$ is

$$r_0(0) = 4h^2 \exp\big\{-(1 + 2\nu_1 - \nu_3 h^2)/\nu_1\big\}\big(1 + O(r/h)\big). \tag{3.2}$$

Formula (3.1) gives an approximate description $\Gamma_0$ of the boundary of the tube $\Gamma = \{r = r_0(z)\}$. The corresponding graphs (see Fig. 3) show very good coincidence of $\Gamma$ and $\Gamma_0$, at least in the middle domain. Moreover, it seems that one has to make a small correction to obtain the description to $\Gamma_0$ near the top of $\Gamma$. This implies that the height of the tube is approximately equal to $h$ (and it is the reason for the separation of the zero level curve into two parts; on the upper domain, we shall construct a small correction near $z = h$).

Now we study the behavior of the boundary near the top ($z = h + \delta z, \delta z \ll h, r \ll h$). We expand the potential $u$ in the neighborhood of the point $z = h, r = 0$. Omitting the correction $O(\delta z/h) + O(r/h)$, we obtain

$$z - h \approx -\frac{1}{2}\kappa r^2 + \frac{1}{2}\frac{1}{\kappa}, \tag{3.3}$$

$$\kappa \equiv \frac{1}{4h}\exp\frac{1 + 2\nu_1 + \frac{8}{3}\nu_3 h^2}{\nu_1 + \nu_3 h^2} > 0. \tag{3.4}$$

The value $(-\kappa)$ is the curvature of the zero-level surface of the function $u$ at the top point.

In the vicinity of the top of the tube, the approximation $\Gamma_{\mathrm{top}}$ (3.3) and the original curve $\Gamma$ practically coincide (see Fig. 3). The surface $\Gamma_0$ gives a good approximation for the middle and the bottom parts of the tube.

At the point $r = 0$, $z|_{r=0} = h + 1/(2\kappa)$.

### *3.2. Computation of Parameters of the Model*

We have 3 model parameters $h, \nu_1, \nu_3$ to simulate a single tube with height $h_{\mathrm{t}}$ and diameter $d_{\mathrm{t}}$. We define "the diameter of the tube" as the diameter of its bottom $2r_0(0)$ and set, for instance, $\kappa \equiv h/d^2$. Then we find $h, \nu_1, \nu_3$ from the following equations:

$$h = h_{\mathrm{t}} - \frac{1}{2\kappa} \approx h_{\mathrm{t}}, \tag{3.5}$$

$$u\big|_{r=0,\ z=h+\frac{1}{2\kappa}} = 0, \tag{3.6}$$

$$\lim_{z\to+0}\big(\frac{u}{z}\big)\big|_{r=d_{\mathrm{t}}/2} = 0. \tag{3.7}$$

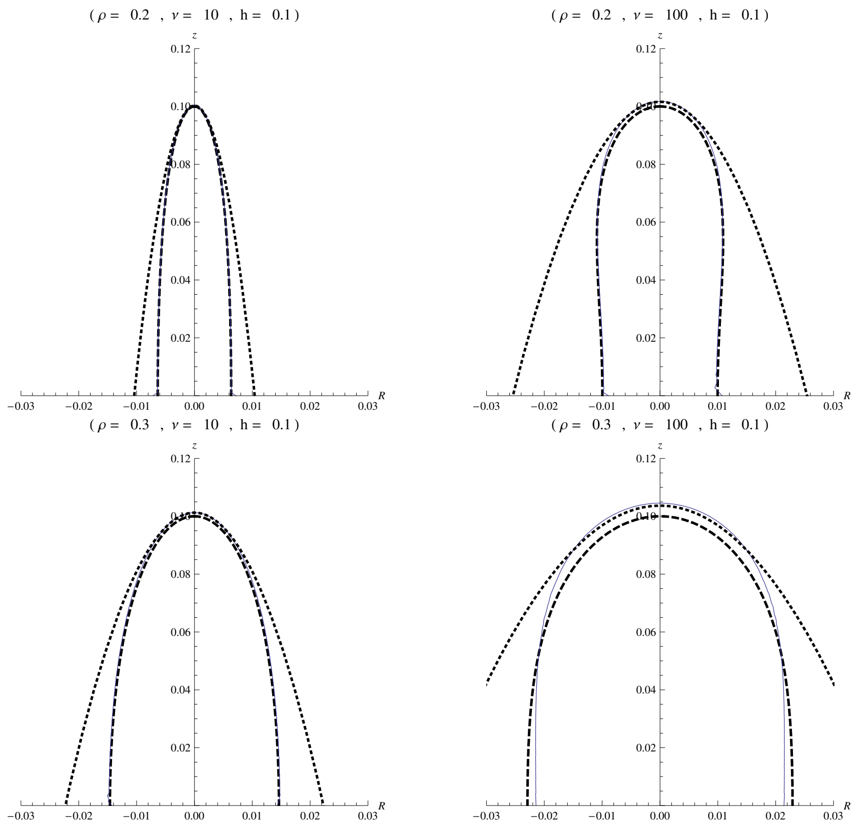


**Fig. 3.** Function $u$ zero-level surface $\Gamma$ (solid) and its asymptotics $\Gamma_0$ (dashed) and $\Gamma_{\rm top}$ (dotted) for different $\nu_1$ and $\nu_3$.

Using formulas (3.3) and (3.2), we find (omitting the corrections $O(d/h)$ and $O(1/\kappa)$):

$$\nu_1=\frac{1}{\mathcal{D}}\Big(\frac{11}{3}-\log(4h\kappa)\Big),\nu_3=\frac{1}{h^2\mathcal{D}}\Big(\log\frac{d^2\kappa}{4h}\Big),\mathcal{D}\equiv(-2+\log(4h\kappa))-\Big(-\frac{8}{3}+\log(4h\kappa)\Big)\Big(-2+\log\frac{16h^2}{d^2}\Big). \tag{3.8}$$

There is some arbitrariness in choosing $\kappa$. For instance, the value $-\kappa=-h/d^2$ is equal to the curvature $\frac{\partial^2 z(x)}{\partial x^2}$ of the ellipse $\frac{z^2}{(h/4)^2}+\frac{x^2}{(d/2)^2}=1$ at the point $x=0,z=h/4$. The value of $\kappa$ of this order gives an appropriate form of the tube given by (2.3) and (2.2).

### *3.3. Enhancement Factor and Emission Current density.*

The field enhancement factor $\beta$ for a single axial-symmetric tube at the point $z=h+\frac{1}{2\kappa},r=0$ is equal to

$$\beta|_{r=0}=1+\int_{-h}^{h}\frac{(\nu_1\zeta+\nu_3\zeta^3)d\zeta}{(h+\frac{1}{2\kappa}-\zeta)^2}=2\kappa(\nu_1h+\nu_3h^3)\Big(1+O\Big(\frac{1}{\kappa}\Big)\Big). \tag{3.9}$$

For its derivatives at the top point, we have

$$\frac{\partial\beta}{\partial x}\Big|_{r=0}=\frac{\partial\beta}{\partial y}\Big|_{r=0}=0,\quad \frac{\partial^2\beta}{\partial x\partial y}\Big|_{r=0}=0,\quad \frac{\partial^2\beta}{\partial x^2}\Big|_{r=0}=\frac{\partial^2\beta}{\partial y^2}\Big|_{r=0}\quad\Rightarrow$$

$$\sqrt{\det\big(-\operatorname{Hess}\beta\big)}\Big|_{r=0}=-\frac{\partial^2\beta}{\partial x^2}\Big|_{r=0}=3(2\kappa)^3(\nu_1h+\nu_3h^3)\times\Big(1+O\Big(\frac{1}{\kappa}\Big)\Big). \tag{3.10}$$

The emission current density of a single tube on an area with square $s^2$ is calculated by using the Laplace method,

$$I = 2\pi \frac{AF_0^3}{B} \times \frac{1}{s^2}\frac{2\kappa}{3}(\nu_1 h + \nu_3 h^3)^3 \exp\Big\{-\frac{B/F_0}{2\kappa(\nu_1 h + \nu_3 h^3)}\Big\} \times \Big(1 + O\Big(\frac{F_0}{B}\Big) + O\Big(\frac{1}{\kappa}\Big)\Big), \qquad (3.11)$$

where parameters $\nu_1$ and $\nu_3$ for a single tube are found from (3.5).

## 4. ANALYSIS FOR AN ARRAY OF TUBES

### *4.1. Computation of the Parameters of the Model*

We characterize an array of tubes by the height $h$ and the diameter $d$ of the tubes and the distances between them, $s$. The potential for the array is defined by formulas (2.1) and depends on four parameters: on the height $h$ and the distances $s$ (explicitly) and on the parameters $\nu_1 > 0$ and $\nu_3 > 0$ (implicitly). As in the case of a single tube, we have a system for $\nu_1, \nu_3$,

$$u|_{x=y=0, z=h+\frac{1}{2\kappa}} = 0, \qquad \lim_{z\to+0} u|_{x^2+y^2=d^2/4} = 0.$$

Using formula (2.2) for $V_0$, we can present this system in the following asymptotic form:

$$1 = \Sigma_1\nu_1 + \Sigma_2\nu_3, \qquad 1 = \Sigma_3\nu_1 + \Sigma_4\nu_3;$$

$$\begin{aligned}
\Sigma_1 &\equiv \sum_{R_{ij}} \Bigg[\frac{\sqrt{R_{ij}^2 + 4h^2} - R_{ij}}{h} + \log\frac{\sqrt{R_{ij}^2 + 4h^2} + 2h}{\sqrt{R_{ij}^2 + \frac{1}{4\kappa^2}} + \frac{1}{2\kappa}}\Bigg] \times \Big(1 + O\Big(\frac{1}{\kappa}\Big)\Big),\\
\Sigma_2 &\equiv \sum_{R_{ij}} \Bigg[\Big(\frac{2}{3}R^2 - \frac{13}{6}h^2\Big)\frac{4h}{\sqrt{R_{ij}^2 + 4h^2} + R} + \frac{5}{6}h\Big(\sqrt{R_{ij}^2 + 4h^2} + R\Big)\\
&\quad + \Big(-\frac{3}{2}R_{ij}^2 + h^2\Big)\log\frac{\sqrt{R_{ij}^2 + 4h^2} + 2h}{\sqrt{R_{ij}^2 + \frac{1}{4\kappa^2}} + \frac{1}{2\kappa}}\Bigg] \times \Big(1 + O\Big(\frac{1}{\kappa}\Big)\Big),\\
\Sigma_3 &\equiv \sum_{i,j} \Bigg[-\frac{2h}{\sqrt{R_{ij,d}^2 + h^2}} + 2\log\frac{\sqrt{R_{ij,d}^2 + h^2} + h}{R_{ij,d}}\Bigg] \times \Big(1 + O\Big(\frac{d}{h}\Big)\Big),\\
\Sigma_4 &\equiv \sum_{i,j} \Bigg[(3R_{ij,d}^2 + h^2)\frac{h}{\sqrt{R_{ij,d}^2 + h^2}} - 3R_{ij,d}^2\log\frac{\sqrt{R_{ij,d}^2 + h^2} + h}{R_{ij,d}}\Bigg] \times \Big(1 + O\Big(\frac{d}{h}\Big)\Big),
\end{aligned} \qquad (4.1)$$

$$R_{ij} \equiv \sqrt{(is)^2 + (js)^2}, \quad R_{ij,d} \equiv \sqrt{(d/2 + is)^2 + (js)^2}, \quad \kappa \equiv \frac{h}{d^2}.$$

This gives asymptotic formulas for $\nu_1, \nu_3$ for given geometrical parameters:

$$\nu_1 = \frac{\Sigma_4 - \Sigma_2}{\Sigma_1\Sigma_4 - \Sigma_3\Sigma_2}, \quad \nu_3 = \frac{\Sigma_1 - \Sigma_3}{\Sigma_1\Sigma_4 - \Sigma_3\Sigma_2}. \qquad (4.2)$$

Figure 4 shows the form of the array of tubes simulated by the suggested model for different distances between tubes. There are $x$-plane cuts to the left and $z$-plane cuts to the right.

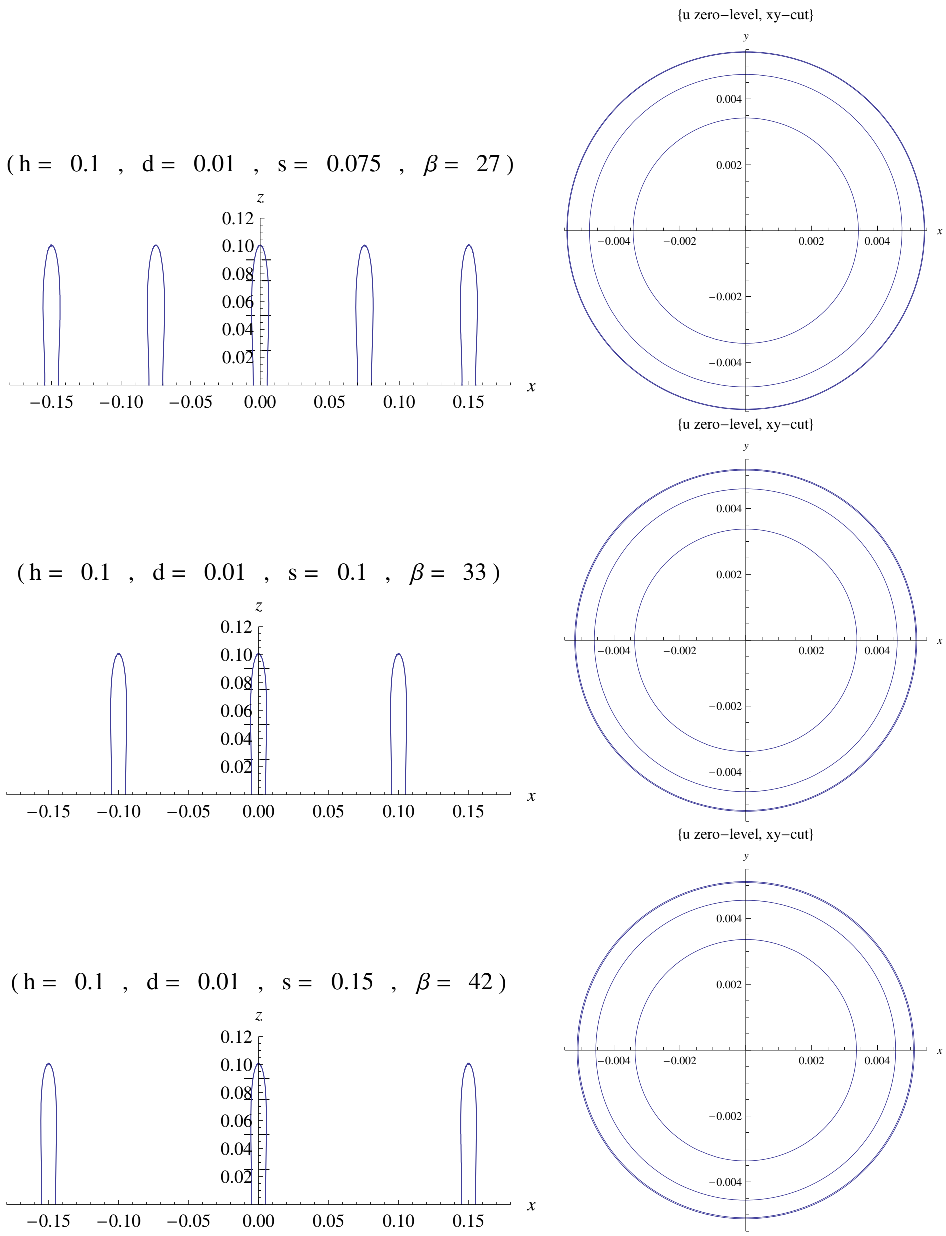


**Fig. 4.** Zero level curves of function $u$ in the plane $y = 0$ (left) and in the planes $z =$const (right) for $z = 0.5h$, $z = 0.75h$, and $z = 0.9h$ for different distances $s$.

Note that $\kappa$ is not a physical value, it is just a parameter of the suggested mathematical model. To obtain the right form of tubes, we take $\kappa \equiv h/d^2$. One can take a slightly different value of $\kappa$ (of the same order $h/d^2$) to find the suitable enhancement factor $\beta$ of a single tube. Then this $\kappa$ should be used to study an array of such tubes.

The order of the enhancement factor is just as that of $\kappa$, namely, $\beta \sim \kappa \sim h/d^2$. It corresponds to the enhancement factor of a multistage tube [4] which is much greater than the enhancement factor for a "one-stage" tube $\beta \sim h/d$. It seems that the consideration of additional terms in the Taylor expansion of the charge density $\rho(\zeta)$ can help to simulate a tube with smaller values of $\kappa$ and $\beta$.

### *4.2. Emission Current*

Having formulas (4.1), (4.2) for $\nu_1$ and $\nu_3$ and formula (2.3) for the solution $u$, we can calculate

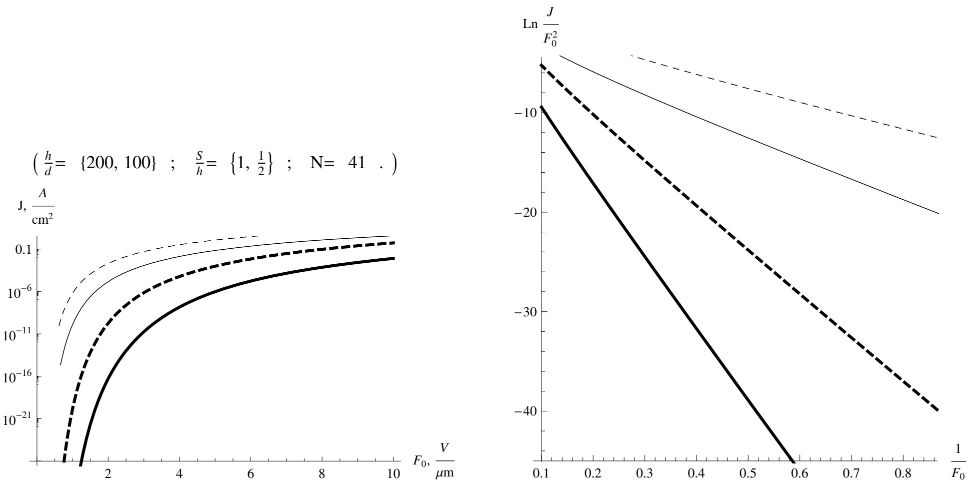


**Fig. 5.** Current–voltage diagram in $(I, V)$ and $\left(\log \frac{I}{V^2}, \frac{1}{V}\right)$ coordinates for the aspect ratio of tubes $h/d = 100$ (thick) and $= 200$ (thin) and the distance between tubes $s/h = 0.5$ (solid) and $= 1$ (dashed).

the enhancement factor and the emission current density for an array of tubes with given geometrical characteristics. For this purpose, it is sufficient to calculate derivatives of the solution ($\beta = \partial u/\partial z$ and Hess$\beta$) at the top point $R = 0, z = h + 1/2\kappa$. At this point, the central tube gives a much larger impact (with respect to $(1/\kappa) \ll 1$) to these derivatives then the other tubes altogether do. This makes it possible to calculate the enhancement factor and its Hessian asymptotically by formulas (3.6) and (3.7) as in the case of a single tube.

**Proposition 3.** *In the case of an array of tubes, the enhancement factor $\beta$ and the emission current density $I$ are equal to*

$$\beta|_{r=0} = 1 + \int_{-h}^{h} \frac{(\nu_1 \zeta + \nu_3 \zeta^3) d\zeta}{(h + \frac{1}{2\kappa} - \zeta)^2} = 2\kappa(\nu_1 h + \nu_3 h^3)\big(1 + O(1/\kappa)\big), \tag{4.3}$$

$$I \approx 2\pi \frac{AF_0^3}{B} \times \frac{1}{s^2} \frac{2\kappa}{3} (\nu_1 h + \nu_3 h^3)^3 \exp\Big\{ - \frac{B/F_0}{2\kappa(\nu_1 h + \nu_3 h^3)} \Big\} \Big(1 + O\Big(\frac{F_0}{B}\Big) + O\Big(1/\kappa\Big)\Big), \tag{4.4}$$

*where parameters $\nu_1$ and $\nu_3$ can be found from* (4.2).

The current-voltage diagram for different aspect ratio $h/d$ of tubes and different distances between tubes is presented on Fig. 5. Here the height of tubes is taken to give $h/D = 0.1$. The dependence of the field enhancement factor $\beta$ and the emission current density $I$ on the distance between tubes $s$ is presented on Fig. 6 ($h/D = 0.1$).

### *4.3. Conclusions*

We propose explicit asymptotic formulas for the emission current density $I$ via geometrical characteristics. These formulas admit easy and fast computer realization and can be used for numerical studies. For example, it is easy to calculate the optimum distance $s$ between tubes that gives the maximum of the emission current density. The knowledge of the optimum distance is important for applications ([1, 10, 11]). The optimum distance is of the order of the height $h$ of tubes. It depends both on the geometry of the array (especially on the tube aspect ratio) and on the applied voltage $V$ (in $V/\mu m$).

In the present paper, we consider regular arrays of identical tubes. For future applications, the study of "irregular" arrays of different tubes or/and with different distances between tubes is of high interest ([1, 8]). For such arrays, the emission current density should be expressed in terms of average geometrical characteristics.

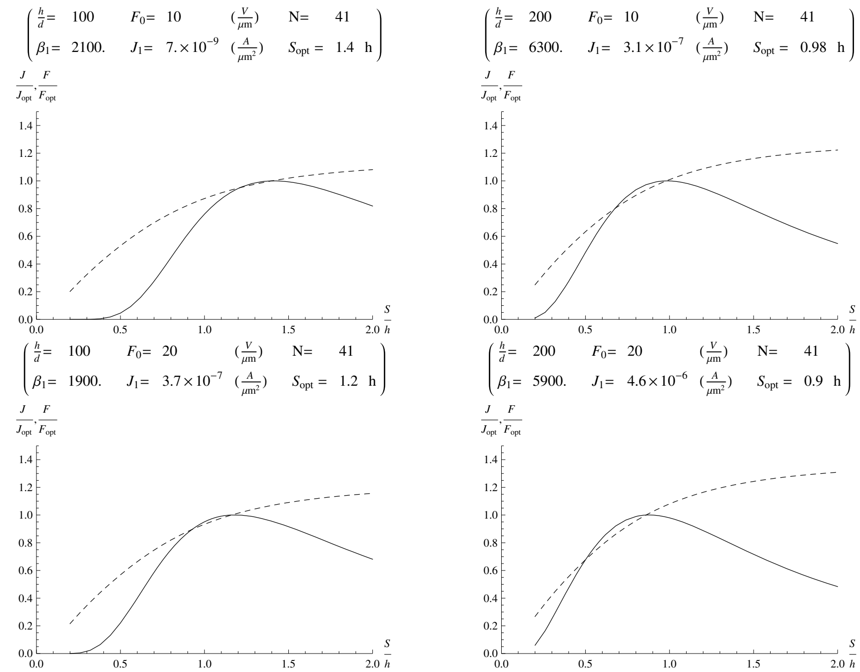


**Fig. 6.** The dependence of field $F$ and the emission current density $I$ on the distance between tubes. Both field and current density are normalized with respect to their values $F_{\mathrm{opt}}$ and $I_{\mathrm{opt}}$ for the optimum distance $s_{\mathrm{opt}}$.

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