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On the gluing formula for the analytic torsion

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Abstract In this paper, we derive the Cheeger–Müller/Bismut–Zhang theorem for manifolds with boundary and the gluing formula for the analytic torsion of flat vector bundles in full generality, i.e., we do not assume that the Hermitian metric on the flat vector bundle is flat nor that the Riemannian metric has product structure near the boundary.

0 Introduction

Given a flat complex vector bundle F of rank $\mathrm{rk}(F)$ with flat connection ∇^F on a compact m-dimensional smooth Riemannian manifold X without boundary, the Ray–Singer analytic torsion [18] is a (weighted) linear combination of the determinants of the Laplacian with values in the differential forms twisted by F, and the Ray–Singer metric on the determinant of the cohomology of F is the product of its L^2 metric and the Ray–Singer analytic torsion. These are geometric invariants which depend on the metrics on F and on the Riemannian manifold.

Let us explain this in greater detail. For a finite dimensional complex vector space E, set $\det E := \Lambda^{\max} E$, and denote by $(\det E)^{-1} := (\det E)^*$ the dual line. If we denote by $H^\bullet(X, F) = \bigoplus_{p=0}^m H^p(X, F)$ the singular cohomology of X with coefficients in F, then the determinant of the cohomology of F is the complex line

$$\det H^\bullet(X, F) = \bigotimes_{p=0}^{m} \left(\det H^p(X, F)\right)^{(-1)^p} . \tag{0.1}$$

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Denote by $\Omega^p(X, F)$ the space of smooth differential p-forms on X with values in F, and let $\Omega(X, F) = \oplus_p \Omega^p(X, F)$. The flat connection ∇^F extends naturally to a differential, d^F, on $\Omega(X, F)$. The de Rham theorem gives us a canonical isomorphism of $H^\bullet(X, F)$ and the cohomology of the de Rham complex $(\Omega(X, F), d^F)$.

Let g^{TX} be a Riemannian metric on X and let h^F be a Hermitian metric on F. Let d^{F*} be the (formal) adjoint operator of d^F associated with g^{TX} and h^F. Then $D := d^F + d^{F*}$ is a first order self-adjoint elliptic operator acting on $\Omega(X, F)$, and the heat semi-group $\exp(-tD^2)$ of D^2 preserves the spaces $\Omega^p(X, F)$ for any p.

Let Γ be the gamma function. For $u \in \mathbb{R}$, $u > m/2$, set

$$\theta(u) := -\frac{1}{\Gamma(u)} \int_0^\infty t^u \sum_{p=0}^m (-1)^p p \Big[\mathrm{Tr}|_{\Omega^p(X,F)}[\exp(-tD^2)] - \dim H^p(X, F) \Big] \frac{dt}{t}. \quad (0.2)$$

The function θ extends to a meromorphic function of $u \in \mathbb{C}$ which is holomorphic at $u = 0$. The Ray–Singer analytic torsion of X with coefficients in F is defined as

$$T(X, g^{TX}, h^F) = \exp\left\{ \frac{1}{2} \frac{\partial \theta}{\partial u}(0) \right\}. \quad (0.3)$$

By identifying $H^\bullet(X, F)$ with the space of harmonic forms, $\mathrm{Ker}(D^2)$, by Hodge theory, $H^\bullet(X, F)$ is naturally equipped with a L^2-metric $h^{H^\bullet(X,F)}$ induced by the L^2-metric of $\Omega(X, F)$. We denote by $|\cdot|^{L^2}_{\det H^\bullet(X,F)}$ the induced metric on $\det H^\bullet(X, F)$. The Ray–Singer metric on the line $\det H^\bullet(X, F)$ is defined by

$$\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,F)} := T(X, g^{TX}, h^F) |\cdot|^{L^2}_{\det H^\bullet(X,F)}. \quad (0.4)$$

If h^F is flat (i.e., (F, ∇^F) is induced by an unitary representation of the fundamental group of X), then the celebrated Cheeger–Müller theorem [7,16] tells us that in this case the Ray–Singer metric can be identified with the so-called Reidemeister metric on $\det H^\bullet(X, F)$ which is a topological invariant of the flat vector bundle F [15]. Bismut and Zhang [4] and Müller [17] simultaneously considered generalizations of this result. Müller [17] extended his result to the case where the dimension of the manifold is odd and only the metric induced on $\det F$ is required to be flat. Bismut and Zhang [4] generalized the original Cheeger–Müller theorem to arbitrary flat vector bundles with arbitrary Hermitian metrics. There are also various extensions to the equivariant case, cf. [11,12,5]. Bismut and Goette [3] obtained a family version of the Bismut–Zhang theorem under the assumption that there exists a fiberwise Morse function for the fibration in question, which generalizes all the above results. See also [8] for a survey on work relevant to this line of approach.

Assume now that X is a manifold with boundary. The corresponding results were studied in [11] and [12,22,9], under the assumption that h^F is flat and that g^{TX} has product structure near the boundary. In particular, these papers establish a Cheeger–Müller type theorem and a gluing formula for the Ray–Singer metric in this setting. In [6, Theorem 0.1], we extended the anomaly formula for Ray–Singer metrics [4, Theorem 0.1] to manifolds with boundary, not assuming that the Hermitian metric on the flat vector bundle is flat nor that the Riemannian metric has product structure near the boundary (more relevant references can be found in [6]). In [13], Ma and Zhang extended the results in [6] to the L^2-analytic torsion.

In this paper, we will establish the Cheeger–Müller/Bismut–Zhang theorem on manifolds with boundary, Theorem 0.1, and the corresponding gluing formula, Theorem 0.3, without any condition on the Riemannian metric of the manifold or the Hermitian metric of the flat vector bundle. To do so, we will derive first a Bismut–Zhang type theorem, Theorem 2.2, which is a special case of Theorem 0.1, for flat vector bundles on manifolds X with boundary under the assumption that g^{TX}, h^F have product structure near the boundary, as an application of [5, Theorem 0.2]. Then we establish the anomaly formula for the analytic torsion, Theorem 3.4, which extends the corresponding result of [6] to the situation where we impose the relative boundary condition on some components of the boundary, and the absolute boundary condition on the complement. Finally, we obtain the gluing formula, Theorem 0.3, for the analytic torsion of flat vector bundles, by combining Theorem 2.2 and Theorem 3.4. Again we do not assume that the Hermitian metric on the flat vector bundle is flat nor that the Riemannian metric has product structure near the boundary.

Let us describe the geometric setting in greater detail. We assume that the manifold X has boundary $\partial X = Y_1 \cup V_1$, where Y_1, V_1 are disjoint (possibly empty) components of ∂X. Let $H^\bullet(X, Y_1, F)$ be the singular cohomology of (X, Y_1) with coefficients in F (cf. (1.6), (1.7)). The corresponding Hodge theorem (cf. Theorem 1.1) tell us that we can still identify $H^\bullet(X, Y_1, F)$ with a certain space of harmonic forms, which satisfy a suitable boundary condition. Then we can define the Ray–Singer analytic torsion $T(X, Y_1, g^{TX}, h^F)$ and the Ray–Singer metric $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F)}$ on the complex determinant line $\det H^\bullet(X, Y_1, F)$ (cf. Def. 1.4) in analogy with (0.2), (0.4).

Let f be a Morse function on X fulfilling the assertion of Lemma 1.5 below with an associated gradient vector field ∇f (which need not be induced by g^{TX}). Then the corresponding Thom–Smale complex [20] computes also $H^\bullet(X, Y_1, F)$, and this induces the Milnor metric $\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}$ on $\det H^\bullet(X, Y_1, F)$ which we define in Definition 1.6 below.

Note that the Milnor metric does not depend on the Riemannian metric g^{TX} on X, it depends only on the gradient vector field ∇f induced by f and the Hermitian metric h^F on F. If h^F is a flat metric on F, then $\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}$ is equal to the Reidemeister metric (which is defined by using any smooth triangulation of X) as in [15, Theorem 9.3], [4, Remark 1.10] (cf. Remark 1.8), thus it is a topological invariant.

Let $\theta(F, h^F)$ be the closed 1-form on X given by (cf. [4, Def. 4.5])

$$\theta(F, h^F) = \mathrm{Tr}\left[(h^F)^{-1}\nabla^F h^F\right]. \tag{0.5}$$

Recall that (F, ∇^F, h^F) is called unimodular if the metric $h^{\det F}$ on $\det F$ induced by h^F is flat. Thus $\theta(F, h^F) = 0$ if and only if (F, ∇^F, h^F) is unimodular.

Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}) and denote by $e(TX, \nabla^{TX})$ the Chern–Weil form of the Euler class of TX associated with ∇^{TX} [cf. (3.5)]. Let $g^{T\partial X}$ be the metric on $T\partial X$ induced by g^{TX}. Let ∇^{TY_1}, ∇^{TV_1} be the Levi-Civita connection on $(TY_1, g^{T\partial X}|_{Y_1})$, $(TV_1, g^{T\partial X}|_{V_1})$. Let $H^\bullet(\partial X, \mathbb{C})$ be the singular cohomology group of ∂X, and

$$\chi(\partial X) = \sum_j (-1)^j \dim H^j(\partial X, \mathbb{C}) \tag{0.6}$$

the Euler characteristic of ∂X.

Denote by δ_X the current of integration on X and by $\pi : TX \to X$ the natural projection. Let $\psi(TX, \nabla^{TX})$ be the Mathai–Quillen current on TX constructed in [4, §3], such that

$$d\psi\left(TX, \nabla^{TX}\right) = \pi^* e\left(TX, \nabla^{TX}\right) - \delta_X. \tag{0.7}$$

We define the currents $\psi(TY_1, \nabla^{TY_1})$, $\psi(TV_1, \nabla^{TV_1})$ on TY_1, TV_1 analogously.

Next we use the inward geodesic flow to identify a neighborhood of the boundary ∂X with the collar $\partial X \times [0, \varepsilon]$, and we identify $\partial X \times \{0\}$ with the boundary ∂X, then there exists a family of metrics, $g^{T\partial X}_{x_m}$, on $T\partial X$ defined by

$$g^{TX}|_{(y,x_m)} = dx_m^2 + g^{T\partial X}_{x_m}(y), \quad (y, x_m) \in \partial X \times [0, \varepsilon). \tag{0.8}$$

Let $\widetilde{g}^{TX}$ be a smooth metric on TX such that $\widetilde{g}^{TX} = g^{TX}$ on ∂X, and $\widetilde{g}^{TX}$ has product structure on $\partial X \times [0, \varepsilon)$,

$$\widetilde{g}^{TX}|_{(y,x_m)} = dx_m^2 + g^{T\partial X}(y). \tag{0.9}$$

Let $\widetilde{\nabla}^{TX}$ be the Levi-Civita connection on $(TX, \widetilde{g}^{TX})$. Let $\widetilde{E}(TX, \widetilde{\nabla}^{TX}, \nabla^{TX})$ be the secondary relative Euler class of TX in the sense of Chern–Simons defined in [6, Theorem 1.9] (cf. Theorem 3.1).

The first main result of this paper extends the Cheeger–Müller/Bismut–Zhang theorem to manifolds with boundary; it reads as follows.

Theorem 0.1 *We have the identity*

$$\begin{aligned}\log\left(\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F)}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}}\right)^2\right) = &-\int_X \theta(F,h^F)(\nabla f)^*\psi(TX,\widetilde{\nabla}^{TX})\\ &+\frac{1}{2}\int_{Y_1}\theta(F,h^F)(\nabla f)^*\psi(TY_1,\nabla^{TY_1})\\ &-\frac{1}{2}\int_{V_1}\theta(F,h^F)(\nabla f)^*\psi(TV_1,\nabla^{TV_1})\\ &+\int_{(X,Y)}\widetilde{E}(TX,\widetilde{\nabla}^{TX},\nabla^{TX})\theta(F,h^F)\\ &+\mathrm{rk}(F)\left(\int_{V_1}+(-1)^{m+1}\int_{Y_1}\right)B(\nabla^{TX})\\ &-\frac{1}{2}\mathrm{rk}(F)\chi(\partial X)\log 2,\end{aligned} \tag{0.10}$$

with the notation (3.10) *for* $\int_{(X,Y)}$ *and with* $B(\nabla^{TX})$ *the secondary characteristic form introduced in [6], [see* (3.6)*], which is zero if* ∂X *is totally geodesic in* (X, g^{TX}).

In particular, if (F, ∇^F, h^F) *is unimodular, then*

$$\begin{aligned}\log\left(\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F)}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}}\right)^2\right) = &\,\mathrm{rk}(F)\left(\int_{V_1}+(-1)^{m+1}\int_{Y_1}\right)B(\nabla^{TX})\\ &-\frac{1}{2}\mathrm{rk}(F)\chi(\partial X)\log 2.\end{aligned} \tag{0.11}$$

In fact, let $\widetilde{e}(TX, \nabla^{TX}_s)$, $\widetilde{e}_b(Y, \nabla^{TX}_s)$ be the forms defined in [6, Definition 1.8] for the path of metrics $g^{TX}_s = (1-s)\widetilde{g}^{TX} + sg^{TX}$, then with $o(TX)$ the orientation line,

$$\widetilde{E}(TX, \widetilde{\nabla}^{TX}, \nabla^{TX}) = (\widetilde{e}(TX, \nabla^{TX}_s), -\widetilde{e}_b(Y, \nabla^{TX}_s)) \in \Omega^{m-1}(X, Y, o(TX)) \tag{0.12}$$

[cf. (3.8)]. Thus by [6, (4.37)], (0.8), (0.9) and (3.15),

$$\begin{aligned} \widetilde{E}(TX, \widetilde{\nabla}^{TX}, \nabla^{TX}) &= 0 \quad \text{if } m \text{ is odd}, \\ \widetilde{e}_b(Y, \nabla_s^{TX}) &= 0 \quad \text{if } m \text{ is even}. \end{aligned} \tag{0.13}$$

Assume now (F, ∇^F, h^F) is unimodular and g^{TX} has product structure near the boundary [i.e., g^{TX} verifies (0.9)], then by Theorem 3.4, $\| \cdot \|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F)}$ does not depend on g^{TX}, and from (0.11), $\| \cdot \|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}$ does not depend on $f, \nabla f$, showing that it is a topological invariant. In fact by the same argument as in [15, Theorem 9.3], $\| \cdot \|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}$ is again equal to the Reidemeister metric (cf. Def. 1.7) on $\det H^\bullet(X, Y_1, F)$ which is a topological invariant. Note that (0.11) can not be directly obtained by passing to the doubled manifold and applying [5], as (2.3) does not hold in general. The anomaly formula, Theorem 3.4, plays a role here.

Remark 0.2 If h^F is flat and g^{TX} has product structure near ∂X, then (2.3) holds for h^F and (0.11) reduces to

$$\log \left(\frac{\| \cdot \|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F)}}{\| \cdot \|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}} \right)^2 = -\frac{1}{2} \mathrm{rk}(F) \chi(\partial X) \log(2). \tag{0.14}$$

This result was established in [12, Theorem 4.5], by passing to the double of X and applying results from [7,16] and [11].

Next we explain the gluing formula for the analytic torsion.

Let Z be a m-dimensional compact manifold with boundary $\partial Z = Y_1 \cup V_1 \cup Y_2 \cup V_2$, where Y_i, V_i are disjoint (possibly empty) components of ∂Z which we introduce to allow different boundary conditions on Y_i and V_i. We suppose that V is a closed hypersurface in the interior of Z such that $Z = Z_1 \cup_V Z_2$, and Z_1, Z_2 are compact manifolds with boundaries $\partial Z_1 = V \cup Y_1 \cup V_1$, $\partial Z_2 = V \cup Y_2 \cup V_2$, respectively, cf. Fig. 1.

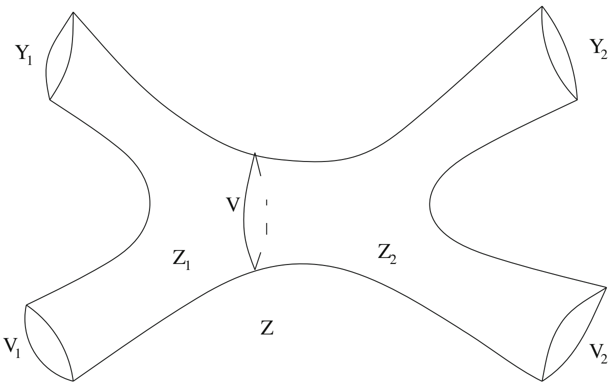

Fig. 1 The setting of the gluing formula

Let F be a flat complex vector bundle on Z with flat connection ∇^F. As above, we define the singular cohomology groups $H^\bullet(Z, Y_1 \cup Y_2, F)$, $H^\bullet(Z_2, Y_2, F)$ and $H^\bullet(Z_1, V \cup Y_1, F)$, and the corresponding complex lines $\det H^\bullet(Z, Y_1 \cup Y_2, F)$, $\det H^\bullet(Z_2, Y_2, F)$ and $\det H^\bullet(Z_1, V \cup Y_1, F)$.

Let $\mathcal{K}_Z$ be a smooth triangulation of Z such that $\mathcal{K}_Z$ induces also a smooth triangulation of V. We denote by $\mathcal{K}_{Z_1}, \mathcal{K}_{Z_2}, \mathcal{K}_{Y_1}, \mathcal{K}_{Y_2}, \mathcal{K}_V$ the smooth triangulation of Z_1, Z_2, Y_1, Y_2, V induced by $\mathcal{K}_Z$. Then by the argument leading to (1.6), we have a short exact sequence of complexes of dual simplicial chains

$$\begin{array}{ccccccccc} & & \uparrow \tilde{\partial} & & \uparrow \tilde{\partial} & & \uparrow \tilde{\partial} & & \\ 0 & \longrightarrow & C^\bullet(\mathcal{K}_{Z_1}, \mathcal{K}_{Y_1} \cup \mathcal{K}_V, F) & \longrightarrow & C^\bullet(\mathcal{K}_Z, \mathcal{K}_{Y_1} \cup \mathcal{K}_{Y_2}, F) & \longrightarrow & C^\bullet(\mathcal{K}_{Z_2}, \mathcal{K}_{Y_2}, F) & \longrightarrow & 0, \end{array} \tag{0.15}$$

which induces an exact sequence in cohomology,

$$\cdots \to H^i(Z, Y_1 \cup Y_2, F) \to H^i(Z_2, Y_2, F) \to H^{i+1}(Z_1, V \cup Y_1, F) \to \cdots. \tag{0.16}$$

Recall that for an exact sequence of finite dimensional vector spaces

$$0 \to E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} E^n \to 0, \tag{0.17}$$

the complex determinant line $\det E := \bigotimes_{j=0}^n (\det E^j)^{(-1)^j}$ has a canonical section σ_δ (in other words, $\det E$ is canonically isomorphic to $\mathbb{C}$) (cf. [15, §3], [2, §1], [4, (1.4)]): we choose $s_{j,k} \in E^j$ such that $\{s_{j,k}\}_{k=1}^{k_j}$ projects to a basis of $E^j/\mathrm{Ker}(\delta|_{E^j})$, then with $\wedge_k s_{j,k} := s_{j,1} \wedge \cdots \wedge s_{j,k_j}$,

$$\sigma_\delta := (\wedge_k s_{0,k}) \otimes \Big((\wedge_k \delta(s_{0,k})) \wedge (\wedge_k s_{1,k}) \Big)^{-1} \otimes \cdots \otimes \big(\wedge_k \delta(s_{n-1,k})\big)^{(-1)^n} \in \det E \tag{0.18}$$

is non-vanishing and does not depend on the choice of $s_{j,k}$.

Any Hermitian metric h^{E^i} on E^i, $i = 1, \cdots, n$, induces a metric $\|\cdot\|_{\det E}$ on $\det E$. If δ^* is the adjoint of δ with respect to $h^E := \oplus_i h^{E^i}$, and $\Delta := (\delta + \delta^*)^2 = \delta^*\delta + \delta\delta^*$, then from [2, Theorem 1.5] (cf. [4, Theorem 1.1]), we know that

$$\|\sigma_\delta\|_{\det E} = \prod_j \big(\det(\Delta|_{E^j})\big)^{(-1)^j j/2} =: T(E, h^E). \tag{0.19}$$

We call $T(E, h^E)$ the analytic torsion of the exact sequence (E, v) associated with h^E.

Thus from (0.16) and (0.18), we get the canonical section ϱ of the complex line

$$\lambda(F) = (\det H^\bullet(Z, Y_1 \cup Y_2, F))^{-1} \otimes \det H^\bullet(Z_1, V \cup Y_1, F) \otimes \det H^\bullet(Z_2, Y_2, F). \tag{0.20}$$

Let g^{TZ} be a Riemannian metric on the tangent bundle TZ of Z, and let h^F be a Hermitian metric on F. Let $\chi(V)$ be the Euler characteristic of V as in (0.6).

Let $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(Z, Y_1 \cup Y_2, F)}$, $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(Z_1, V \cup Y_1, F)}$, $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(Z_2, Y_2, F)}$ be the Ray–Singer metrics on $\det H^\bullet(Z, Y_1 \cup Y_2, F)$, $\det H^\bullet(Z_1, V \cup Y_1, F)$, $\det H^\bullet(Z_2, Y_2, F)$ induced by g^{TZ}, h^F (cf. Def. 1.4). Let $\|\cdot\|^{\mathrm{RS}}_{\lambda(F)}$ be the corresponding Ray–Singer metric on $\lambda(F)$.

The second main result of this paper is the following gluing formula for the analytic torsion.

Theorem 0.3 *We have the identity*

$$\log\Big(\|\varrho\|^{\mathrm{RS},2}_{\lambda(F)}\Big) = -\mathrm{rk}(F)\chi(V)\log(2) + 2\,(-1)^{m+1}\,\mathrm{rk}(F)\int_V B(\nabla^{TZ_1}), \tag{0.21}$$

with $B(\nabla^{TZ_1})$ the secondary characteristic form introduced in [6], [see (3.6)], which is zero if V is totally geodesic in (Z, g^{TZ}).

Let $T(Z, Y_1 \cup Y_2, g^{TZ}, h^F)$, $T(Z_1, V \cup Y_1, g^{TZ}, h^F)$, $T(Z_2, Y_2, g^{TZ}, h^F)$ be the associated analytic torsions which will be defined in Definition 1.3. We denote by $T(\mathscr{H}, h^{\mathscr{H}})$ the analytic torsion (in the sense of (0.19)) of the exact sequence (0.16) with L^2-metrics induced by Theorem 1.1 (d) and $\mathscr{H}^0 = H^0(Z_1, V \cup Y_1, F)$. Then (0.21) can be reformulated as

$$\begin{aligned} &T(\mathscr{H}, h^{\mathscr{H}}) \cdot T(Z_1, V \cup Y_1, g^{TZ}, h^F) \cdot T(Z_2, Y_2, g^{TZ}, h^F) \\ &\quad = T(Z, Y_1 \cup Y_2, g^{TZ}, h^F) \cdot 2^{-\frac{1}{2}\mathrm{rk}(F)\chi(V)} e^{(-1)^{m+1}\mathrm{rk}(F)\int_V B(\nabla^{TZ_1})}. \end{aligned} \tag{0.22}$$

In the same way, from (0.18), (1.7), we get the canonical section $\widetilde{\varrho}$ of the complex line

$$\widetilde{\lambda}(F) = \big(\det H^\bullet(Z, F)\big)^{-1} \otimes \det H^\bullet(Z, Y_1, F) \otimes \det H^\bullet(Y_1, F). \tag{0.23}$$

Analogously, we obtain the following result.

Theorem 0.4 *We have the identity*

$$\log\left(\|\widetilde{\varrho}\|^{\mathrm{RS},2}_{\widetilde{\lambda}(F)}\right) = \big((-1)^{m+1} - 1\big)\,\mathrm{rk}(F) \int_{Y_1} B(\nabla^{TZ}), \tag{0.24}$$

with $B(\nabla^{TZ})$ the secondary characteristic form introduced in [6], [see (3.6)]. This expression is zero if Y_1 is totally geodesic in (Z, g^{TZ}).

Note that we do not assume that the Hermitian metric h^F is flat nor that g^{TZ} has product structure near the boundary ∂Z nor near V. If h^F is flat and g^{TZ} has product structure near the boundary ∂Z and near V, then Theorems 0.3, 0.4 have been established first in [12, Theorem 5.9] (cf. [22, Theorems 1.1, 1.2]).

This paper is organized as follows. In Sect. 1, we construct Ray–Singer metrics and Milnor metrics for a flat vector bundle. In Sect. 2, we establish the comparison formula for Ray–Singer metrics and Milnor metrics, Theorem 2.2, when the metrics have product structure near the boundary which is a special case of Theorem 0.1. In Sect. 3, we establish first the anomaly formula of the analytic torsion, Theorem 3.4, then we prove Theorem 0.1 by combining Theorem 2.2 with Theorem 3.4, and finally the gluing formula for the analytic torsion, Theorems 0.3, 0.4.

A preliminary version of this paper was written in 2006, for some recent related works see [10,21].

1 Ray–Singer metrics and Milnor metrics

This Section is organized as follows: we recall in Sect. 1.1 the construction of a simplicial complex from a smooth trivialization. Then we explain in detail the definition of Ray–Singer and Milnor metrics for a flat vector bundle on a compact manifold with boundary in Sects. 1.2 and 1.3, respectively.

1.1 Singular cohomology

From now on let X be a m-dimensional compact manifold with smooth boundary $\partial X = Y_1 \cup V_1$, where Y_1 and V_1 are disjoint (possibly empty) components of ∂X. Let F be a flat complex vector bundle on X with flat connection ∇^F. The following presentation follows [4, §1 b)].

Let $H_\bullet(X, F^*) = \oplus_j H_j(X, F^*)$ (resp. $H_\bullet(X, Y_1, F^*)$) denote the singular homology of X (resp.(X, Y_1)) with coefficients in F^*, and let $H^\bullet(X, F) = \oplus_j H^j(X, F)$ (resp.

$H^\bullet(X, Y_1, F)$) denote the singular cohomology of X (resp. (X, Y_1)) with coefficients in F. Then for $0 \leqslant j \leqslant m$, we have canonical identifications

$$H_j(X, F^*) = (H^j(X, F))^*, \quad H_j(X, Y_1, F^*) = (H^j(X, Y_1, F))^*. \tag{1.1}$$

Let $\mathcal{K}$ be a smooth triangulation of X, then $\mathcal{K}_1 := \mathcal{K}|_{Y_1}$ also is a smooth triangulation of Y_1. $\mathcal{K}$ consists of a finite set of simplexes, $\mathfrak{a}$, each with a fixed orientation. Let B be the finite subset of X formed by the barycenters of the simplexes in $\mathcal{K}$. Let $b : \mathcal{K} \to B$ denote the obvious one-to-one map. For $0 \leqslant i \leqslant m$, let $\mathcal{K}^i$ be the union of the simplexes in $\mathcal{K}$ of dimension $\leqslant i$, such that for $0 \leqslant i \leqslant m$, $\mathcal{K}^i \setminus \mathcal{K}^{i-1}$ is the union of simplexes of dimension i.

If $\mathfrak{a} \in \mathcal{K}$, let $[\mathfrak{a}]$ be the real line generated by $\mathfrak{a}$. Let $(C_\bullet(\mathcal{K}, F^*), \partial)$ be the complex of simplicial chains in $\mathcal{K}$ with values in F^*. For $0 \leqslant i \leqslant m$, we define

$$C_i(\mathcal{K}, F^*) := \sum_{\mathfrak{a} \in \mathcal{K}^i \setminus \mathcal{K}^{i-1}} [\mathfrak{a}] \otimes_{\mathbb{R}} F^*_{b(\mathfrak{a})}. \tag{1.2}$$

∂ maps $C_i(\mathcal{K}, F^*)$ into $C_{i-1}(\mathcal{K}, F^*)$. Also, the homologies of the complexes $(C_\bullet(\mathcal{K}, F^*), \partial)$ and $(C_\bullet(\mathcal{K}, F^*)/C_\bullet(\mathcal{K}_1, F^*), \partial)$ are canonically identified with the singular homologies $H_\bullet(X, F^*)$ and $H_\bullet(X, Y_1, F^*)$, respectively. We also have the short exact sequence of complexes

$$\begin{array}{ccccccccc} & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial & & \\ 0 & \longrightarrow & C_\bullet(\mathcal{K}_1, F^*) & \longrightarrow & C_\bullet(\mathcal{K}, F^*) & \longrightarrow & C_\bullet(\mathcal{K}, F^*)/C_\bullet(\mathcal{K}_1, F^*) & \longrightarrow & 0. \end{array} \tag{1.3}$$

The long exact sequence induced by (1.3) in homology is canonically identified with the exact sequence of singular homologies,

$$\cdots \to H_i(X, F^*) \to H_i(X, Y_1, F^*) \to H_{i-1}(Y_1, F^*) \to \cdots . \tag{1.4}$$

If $\mathfrak{a} \in \mathcal{K}$, let $[\mathfrak{a}]^*$ be the line dual to the line $[\mathfrak{a}]$. Let $(C^\bullet(\mathcal{K}, F), \widetilde{\partial})$ be the complex dual to the complex $(C_\bullet(\mathcal{K}, F^*), \partial)$. In particular, for $0 \leqslant i \leqslant m$, we have the identity

$$C^i(\mathcal{K}, F) = \sum_{\mathfrak{a} \in \mathcal{K}^i \setminus \mathcal{K}^{i-1}} [\mathfrak{a}]^* \otimes_{\mathbb{R}} F_{b(\mathfrak{a})}. \tag{1.5}$$

Let $(C^\bullet(\mathcal{K}, \mathcal{K}_1, F), \widetilde{\partial})$ be the dual complex of $(C_\bullet(\mathcal{K}, F^*)/C_\bullet(\mathcal{K}_1, F^*), \partial)$. The cohomology of the complexes $(C^\bullet(\mathcal{K}, F), \widetilde{\partial})$ and $(C^\bullet(\mathcal{K}, \mathcal{K}_1, F), \widetilde{\partial})$ is canonically identified with $H^\bullet(X, F)$ and $H^\bullet(X, Y_1, F)$, respectively. Then the short exact sequence of complexes

$$\begin{array}{ccccccccc} & & \uparrow \widetilde{\partial} & & \uparrow \widetilde{\partial} & & \uparrow \widetilde{\partial} & & \\ 0 & \longrightarrow & C^\bullet(\mathcal{K}, \mathcal{K}_1, F) & \longrightarrow & C^\bullet(\mathcal{K}, F) & \longrightarrow & C^\bullet(\mathcal{K}_1, F) & \longrightarrow & 0. \end{array} \tag{1.6}$$

induces the long exact sequence of singular cohomologies,

$$\cdots \to H^i(X, F) \to H^i(Y_1, F) \to H^{i+1}(X, Y_1, F) \to \cdots . \tag{1.7}$$

1.2 Ray–Singer metrics

Denote by $\Omega(X, F) := \oplus_{p=0}^m \Omega^p(X, F) := \oplus_{p=0}^m C^\infty(X, \Lambda^p(T^*X) \otimes F)$ the space of smooth differential forms on X with values in F. The flat connection ∇^F extends naturally to a differential, d^F, on $\Omega(X, F)$.

Let g^{TX} be a Riemannian metric on the tangent bundle TX of X, and let h^F be a Hermitian metric on F. Let $o(TX)$ be the orientation bundle of TX, which is a flat real line bundle on X. Let dv_X be the Riemannian volume element on (X, g^{TX}), then we can view dv_X as a section of $\Lambda^m(T^*X) \otimes o(TX)$. We define a Hermitian product on $\Omega(X, F)$ by

$$\langle \sigma, \sigma' \rangle := \int_X \langle \sigma, \sigma' \rangle_{\Lambda(T^*X)\otimes F} dv_X, \tag{1.8}$$

for $\sigma, \sigma' \in \Omega(X, F)$; we denote by $L^2(X, \Lambda(T^*X) \otimes F)$ the Hilbert space obtained by completion. Let d^{F*} be the formal adjoint of d^F with respect to (1.8). Set

$$D = d^F + d^{F*}. \tag{1.9}$$

Then for any p,

$$D^2 = d^F d^{F*} + d^{F*} d^F : \Omega^p(X, F) \to \Omega^p(X, F) \tag{1.10}$$

is the Hodge Laplacian associated with the pair of metrics g^{TX} and h^F.

Next we need to define self-adjoint extensions of D by elliptic boundary conditions. To do so, we use the metric on X to identify the normal bundle $\mathfrak{n}$ to ∂X in X with the orthogonal complement of $T\partial X$ in $TX|_{\partial X}$. Denote by $e_{\mathfrak{n}}$ the inward pointing unit normal vector field along ∂X, and by $e^{\mathfrak{n}}$ its dual vector. We use $i(\cdot)$ for interior and $w(\cdot)$ for exterior multiplication.

Let $\Omega^p_{\mathrm{bd}}(X, Y_1, F)$ be the subspace of $\Omega^p(X, F)$ defined by

$$\Omega^p_{\mathrm{bd}}(X, Y_1, F) := \left\{\sigma \in \Omega^p(X, F); \quad w(e^{\mathfrak{n}})\sigma = 0 \text{ on } Y_1, \quad i(e_{\mathfrak{n}})\sigma = 0 \text{ on } V_1\right\}. \tag{1.11}$$

Then the restriction of the operator D on $\Omega^p_{\mathrm{bd}}(X, Y_1, F)$ is self-adjoint.

In the same way, we define $\Omega^p_{\mathrm{bd},D^2}(X, Y_1, F)$ the subspace of $\Omega^p(X, F)$ by

$$\begin{aligned}\Omega^p_{\mathrm{bd},D^2}(X, Y_1, F) := \Big\{\sigma \in \Omega^p(X, F);\ & w(e^{\mathfrak{n}})\sigma = w(e^{\mathfrak{n}})(d^{F*}\sigma) = 0 \text{ on } Y_1, \\ & i(e_{\mathfrak{n}})\sigma = i(e_{\mathfrak{n}})(d^F\sigma) = 0 \text{ on } V_1\Big\}.\end{aligned} \tag{1.12}$$

Set

$$D^2_{\mathrm{bd}} = D^2|_{\Omega_{\mathrm{bd},D^2}(X, Y_1, F)}. \tag{1.13}$$

Thus D^2_{bd} is the operator D^2 with the relative boundary condition on Y_1 and the absolute boundary condition on V_1, and it is essentially self-adjoint.

We define the space of harmonic forms $\mathscr{H}^p(X, Y_1, F)$ by

$$\mathscr{H}^p(X, Y_1, F) = \left\{\sigma \in \Omega^p_{\mathrm{bd},D^2}(X, Y_1, F); D^2\sigma = 0\right\}. \tag{1.14}$$

Let $\mathcal{K}$ be a smooth triangulation of X as in Sect. 1.1. We define the de Rham map $P_\infty : \Omega(X, F) \to C^\bullet(\mathcal{K}, F)$ by

$$P_\infty(\sigma)(\mathfrak{a}) = \int_{\mathfrak{a}} \sigma \quad \text{for } \sigma \in \Omega(X, F), \mathfrak{a} \in C_\bullet(\mathcal{K}, F^*). \tag{1.15}$$

Theorem 1.1 (*Hodge decomposition theorem*)

(*a*) *We have*

$$\mathscr{H}^p(X, Y_1, F) = \mathrm{Ker}(d^F) \cap \mathrm{Ker}(d^{F*}) \cap \Omega^p_{\mathrm{bd}}(X, Y_1, F). \tag{1.16}$$

(*b*) *The spaces* $\mathscr{H}^p(X, Y_1, F)$ *are finite dimensional.*
(*c*) *We have the orthogonal decompositions*

$$\Omega^p_{\mathrm{bd}}(X, Y_1, F) = \mathscr{H}^p(X, Y_1, F) \oplus d^F\left(\Omega^{p-1}_{\mathrm{bd}}(X, Y_1, F)\right) \oplus d^{F*}\left(\Omega^{p+1}_{\mathrm{bd}}(X, Y_1, F)\right), \tag{1.17a}$$

$$L^2(X, \Lambda^p(T^*X) \otimes F) = \mathscr{H}^p(X, Y_1, F) \oplus \overline{d^F\left(\Omega^{p-1}_{\mathrm{bd}}(X, Y_1, F)\right)} \oplus \overline{d^{F*}\left(\Omega^{p+1}_{\mathrm{bd}}(X, Y_1, F)\right)}. \tag{1.17b}$$

Here $\overline{}$ *denotes the* L^2*-closure.*

(*d*) *The inclusion* $\iota : \mathscr{H}^p(X, Y_1, F) \to \mathrm{Ker}(d^F) \cap \Omega^p_{\mathrm{bd}}(X, Y_1, F)$ *composed with the de Rham map* P_∞ *maps into the space of cocycles in* $C^p(\mathcal{K}, \mathcal{K}_1, F)$*, and we obtain an isomorphism*

$$P_\infty : \mathscr{H}^p(X, Y_1, F) \to H^p(X, Y_1, F). \tag{1.18}$$

Proof If h^F is flat, this result was proved in [18, Prop. 4.2, Corollary 5.7] (cf. [16, p. 239], [12, Theorem 1.10]), the same proof works in the general case here. □

For $\lambda \in \mathbb{R}$, $0 \leqslant p \leqslant m$, set

$$E^p_\lambda(X, Y_1, F) = \left\{\sigma \in \Omega^p_{\mathrm{bd}, D^2}(X, Y_1, F);\, D^2\sigma = \lambda\sigma\right\}. \tag{1.19}$$

Let $P_{\mathscr{H}}$ be the orthogonal projection from $\Omega(X, F)$ onto $\mathscr{H}(X, Y_1, F)$ with respect to the Hermitian product (1.8). Set $P^\perp_{\mathscr{H}} = 1 - P_{\mathscr{H}}$. Let N be the number operator of $\Omega(X, F)$, i.e., N acts as multiplication by p on $\Omega^p(X, F)$. We denote the supertrace by $\mathrm{Tr}_s[\cdot] := \mathrm{Tr}[(-1)^N \cdot]$.

Let $\exp(-tD^2_{\mathrm{bd}})$ be the heat semi-group of D^2_{bd}, with D^2_{bd} from (1.13).

Definition 1.2 For $u \in \mathbb{C}$, $\mathrm{Re}(u) > \frac{1}{2}m$, set

$$\begin{aligned} \theta^F(u) &:= -\mathrm{Tr}_s\left[N(D^2_{\mathrm{bd}})^{-u} P^\perp_{\mathscr{H}}\right] \\ &= -\frac{1}{\Gamma(u)} \int_0^\infty t^u\, \mathrm{Tr}_s\left[N \exp(-tD^2_{\mathrm{bd}}) P^\perp_{\mathscr{H}}\right] \frac{dt}{t}. \end{aligned} \tag{1.20}$$

$\theta^F(u)$ extends to a meromorphic function of $u \in \mathbb{C}$ which is holomorphic at $u = 0$ in view of Theorem 3.2 (cf. also [19]).

Definition 1.3 The Ray–Singer analytic torsion $T(X, Y_1, g^{TX}, h^F)$ of F (with the relative boundary condition on Y_1 and the absolute boundary condition on V_1) is defined by

$$T(X, Y_1, g^{TX}, h^F) := \exp\left(\frac{1}{2}\frac{\partial\theta^F}{\partial u}(0)\right). \tag{1.21}$$

Let $\det H^\bullet(X, Y_1, F)$ be the complex line defined by

$$\det H^\bullet(X, Y_1, F) := \bigotimes_{j=0}^m \left(\det H^j(X, Y_1, F)\right)^{(-1)^j}. \tag{1.22}$$

By the identification in Theorem 1.1 (d), $H^\bullet(X, Y_1, F)$ inherits a L^2-metric $h^{H^\bullet(X, Y_1, F)}$ from the Hermitian product (1.8) on $\Omega(X, F)$. Let $|\cdot|^{L^2}_{\det H^\bullet(X, Y_1, F)}$ be the corresponding metric on $\det H^\bullet(X, Y_1, F)$.

Definition 1.4 The *Ray–Singer metric* $\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}(X,Y_1,F)}$ on the complex line $\det H^{\bullet}(X,Y_1,F)$ is defined by

$$\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}(X,Y_1,F)} := |\cdot|^{L^2}_{\det H^{\bullet}(X,Y_1,F)} T(X, Y_1, g^{TX}, h^F). \tag{1.23}$$

1.3 Milnor metrics and Reidemeister metrics

Let f be a Morse function on X. Let $f|_{\partial X}$ be the restriction of f on ∂X. Set

$$B = \{x \in X; df(x) = 0\}, \quad B_\partial = \{x \in \partial X; d(f|_{\partial X})(x) = 0\}. \tag{1.24}$$

For $x \in B$, let $\mathrm{ind}(x)$ be the index of f at x, i.e., the number of negative eigenvalues of the quadratic form $d^2 f(x)$ on $T_x X$.

Consider the differential equation

$$\frac{\partial y}{\partial t} = -\nabla f(y), \tag{1.25}$$

and denote by (ψ_t) the associated flow. For $x \in B$, the unstable cell $W^u(x)$ and the stable cell $W^s(x)$ of x are defined by

$$\begin{aligned} W^u(x) &= \left\{ y \in X; \lim_{t\to -\infty} \psi_t(y) = x \right\}, \\ W^s(x) &= \left\{ y \in X; \lim_{t\to +\infty} \psi_t(y) = x \right\}. \end{aligned} \tag{1.26}$$

The Smale transversality conditions [20] require that

$$\text{for } x, y \in B, x \neq y, W^u(x) \text{ and } W^s(y) \text{ intersect transversally.} \tag{1.27}$$

Lemma 1.5 *There exists a Morse function f on X such that $f|_{\partial X}$ is a Morse function on ∂X, $B_\partial = B \cap \partial X$, and for $x \in B_\partial$, the restriction of $d^2 f(x)$ to the normal bundle $\mathfrak{n}$ to ∂X in X verifies $d^2 f(x)|_{\mathfrak{n}} > 0$. Moreover, there exists a gradient vector field ∇f of f (defined by some metric $\widetilde{g}^{TX}$ on X), verifying the Smale transversality conditions (1.27) and $\nabla f|_{\partial X} \in T\partial X$.*

Proof We have a natural $\mathbb{Z}_2$-action on $\overline{X} := X \cup_{\partial X} X$, and ∂X is the fixed point set of the $\mathbb{Z}_2$-action. By the proof of [5, Theorem 1.10], we can construct a $\mathbb{Z}_2$-invariant Morse function $f : \overline{X} \to \mathbb{R}$ and a $\mathbb{Z}_2$-invariant metric $g_0^{T\overline{X}}$ on $T\overline{X}$ such that if ∇f is the corresponding gradient vector field of f, then ∇f verifies the Smale transversality conditions and $\nabla f|_{\partial X} \in T\partial X$, moreover, if $x \in \partial X$ is a critical point of f, then $d^2 f(x)|_{\mathfrak{n}} > 0$. So $f|_{\partial X}$ is also a Morse function on ∂X. □

From now on, we fix a Morse function f on X fulfilling the conditions of Lemma 1.5. For $x \in B \cap Y_1$, set

$$W^u_{Y_1}(x) = W^u(x) \cap Y_1, \quad W^s_{Y_1}(x) = W^s(x) \cap Y_1. \tag{1.28}$$

As $d^2 f(x)|_{\mathfrak{n}} > 0$, for $x \in B_\partial$, and $\nabla f|_{\partial X} \in T\partial X$, we know that $\nabla f|_{\partial X}$ verifies also the Smale transversality conditions (1.27).

For $x \in B$, we denote (as in Sect. 1.1) by $[W^u(x)]$ the real line generated by $W^u(x)$, and by $[W^u(x)]^*$ the dual line. Set

$$\begin{aligned} C_j(W^u, F) &= \bigoplus_{x\in B,\, \mathrm{ind}(x)=j} [W^u(x)] \otimes F_x^*, \\ C_j(W^u_{Y_1}, F) &= \bigoplus_{x\in B\cap Y_1,\, \mathrm{ind}(x)=j} [W^u(x)] \otimes F_x^*. \end{aligned} \tag{1.29}$$

There is a map $\partial : C_j(W^u, F^*) \to C_{j-1}(W^u, F^*)$ with $\partial^2 = 0$, which defines the Thom–Smale complex $(C_\bullet(W^u, F^*), \partial)$ (cf. [20], [1, Chap. 7], [4, (1.30)]). This complex calculates the homology $H_\bullet(X, F^*)$ (cf. [4, Theorem 1.6]). As $d^2 f(x)|_{\mathfrak{n}} > 0$, for $x \in B_\partial$, and $\nabla f|_{\partial X} \in T\partial X$, we know that

$$\partial C_j(W^u_{Y_1}, F^*) \subset C_{j-1}(W^u_{Y_1}, F^*), \tag{1.30}$$

thus the Thom–Smale complex $(C_\bullet(W^u_{Y_1}, F^*), \partial)$ is a sub-complex of $(C_\bullet(W^u, F^*), \partial)$. As in Sect. 1.1, let $(C^\bullet(W^u, F), \widetilde{\partial})$ and $(C^\bullet(W^u_{Y_1}, F), \widetilde{\partial})$, be the dual complex of $(C_\bullet(W^u, F^*), \partial)$ and $(C_\bullet(W^u_{Y_1}, F^*), \partial)$, respectively. Then the complexes $(C^\bullet(W^u, F), \widetilde{\partial})$ and $(C^\bullet(W^u_{Y_1}, F), \widetilde{\partial})$ calculate the cohomology $H^\bullet(X, F)$ and $H^\bullet(Y_1, F)$.

Let $\jmath$ be the natural morphism of complexes

$$\jmath : C^\bullet(W^u, F) \to C^\bullet(W^u_{Y_1}, F). \tag{1.31}$$

Define

$$C^\bullet(W^u/W^u_{Y_1}, F) := \mathrm{Ker}\, \jmath, \tag{1.32}$$

and denote by $H^\bullet(C^\bullet(W^u/W^u_{Y_1}, F), \widetilde{\partial})$ the cohomology of the complex $(C^\bullet(W^u/W^u_{Y_1}, F), \widetilde{\partial})$. Then by standard arguments we get a canonical isomorphism

$$H^\bullet(C^\bullet(W^u/W^u_{Y_1}, F), \widetilde{\partial}) \simeq H^\bullet(X, Y_1, F). \tag{1.33}$$

The metric h^F_x $(x \in B)$ determines a metric on $C^\bullet(W^u, F)$ such that the elements $[W^u(x)]^* \otimes F_x$ are mutually orthogonal in $C^\bullet(W^u, F)$, and if $x \in B$, $w \in F_x$

$$\left| W^u(x)^* \otimes w \right| = |w|_{h^{F_x}}. \tag{1.34}$$

Let $h^{C^\bullet(W^u/W^u_{Y_1}, F)}$ be the metric on $C^\bullet(W^u/W^u_{Y_1}, F)$ induced by the metric on $C^\bullet(W^u, F)$. Finally, let $\|\cdot\|_{\det C^\bullet(W^u/W^u_{Y_1}, F)}$ be the metric on the complex line

$$\det C^\bullet(W^u/W^u_{Y_1}, F) = \bigotimes_{j=0}^{m} \left(\det C^j(W^u/W^u_{Y_1}, F)\right)^{(-1)^j} \tag{1.35}$$

associated with the metric h^F (cf. also [4, §1a)]).

Consider now a complex of finite dimensional vector spaces over $\mathbb{C}$

$$0 \to E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} E^n \to 0, \tag{1.36}$$

with cohomology groups $H^j(E) := H^j(E, \delta) := \mathrm{Ker}(\delta|_{E^j})/\mathrm{Im}(\delta|_{E^{j-1}})$. As in the special (0.17), there is a canonical isomorphism

$$\det E := \bigotimes_{j=0}^{n} (\det E^j)^{(-1)^j} \simeq \det H^\bullet(E) := \bigotimes_{j=0}^{n} (\det H^j(E))^{(-1)^j} \tag{1.37}$$

which is defined as follow: if $0 \neq s_j \in \det(E^j/\mathrm{Ker}(\delta|_{E^j}))$, $0 \neq \mu_j \in \det H^j(E)$, then

$$\begin{aligned}\det E \ni s_0 \otimes \mu_0 \otimes (\delta(s_0) \wedge \mu_1 \wedge s_1)^{-1} \otimes \cdots \otimes (\delta(s_{n-1}) \wedge \mu_n)^{(-1)^n} \\ \longrightarrow \mu_0 \otimes \mu_1^{-1} \otimes \cdots \otimes (\mu_n)^{(-1)^n} \in \det H^\bullet(E).\end{aligned} \tag{1.38}$$

As in [4, Definiton 1.9], we can define the Milnor metric.

Definition 1.6 Let $\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}$ be the metric on $\det H^\bullet(X, Y_1, F)$ corresponding to $\|\cdot\|_{\det C^\bullet(W^u/W^u_{Y_1},F)}$ via the canonical isomorphism in (1.38). The metric $\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}$ will be called a *Milnor metric*.

In the same way, for the complex $(C^\bullet(\mathcal{K}, \mathcal{K}_1, F), \widetilde{\partial})$ in Sect. 1.1, let $\|\cdot\|_{\det C^\bullet(\mathcal{K},\mathcal{K}_1,F)}$ be the metric on the complex line

$$\det C^\bullet(\mathcal{K}, \mathcal{K}_1, F) = \bigotimes_{j=0}^{m} (\det C^j(\mathcal{K}, \mathcal{K}_1, F))^{(-1)^j} \tag{1.39}$$

associated with the metric h^F (cf. also [4, §1a)]). As in [4, Definiton 1.4], we can define the Reidemeister metric.

Definition 1.7 The Reidemeister metric $\|\cdot\|^{R,\mathcal{K}}_{\det H^\bullet(X,Y_1,F)}$ on $\det H^\bullet(X, Y_1, F)$ is the metric corresponding to $\|\cdot\|_{\det C^\bullet(\mathcal{K},\mathcal{K}_1,F)}$ via the canonical isomorphism in (1.38).

Remark 1.8 (i) The Milnor metric does not depend on the choice of Riemannian metric g^{TX} on X, it depends only on the vector field ∇f and the Hermitian metric h^F on F.
(ii) If h^F is unimodular, then as in [4, Remark 1.10], [15, Theorem 9.3], $\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}$ does not depend on ∇f and is equal to the Reidemeister metric $\|\cdot\|^{R,\mathcal{K}}_{\det H^\bullet(X,Y_1,F)}$ which does not depend on the choice of the triangulation $\mathcal{K}$.

2 Comparison of Ray–Singer metrics and Milnor metrics

This Section is organized as follows: we explain first the doubling formula for the analytic torsion and for the Milnor metric in Sects. 2.1, 2.2, respectively. In Sect. 2.3, we compare the Ray–Singer and Milnor metrics by applying [5, Theorem 0.2] to the doubled manifold, thus we establish Theorem 0.1 under the assumptions (2.1) and (2.3).

We use the notation introduced in Sect. 1.

2.1 Doubling formula for the analytic torsion

We assume that g^{TX} has *product structure* near the boundary ∂X, i.e., there exists a neighborhood U_ε of ∂X and an identification $\partial X \times [0, \varepsilon[\to U_\varepsilon$, such that for $(y, x_m) \in \partial X \times [0, \varepsilon[$,

$$g^{TX}|_{(y,x_m)} = dx_m^2 \oplus g^{T\partial X}(y). \tag{2.1}$$

This condition insures that the manifold $\widetilde{X} := X \cup_{Y_1} X$ has the canonical Riemannian metric $g^{T\widetilde{X}} = g^{TX} \cup_{Y_1} g^{TX}$. The natural involution on $\widetilde{X}$ will be denoted by ϕ, it generates a $\mathbb{Z}_2$-action on $\widetilde{X}$. Let $j_k : X \to \widetilde{X}$ be the natural inclusion into the k-th factor, $k = 1, 2$, which identifies X with $j_k(X)$. For simplicity, we will write $X := j_1(X)$

We trivialize F on U_ε using the parallel transport along the curve $[0, 1[\ni u \to (y, u\varepsilon)$ defined by the connection ∇^F, then, as ∇^F is flat, we have

$$(F, \nabla^F)|_{U_\varepsilon} = \pi_\varepsilon^*(F|_{\partial X}, \nabla^F|_{\partial X}), \tag{2.2}$$

where $\pi_\varepsilon : \partial X \times [0, \varepsilon[\to \partial X$ is the obvious projection. We assume that

$$h^F = \pi_\varepsilon^* h^F|_{\partial X} \quad \text{on } U_\varepsilon, \tag{2.3}$$

which is true for flat h^F. However, (2.3) does not follow from the product structure (2.1) alone.

Denote by $\mathbb{C}^+, \mathbb{C}^-$ the trivial and the nontrivial one dimensional complex $\mathbb{Z}_2$-representation, respectively, and let $1_{\mathbb{C}^+}, 1_{\mathbb{C}^-}$ be their unit elements.

Let $\widetilde{F} := F \cup_{Y_1} F$ be the flat complex vector bundle with Hermitian metric $h^{\widetilde{F}}$ on $\widetilde{X} := X \cup_{Y_1} X$ induced by (F, h^F). Consider $E^p_\lambda(\widetilde{X}, \widetilde{F})$ defined in analogy with (1.19) as a $\mathbb{Z}_2$-space under the $\mathbb{Z}_2$-action induced by ϕ. As in [5, §2a)], we can then define the $\mathbb{Z}_2$-equivariant analytic torsion $T(\widetilde{X}, g^{T\widetilde{X}}, h^{\widetilde{F}})(g)$ for $g \in \mathbb{Z}_2$ by replacing $\theta^F(u)$ in (1.20) by

$$\theta_g^{\widetilde{F}}(u) := -\mathrm{Tr}_s\left[gN(\widetilde{D}^2_{\mathrm{bd}})^{-u} P^\perp_{\widetilde{\mathscr{H}}}\right], \tag{2.4}$$

where $\widetilde{D}^2_{\mathrm{bd}}$, $P^\perp_{\widetilde{\mathscr{H}}}$ are the corresponding operator and orthogonal projector on $\widetilde{X}$ with the absolute boundary condition on $\partial\widetilde{X}$.

Let $T(X, g^{TX}, h^F) := T(X, \emptyset, g^{TX}, h^F)$ be the analytic torsion of F (with the absolute boundary condition on ∂X).

Proposition 2.1 (*Doubling formula for the analytic torsion*). *For $\lambda \in \mathbb{R}$, we have a $\mathbb{Z}_2$-equivariant isometry*

$$\begin{aligned} &\widetilde{\phi} : E^p_\lambda(\widetilde{X}, \widetilde{F}) \longrightarrow E^p_\lambda(X, F) \otimes \mathbb{C}^+ \oplus E^p_\lambda(X, Y_1, F) \otimes \mathbb{C}^-, \\ &\widetilde{\phi}(\sigma) = \frac{\sqrt{2}}{2}\left(\sigma + \phi^*\sigma\right)|_X \otimes 1_{\mathbb{C}^+} + \frac{\sqrt{2}}{2}\left(\sigma - \phi^*\sigma\right)|_X \otimes 1_{\mathbb{C}^-}. \end{aligned} \tag{2.5}$$

In particular, with χ the nontrivial character of $\mathbb{Z}_2$, we have for $g \in \mathbb{Z}_2$,

$$\log T(\widetilde{X}, g^{T\widetilde{X}}, h^{\widetilde{F}})(g) = \log T(X, g^{TX}, h^F) + \chi(g) \log T(X, Y_1, g^{TX}, h^F). \tag{2.6}$$

Proof It is easy to see that $\widetilde{\phi}$ is well defined and injective from (1.12). To prove the surjectivity, we need to show that for $\omega \in E^p_\lambda(X, F)$, $\widetilde{\omega} = \omega$ on X, and $\widetilde{\omega} = \phi^*\omega$ on $\phi(X)$ is a smooth form on $\widetilde{X}$ with coefficients in $\widetilde{F}$, and thus $\widetilde{\omega} \in E^p_\lambda(\widetilde{X}, \widetilde{F})$. If h^F is flat, this result was proved in [12, Proposition 1.27], the same proof works in the general case. □

2.2 Doubling formula for the Milnor metric

Let f be a Morse function on X which is induced by a $\mathbb{Z}_2$-equivariant Morse function f on $\overline{X} = X \cup_{\partial X} X$, as in the proof of Lemma 1.5, such that f induces a Morse function on $\widetilde{X}$ with critical set $\widetilde{B} = \{x \in \widetilde{X}; df(x) = 0\}$. Let $\widetilde{W}^u(x)$ be the unstable set of $x \in \widetilde{B} \subset \widetilde{X}$. We also have a $\mathbb{Z}_2$-equivariant isomorphism of complexes

$$\gamma : C^\bullet(W^u, F) \otimes \mathbb{C}^+ \oplus C^\bullet(W^u/W^u_{Y_1}, F) \otimes \mathbb{C}^- \to C^\bullet(\widetilde{W}^u, \widetilde{F}), \tag{2.7}$$

given by

$$\gamma(\mathfrak{a}^* \otimes 1_{\mathbb{C}^+} \oplus \mathfrak{b}^* \otimes 1_{\mathbb{C}^-}) = \frac{\sqrt{2}}{2}\Big((j_1^{-1})^*\mathfrak{a}^* + (j_2^{-1})^*\mathfrak{a}^*\Big) + \frac{\sqrt{2}}{2}\Big((j_1^{-1})^*\mathfrak{b}^* - (j_2^{-1})^*\mathfrak{b}^*\Big), \tag{2.8}$$

which induces a $\mathbb{Z}_2$-isomorphism

$$\gamma : H^\bullet(X, F) \otimes \mathbb{C}^+ \oplus H^\bullet(X, Y_1, F) \otimes \mathbb{C}^- \longrightarrow H^\bullet(\widetilde{X}, \widetilde{F}). \tag{2.9}$$

Note that as complex vector spaces, we have

$$
\begin{aligned}
C^j(W^u, F) &= \bigoplus_{x\in B,\, \mathrm{ind}(x)=j} [W^u(x)]^* \otimes F_x,\\
C^j(W^u/W^u_{Y_1}, F) &= \bigoplus_{x\in B\setminus Y_1,\, \mathrm{ind}(x)=j} [W^u(x)]^* \otimes F_x.
\end{aligned} \tag{2.10}
$$

By (1.34), (2.8), γ is an isometry from $C^j(W^u/W^u_{Y_1}, F)\otimes\mathbb{C}^+ \oplus C^j(W^u/W^u_{Y_1}, F)\otimes\mathbb{C}^-$ into $C^\bullet(\widetilde{W}^u, \widetilde{F})$ such that for $\mathfrak{a}^* \in [W^u(x)]^*\otimes F_x$ with $x\in Y_1$, we have

$$
\gamma(\mathfrak{a}^*\otimes 1_{\mathbb{C}^+}) = \sqrt{2}\mathfrak{a}^*. \tag{2.11}
$$

Thus the linear map γ in (2.8) is not an isometry.

Let $C^\bullet(\widetilde{W}^u, \widetilde{F})^\pm$ and $H^\bullet(\widetilde{X}, \widetilde{F})^\pm$ be the ±1-eigenspaces of the $\mathbb{Z}_2$-action induced by ϕ on $C^\bullet(\widetilde{W}^u, \widetilde{F})$ and $H^\bullet(\widetilde{X}, \widetilde{F})$; then $H^\bullet(\widetilde{X}, \widetilde{F})^\pm$ is the cohomology of the complex $(C^\bullet(\widetilde{W}^u, \widetilde{F})^\pm, \widetilde{\partial})$. Following [5, (1.10)], we define

$$
\det\left(H^\bullet(\widetilde{X}, \widetilde{F}), \mathbb{Z}_2\right) = \det\left(H^\bullet(\widetilde{X}, \widetilde{F})^+\right)\otimes\mathbb{C}^+ \oplus \det\left(H^\bullet(\widetilde{X}, \widetilde{F})^-\right)\otimes\mathbb{C}^-. \tag{2.12}
$$

Let $\|\cdot\|_{\det C^\bullet(\widetilde{W}^u,\widetilde{F})^\pm}$ be the metric on $\det H^\bullet(\widetilde{X}, \widetilde{F})^\pm$ defined as in Definition 1.6. For $\mu = (\mu_1, \mu_2) \in \det(H^\bullet(\widetilde{X}, \widetilde{F}), \mathbb{Z}_2)$, $g\in\mathbb{Z}_2$, and χ the nontrivial character of $\mathbb{Z}_2$, we introduce the equivariant Milnor metric by (cf. Definition 1.6 and [5, Definition 1.1])

$$
\log(\|\mu\|^{M,\nabla f}_{\det(H^\bullet(\widetilde{X},\widetilde{F}),\mathbb{Z}_2)})(g) = \log\|\mu_1\|_{\det C^\bullet(\widetilde{W}^u,\widetilde{F})^+} + \chi(g)\log\|\mu_2\|_{\det C^\bullet(\widetilde{W}^u,\widetilde{F})^-}. \tag{2.13}
$$

Now $\widetilde{\phi}$, γ in (2.5), (2.9) induce isomorphisms

$$
\begin{aligned}
\widetilde{\phi}_1,\ \gamma_1^{-1} &: H^\bullet(\widetilde{X}, \widetilde{F})^+ \to H^\bullet(X, F),\\
\widetilde{\phi}_2,\ \gamma_2^{-1} &: H^\bullet(\widetilde{X}, \widetilde{F})^- \to H^\bullet(X, Y_1, F).
\end{aligned} \tag{2.14}
$$

From Definition 1.6, (2.11), (2.13) and (2.14), we get for $\mu = (\mu_1, \mu_2) \in \det(H^\bullet(\widetilde{X}, \widetilde{F}), \mathbb{Z}_2)$, $g\in\mathbb{Z}_2$,

$$
\begin{aligned}
&\log\left(\|\mu\|^{M,\nabla f}_{\det(H^\bullet(\widetilde{X},\widetilde{F}),\mathbb{Z}_2)}\right)(g)\\
&= \frac{1}{2}\log(2)\sum_{x\in B\cap Y_1}(-1)^{\mathrm{ind}(x)}\mathrm{rk}(F)\\
&\quad + \log\|\gamma_1^{-1}\mu_1\|^{M,\nabla f}_{\det H^\bullet(X,F)} + \chi(g)\log\|\gamma_2^{-1}\mu_2\|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}.
\end{aligned} \tag{2.15}
$$

2.3 Comparison of Ray–Singer metrics and Milnor metrics

We use the notation explained in the Sect. 0. Let $\chi(\partial X)$, $\chi(Y_1)$, and $\chi(V_1)$ be the Euler characteristics of ∂X, Y_1, and V_1, respectively, and define by

$$
\chi(Y_1, F) := \sum_j (-1)^j \dim H^j(Y_1, F), \tag{2.16}
$$

the Euler characteristic of Y_1 with coefficients in F. Then from (1.5) (cf. also [6, Theorem 3.2]), we deduce that

$$
\chi(Y_1, F) = \mathrm{rk}(F)\chi(Y_1) = \mathrm{rk}(F)\sum_{x\in B\cap Y_1}(-1)^{\mathrm{ind}(x)}. \tag{2.17}
$$

As an application of [5, Theorem 0.2], we will establish the following formula which is a special case of Theorem 0.1.

Theorem 2.2 *If g^{TX} and h^F have product structure near the boundary ∂X,* [*i.e.,* (2.1)*,* (2.3) *are verified*]*, we have the identity*

$$\log\left(\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}(X,Y_1,F)}}{\|\cdot\|^{M,\nabla f}_{\det H^{\bullet}(X,Y_1,F)}}\right)^2\right) = -\int_X \theta(F,h^F)(\nabla f)^*\psi(TX,\nabla^{TX}) + \frac{1}{2}\int_{Y_1}\theta(F,h^F)(\nabla f)^*\psi(TY_1,\nabla^{TY_1}) - \frac{1}{2}\int_{V_1}\theta(F,h^F)(\nabla f)^*\psi(TV_1,\nabla^{TV_1}) - \frac{1}{2}\mathrm{rk}(F)\chi(\partial X)\log 2. \tag{2.18}$$

Proof Assume first that f is a Morse function on X induced by a $\mathbb{Z}_2$-equivariant Morse function f on $\overline{X} = X\cup_{\partial X} X$ as in the proof of Lemma 1.5.

By Proposition 2.1 for $\lambda = 0$, we have a natural isometry of $\mathbb{Z}_2$-vector spaces

$$\begin{aligned}&\widetilde{\phi} : \mathscr{H}(\widetilde{X},\widetilde{F}) \longrightarrow \mathscr{H}(X,F)\otimes\mathbb{C}^+ \oplus \mathscr{H}(X,Y_1,F)\otimes\mathbb{C}^-,\\ &\widetilde{\phi}(\sigma) = \frac{\sqrt{2}}{2}\cdot\left(\sigma+\phi^*\sigma\right)|_X + \frac{\sqrt{2}}{2}\cdot\left(\sigma-\phi^*\sigma\right)|_X.\end{aligned} \tag{2.19}$$

By Theorem 1.1 (d), we can canonically identify the three terms in (2.19) with the corresponding elements in the cohomology groups $H^\bullet(\widetilde{X},\widetilde{F})$, $H^\bullet(X,F)$ and $H^\bullet(X,Y_1,F)$.

We denote by $C_\bullet(\widetilde{W}^u/W^u_{Y_1},\widetilde{F}^*) = \bigoplus_{x\in\widetilde{B}\setminus Y_1}[W^u(x)]\otimes\widetilde{F}^*_x$. We use the notation P_∞ for the de Rham map (cf. [4, Definition 2.8, Theorem 2.9]) in the identification (1.18) which commutes with ϕ^*. Thus by (2.9), (2.14) and (2.19), for $\sigma\in H^\bullet(\widetilde{X},\widetilde{F})$,

$$\begin{aligned}(\gamma\circ P_\infty\circ\widetilde{\phi}_1\circ P_\infty^{-1})(\sigma)|_{C_\bullet(W^u_{Y_1},F^*)} &= \frac{\sqrt{2}}{2}\gamma(\sigma+\phi^*\sigma)|_{C_\bullet(W^u_{Y_1},F^*)} = 2\sigma|_{C_\bullet(W^u_{Y_1},F^*)},\\ (\gamma\circ P_\infty\circ\widetilde{\phi}\circ P_\infty^{-1})(\sigma)|_{C_\bullet(\widetilde{W}^u/W^u_{Y_1},\widetilde{F}^*)} &= \sigma|_{C_\bullet(\widetilde{W}^u/W^u_{Y_1},\widetilde{F}^*)}.\end{aligned} \tag{2.20}$$

Set

$$\tau_\pm = \gamma\circ P_\infty\circ\widetilde{\phi}\circ P_\infty^{-1} \;: H^\bullet(\widetilde{X},\widetilde{F})^\pm \to H^\bullet(\widetilde{X},\widetilde{F})^\pm. \tag{2.21}$$

By (2.20), we get

$$\prod_{j=0}^m\left(\det\tau_+|_{H^j(\widetilde{X},\widetilde{F})^+}\right)^{(-1)^j} = 2^{\chi(Y_1)\mathrm{rk}(F)},\quad \prod_{j=0}^m\left(\det\tau_-|_{H^j(\widetilde{X},\widetilde{F})^-}\right)^{(-1)^j} = 1. \tag{2.22}$$

Let $|\cdot|_{\det(H^\bullet(\widetilde{X},\widetilde{F})^\pm)}$ be the L^2-metric on $\det\left(H^\bullet(\widetilde{X},\widetilde{F})^\pm\right)$. Then according to [5, Definition 2.3], for $\mu = (\mu_1,\mu_2)\in\det(H^\bullet(\widetilde{X},\widetilde{F}),\mathbb{Z}_2)$ (as in (2.12)), $g\in\mathbb{Z}_2$, the equivariant Ray–Singer metric is defined by

$$\log\left(\|\mu\|^{\mathrm{RS}}_{\det(H^\bullet(\widetilde{X},\widetilde{F}),\mathbb{Z}_2)}\right)(g) = \log|\mu_1|_{\det(H^\bullet(\widetilde{X},\widetilde{F})^+)} + \chi(g)\log|\mu_2|_{\det(H^\bullet(\widetilde{X},\widetilde{F})^-)} + \log T\left(\widetilde{X},g^{T\widetilde{X}},h^{\widetilde{F}}\right)(g). \tag{2.23}$$

By (2.6), (2.19) and (2.23), for $\mu = (\mu_1, \mu_2) \in \det(H^\bullet(\widetilde{X}, \widetilde{F}), \mathbb{Z}_2)$, $g \in \mathbb{Z}_2$,

$$\begin{aligned}\log\left(\|\mu\|^{\mathrm{RS}}_{\det(H^\bullet(\widetilde{X},\widetilde{F}),\mathbb{Z}_2)}\right)(g) &= \log\|\widetilde{\phi}_1\mu_1\|^{\mathrm{RS}}_{\det H^\bullet(X,F)} \\ &\quad + \chi(g)\log\|\widetilde{\phi}_2\mu_2\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F)}.\end{aligned} \tag{2.24}$$

By (2.15), (2.17), (2.20), (2.22) and (2.24), for $g \in \mathbb{Z}_2$ and χ the nontrivial character,

$$\begin{aligned}\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det(H^\bullet(\widetilde{X},\widetilde{F}),\mathbb{Z}_2)}}{\|\cdot\|^{M,\nabla f}_{\det(H^\bullet(\widetilde{X},\widetilde{F}),\mathbb{Z}_2)}}\right)^2(g) &= \chi(Y_1)\mathrm{rk}(F)\log(2) + \log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,F)}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,F)}}\right)^2 \\ &\quad + \chi(g)\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F)}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}}\right)^2.\end{aligned} \tag{2.25}$$

As in (0.7), we denote by $\psi\left(T\widetilde{X}, \nabla^{T\widetilde{X}}\right)$, $\psi(T\partial X, \nabla^{T\partial X})$ the Mathai–Quillen current on $T\widetilde{X}$, $T\partial X$, respectively.

Assume first $\partial X = Y_1$. By [4, Theorem 0.2] and [5, Theorem 0.2], we get

$$\begin{aligned}\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(\widetilde{X},\widetilde{F})}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(\widetilde{X},\widetilde{F})}}\right)^2 &= -\int_{\widetilde{X}}\theta\left(\widetilde{F}, h^{\widetilde{F}}\right)(\nabla f)^*\psi\left(T\widetilde{X}, \nabla^{T\widetilde{X}}\right), \\ \log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det(H^\bullet(\widetilde{X},\widetilde{F}),\mathbb{Z}_2)}}{\|\cdot\|^{M,\nabla f}_{\det(H^\bullet(\widetilde{X},\widetilde{F}),\mathbb{Z}_2)}}\right)^2(\phi) &= -\int_{Y_1}\theta(F, h^F)(\nabla f)^*\psi(TY_1, \nabla^{TY_1}) \\ &\quad - \frac{1}{4}\sum_{x\in B\cap Y_1}\mathrm{Tr}[\phi|_{F_x}](-1)^{\mathrm{ind}(x)}\Big[2\frac{\Gamma'}{\Gamma}\Big(\frac{1}{2}\Big) - 2\Gamma'(1)\Big].\end{aligned} \tag{2.26}$$

By [14] (cf. [5, (5.53)]), we know

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) - \Gamma'(1) = -2\log(2). \tag{2.27}$$

As ϕ acts as Id on F_x for $x \in Y_1$, the last term in (2.26) is $\mathrm{rk}(F)\chi(Y_1)\log(2)$.

By (2.25) and (2.26), we get (2.18), and

$$\begin{aligned}\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,F)}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,F)}}\right)^2 &= -\int_X\theta(F, h^F)(\nabla f)^*\psi(TX, \nabla^{TX}) \\ &\quad - \frac{1}{2}\int_{\partial X}\theta(F, h^F)(\nabla f)^*\psi(T\partial X, \nabla^{T\partial X}) - \frac{1}{2}\mathrm{rk}(F)\chi(\partial X)\log(2).\end{aligned} \tag{2.28}$$

Finally, we treat *the general case*, i.e., $\partial X = Y_1 \cup V_1$. As $\partial\widetilde{X} = V_1 \cup V_1$, by (2.28), we get

$$\begin{aligned}\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(\widetilde{X},\widetilde{F})}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(\widetilde{X},\widetilde{F})}}\right)^2 &= -\int_{\widetilde{X}}\theta\left(\widetilde{F}, h^{\widetilde{F}}\right)(\nabla f)^*\psi\left(T\widetilde{X}, \nabla^{T\widetilde{X}}\right) \\ &\quad - \int_{V_1}\theta(F, h^F)(\nabla f)^*\psi(TV_1, \nabla^{TV_1}) - \mathrm{rk}(F)\chi(V_1)\log(2).\end{aligned} \tag{2.29}$$

By (2.25) for $g = 1$, (2.28), and (2.29), we get (2.18).

We established until now Theorem 2.2 for a special Morse function f on X induced by a $\mathbb{Z}_2$-equivariant Morse function f on $\widetilde{X}$. By combining this with the argument in [4, §16], we know that Theorem 2.2 holds for any f verifying Lemma 1.5. □

3 Gluing formula for analytic torsion

This Section is organized as follows: in Sect. 3.1, we review the relative Euler class for manifolds with boundary as introduced in [6, §1]. In Sect. 3.2, we establish the anomaly formula for the analytic torsion for manifolds with boundary, Theorem 3.4. We explain that Theorem 3.4 is also a consequence of Theorem 2.2 if two couples of metrics (g_0^{TX}, h_0^F) and (g_1^{TX}, h_1^F) on TX and F are product metrics near the boundary. In Sect. 3.3, we establish Theorem 0.1. In Sects. 3.4 and 3.5, we prove Theorem 0.3, and in Sect. 3.6, we prove Theorem 0.4.

3.1 The Euler class for manifolds with boundary

We use the same terminology as in Sect. 1.2, and we use freely the notation introduced in [6, §1].

For $\mathbb{Z}_2$-graded algebras $\mathcal{A}, \mathcal{B}$ with identity, we introduce the $\mathbb{Z}_2$-graded tensor product $\mathcal{A}\widehat{\otimes}\mathcal{B}$ and define $\mathcal{A} := \mathcal{A}\widehat{\otimes}I$, and $\widehat{\mathcal{B}} := I\widehat{\otimes}\mathcal{B}$, and we write $\wedge := \widehat{\otimes}$ such that $\mathcal{A}\widehat{\otimes}\mathcal{B} = \mathcal{A}\wedge\widehat{\mathcal{B}}$; the canonical isomorphism $\mathcal{B} \to \widehat{\mathcal{B}}$ will be written as $\omega \to \hat{\omega} = 1\widehat{\otimes}\omega$. Let E and V be finite dimensional real vector spaces of dimension n and l, respectively. Assume that E is Euclidean and oriented, with oriented orthonormal basis $\{f_i\}_{i=1}^n$ and dual basis $\{f^i\}_{i=1}^n$ with respect to the Euclidean metric h^E, and denote by ΛE^* the exterior algebra of E^*. Then the Berezin integral we use is the linear map

$$\int^B : \Lambda V^* \wedge \widehat{\Lambda E^*} \to \Lambda V^*, \quad \alpha \wedge \hat{\beta} \mapsto c_B\ \beta(f_1, \cdots, f_n)\alpha, \tag{3.1}$$

where the normalizing constant is given by $c_B := (-1)^{n(n+1)/2}\pi^{-n/2}$. More generally, for any Euclidean vector space E with orientation line $o(E)$, the Berezin integral maps $\Lambda V^* \wedge \widehat{\Lambda E^*}$ into $\Lambda V^* \otimes o(E)$.

Let X be a m-dimensional compact manifold with boundary $\partial X := Y = Y_1 \cup V_1$. Let $\jmath : Y \hookrightarrow X$ be the natural injection. Let g^{TX} be a metric on TX and denote by g^{TY} the metric on TY induced by g^{TX}. Let ∇^{TX} and ∇^{TY} be the Levi-Civita connection on (TX, g^{TX}) and (TY, g^{TY}), respectively.

We only consider orthonormal frames $\{e_i\}_{i=1}^m$ of TX with the property that near the boundary Y, $e_m =: e_{\mathfrak{n}}$ is the inward pointing unit normal at any boundary point. We will use greek indices to specify the induced frame of TY, such that $\{e_\alpha\}_{\alpha=1}^{m-1}$ denotes a local orthonormal frame for TY.

Thus if ω is a smooth section of $\Lambda(T^*X)$ we identify ω with the section $\omega\widehat{\otimes}1$ of $\Lambda(T^*X)\widehat{\otimes}\widehat{\Lambda(T^*X)}$, and $\widehat{\omega}$ will denote the section $1\widehat{\otimes}\widehat{\omega}$ of $\Lambda(T^*X)\widehat{\otimes}\widehat{\Lambda(T^*X)}$ as before. We will apply the Berezin integral from (3.1) to $\Lambda(T^*X)\widehat{\otimes}\widehat{\Lambda(T^*X)}$ and $\Lambda(T^*Y)\widehat{\otimes}\widehat{\Lambda(T^*Y)}$, and, for convenience, we will denote this operation by $\int^{B_X}$ and $\int^{B_Y}$, respectively, cf. [6, (1.14)].

Let $\{e_i\}_{i=1}^m$ be an orthonormal frame of (TX, g^{TX}) and let $\{e^i\}$ be the corresponding dual frame of T^*X. Set

$$\dot{R}^{TX} := \frac{1}{2} \sum_{1 \leqslant i,j \leqslant m} \left\langle e_i, R^{TX} e_j \right\rangle \widehat{e^i} \wedge \widehat{e^j} \in \Lambda^2(T^*X) \widehat{\otimes} \widehat{\Lambda^2(T^*X)}. \tag{3.2}$$

We define

$$\begin{aligned} \dot{S} &:= \frac{1}{2} \jmath^* \nabla^{TX} \widehat{e^m} = \frac{1}{2} \sum_{\beta=1}^{m-1} \left\langle (\jmath^* \nabla^{TX}) e_{\mathfrak{n}}, e_\beta \right\rangle \widehat{e^\beta} && \in T^*Y \widehat{\otimes} \widehat{\Lambda^1(T^*Y)}, \\ \dot{R}^{TX}|Y &:= \frac{1}{2} \sum_{1 \leqslant \gamma,\delta \leqslant m-1} \left\langle e_\gamma, \jmath^* R^{TX} e_\delta \right\rangle \widehat{e^\gamma} \wedge \widehat{e^\delta} && \in \Lambda^2(T^*Y) \widehat{\otimes} \widehat{\Lambda^2(T^*Y)}, \\ \dot{R}^{TY} &:= \frac{1}{2} \sum_{1 \leqslant \gamma,\delta \leqslant m-1} \left\langle e_\gamma, R^{TY} e_\delta \right\rangle \widehat{e^\gamma} \wedge \widehat{e^\delta} && \in \Lambda^2(T^*Y) \widehat{\otimes} \widehat{\Lambda^2(T^*Y)}. \end{aligned} \tag{3.3}$$

By [6, (1.16)], we have

$$\dot{R}^{TY} = \dot{R}^{TX}|Y - 2\dot{S}^2. \tag{3.4}$$

The Chern–Weil forms

$$e(TX, \nabla^{TX}) := \int^{B_X} \exp\left(-\frac{\dot{R}^{TX}}{2}\right), \quad e(TY, \nabla^{TY}) := \int^{B_Y} \exp\left(-\frac{\dot{R}^{TY}}{2}\right), \tag{3.5}$$

are closed and $e(TX, \nabla^{TX})$ is an $o(TX)$-valued m-form on X which represents the Euler class of TX. On Y, we further introduce

$$\begin{aligned} e_b(Y, \nabla^{TX}) &:= (-1)^{m-1} \int^{B_Y} \exp\left(-\frac{1}{2}(\dot{R}^{TX}|Y)\right) \sum_{k=0}^{\infty} \frac{\dot{S}^k}{2\Gamma(\frac{k}{2}+1)}, \\ B(\nabla^{TX}) &:= -\int_0^1 \frac{du}{u} \int^{B_Y} \exp\left(-\frac{1}{2}\dot{R}^{TY} - u^2 \dot{S}^2\right) \sum_{k=1}^{\infty} \frac{(u\dot{S})^k}{2\Gamma(\frac{k}{2}+1)}. \end{aligned} \tag{3.6}$$

Then $e_b(Y, \nabla^{TX})$, $B(\nabla^{TX})$ are $(m-1)$-forms on Y with values in the orientation line bundle $o(TY)$. If $\dim X = m$ is odd, then by (3.4) and (3.6),

$$\begin{aligned} e_b(Y, \nabla^{TX}) &= \frac{1}{2} e(TY, \nabla^{TY}), \quad e(TX, \nabla^{TX}) = 0, \\ B(\nabla^{TX}) &= \int^{B_Y} \exp\left(-\frac{1}{2}\dot{R}^{TY}\right) \sum_{k=1}^{\infty} \frac{(-\dot{S}^2)^k}{4k\Gamma(k+1)}. \end{aligned} \tag{3.7}$$

Let $\Omega(X, o(TX))$, $\Omega(Y, o(TX))$ be the $o(TX)$-valued C^∞ forms on X, Y. The algebraic mapping cone of $\jmath^* : \Omega(X, o(TX)) \to \Omega(Y, o(TX))$ is defined as the following object: we put

$$\Omega^p(X, Y, o(TX)) = \Omega^p(X, o(TX)) \oplus \Omega^{p-1}(Y, o(TX)), \tag{3.8}$$

and define the differential by

$$d(\sigma_1, \sigma_2) = (d^X \sigma_1, \jmath^* \sigma_1 - d^Y \sigma_2); \tag{3.9}$$

then the complex $(\Omega(X, Y, o(TX)), d)$ calculates the relative cohomology $H^\bullet(X, Y, o(TX))$. For $(\sigma_1, \sigma_2) \in \Omega(X, Y, o(TX))$, $\sigma_3 \in \Omega(X)$, we define a nonsingular pairing

$$\int_{(X,Y)} (\sigma_1, \sigma_2) \wedge \sigma_3 := \int_X \sigma_1 \wedge \sigma_3 - \int_Y \sigma_2 \wedge \jmath^* \sigma_3; \tag{3.10}$$

this induces the Poincaré duality $H^\bullet(X, Y, o(TX)) \times H^\bullet(X, \mathbb{R}) \to \mathbb{R}$.

We define the *relative Euler form* of TX associated with ∇^{TX}

$$E(TX, \nabla^{TX}) := (e(TX, \nabla^{TX}), e_b(Y, \nabla^{TX})) \in \Omega^m(X, Y, o(TX)). \tag{3.11}$$

The following result was established in [6, Theorem 1.9].

Theorem 3.1 (1) *$E(TX, \nabla^{TX})$ is closed in the complex $(\Omega(X, Y, o(TX)), d)$ and, modulo exact forms, it does not depend on the choice of g^{TX}, i.e., the cohomology class $E(TX) = [E(TX, \nabla^{TX})] \in H^m(X, Y, o(TX))$ does not depend on g^{TX}.*

(2) *For two metrics g_0^{TX}, g_1^{TX} on TX, there exists a canonically defined secondary relative Euler class*

$$\widetilde{E}(TX, \nabla_0^{TX}, \nabla_1^{TX}) \in \Omega^{m-1}(X, Y, o(TX))/d\Omega^{m-2}(X, Y, o(TX))$$

of TX such that

$$d\,\widetilde{E}(TX, \nabla_0^{TX}, \nabla_1^{TX}) = E(TX, \nabla_1^{TX}) - E(TX, \nabla_0^{TX}). \tag{3.12}$$

In particular, for three metrics g_0^{TX}, g_1^{TX}, g_2^{TX} on TX, we have

$$\widetilde{E}(TX, \nabla_0^{TX}, \nabla_2^{TX}) = \widetilde{E}(TX, \nabla_0^{TX}, \nabla_1^{TX}) + \widetilde{E}(TX, \nabla_1^{TX}, \nabla_2^{TX}). \tag{3.13}$$

If $Y = \emptyset$, then $\widetilde{E}$ is the Chern–Simons form associated with the Euler class of TX, as defined in [4, (4.53)].

We will use the subscripts 0 and 1 to distinguish various objects associated with these metrics. For example, $\widetilde{e}(TY, \nabla_0^{TY}, \nabla_1^{TY})$ denotes the Chern–Simons class of smooth $(m-2)$-forms on Y with values in $o(TY)$, which is defined modulo exact forms and satisfies

$$d\,\widetilde{e}(TY, \nabla_0^{TY}, \nabla_1^{TY}) = e(TY, \nabla_1^{TY}) - e(TY, \nabla_0^{TY}). \tag{3.14}$$

If $\dim X$ is odd, then we derive from (3.7), [6, (1.47)]:

$$\widetilde{E}(TX, \nabla_0^{TX}, \nabla_1^{TX}) = \Big(0, -\tfrac{1}{2}\widetilde{e}(TY, \nabla_0^{TY}, \nabla_1^{TY})\Big). \tag{3.15}$$

3.2 Anomaly formula

Recall that X is a m-dimensional compact manifold with boundary $\partial X := Y = Y_1 \cup V_1$ with operator D^2_{bd} as defined in (1.13). The Euler number of X relative to Y_1 is defined by

$$\chi(X, Y_1) = \sum_j (-1)^j \dim H^j(X, Y_1, \mathbb{C}). \tag{3.16}$$

Note that by the Gauss–Bonnet–Chern theorem, if m is even then

$$\chi(X, Y_1) = \int_{(X,Y)} E(TX, \nabla^{TX}), \tag{3.17}$$

and if m is odd, then

$$\chi(X,Y_1)=\frac{1}{2}\int_{V_1} e(TY,\nabla^{TY})-\frac{1}{2}\int_{Y_1} e(TY,\nabla^{TY}). \tag{3.18}$$

The following result insuring that $\theta^F(u)$ is holomorphic at 0 [cf. (1.20)], extends [4, Theorem 7.10] where $Y=\emptyset$, and [6, Theorem 4.2] where $Y_1=\emptyset$.

Theorem 3.2 *When $t\to 0$, we have for any $k\in\mathbb{N}$,*

$$\mathrm{Tr}_s\left[N\exp(-t^2D^2_{\mathrm{bd}})\right]=\sum_{j=-1}^{k} c_j t^j+\mathcal{O}(t^{k+1}), \tag{3.19}$$

where

$$\begin{aligned} c_{-1}=&\frac{1}{2}\mathrm{rk}(F)\int_X\int^{B_X}\sum_{i=0}^{m} e^i\wedge\widehat{e^i}\exp\left(-\frac{1}{2}\dot{R}^{TX}\right)\\ &+\frac{1}{2}\mathrm{rk}(F)\Big(\int_{V_1}+(-1)^{m+1}\int_{Y_1}\Big)\int^{B_Y}\sum_{\alpha=1}^{m-1} e^\alpha\wedge\widehat{e^\alpha}\\ &\cdot\sum_{k=0}^{\infty}\frac{\dot{S}^k}{2\Gamma(\frac{k}{2}+1)}\exp\left(-\frac{1}{2}(\dot{R}^{TX}|Y)\right),\\ c_0=&\frac{m}{2}\mathrm{rk}(F)\chi(X,Y_1). \end{aligned} \tag{3.20}$$

Let $*^F:\Lambda(T^*X)\otimes F\to\Lambda(T^*X)\otimes F^*\otimes o(TX)$ be the Hodge operator defined by

$$(\sigma\wedge *^F\sigma')_F:=\langle\sigma,\sigma'\rangle_{\Lambda(T^*X)\otimes F}\,dv_X. \tag{3.21}$$

Let $*$ be the usual Hodge operator on $\Lambda(T^*X)$ associated with g^{TX}.

Let $(g^{TX}_s,h^F_s)_{s\in\mathbb{R}}$ be a smooth family of metrics on TX and F. We add the subscript s to denote the objects we considered which attached to (g^{TX}_s,h^F_s). For example, $\|\cdot\|_{\det F,s}$ is the metric on the line bundle $\det F$ induced by h^F_s.

The following result is an extension of [4, Theorem 4.14], where the case $\partial X=Y=\emptyset$ was treated, and of [6, Theorem 4.5] dealing with the case $Y_1=\emptyset$.

Theorem 3.3 *As $t\to 0$, for any $k\in\mathbb{N}$, we have the asymptotic estimate*

$$\mathrm{Tr}_s\left[\left(*_s^{-1}\frac{\partial *_s}{\partial s}+(h^F_s)^{-1}\frac{\partial h^F_s}{\partial s}\right)e^{-tD^2_{s,\mathrm{bd}}}\right]=\sum_{j=-m}^{k} M_{j,s}t^{j/2}+\mathcal{O}(t^{(k+1)/2}), \tag{3.22}$$

with

$$M_{0,s}=\frac{\partial}{\partial s}\log\left(\left(\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F),s}\right)^2\right). \tag{3.23}$$

Proof By [6, (3.35)], as in [6, (4.10)], we have

$$w(e^m)d^F\sigma|_{Y_1}=0\quad\text{if } w(e^m)\sigma|_{Y_1}=0. \tag{3.24}$$

With (3.24), the argument given in the proof of [6, §4] and [13, §3.4] works here as well. Thus we get Theorem 3.3. □

Theorem 3.4 *Let (g_0^{TX}, h_0^F) and (g_1^{TX}, h_1^F) be two couples of metrics on TX and F. If m is even, then*

$$\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F),1}}{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F),0}}\right)^2 = \int_{(X,Y)} \log\left(\frac{\|\cdot\|_{\det F,1}}{\|\cdot\|_{\det F,0}}\right)^2 E(TX,\nabla_0^{TX}) + \int_{(X,Y)} \widetilde{E}(TX,\nabla_0^{TX},\nabla_1^{TX})\theta(F,h_1^F) + \mathrm{rk}(F)\left[\left(\int_{V_1}-\int_{Y_1}\right)B(\nabla_1^{TX}) - \left(\int_{V_1}-\int_{Y_1}\right)B(\nabla_0^{TX})\right]. \tag{3.25}$$

If m is odd, then

$$\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F),1}}{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F),0}}\right)^2 = \frac{1}{2}\left(\int_{V_1}-\int_{Y_1}\right)\log\left(\frac{\|\cdot\|_{\det F,1}}{\|\cdot\|_{\det F,0}}\right)^2 e(TY,\nabla_0^{TY}) + \frac{1}{2}\left(\int_{V_1}-\int_{Y_1}\right)\widetilde{e}(TY,\nabla_0^{TY},\nabla_1^{TY})\theta(F,h_1^F) + \mathrm{rk}(F)\left[\int_Y B(\nabla_1^{TX}) - \int_Y B(\nabla_0^{TX})\right]. \tag{3.26}$$

Proof of Theorems 3.2 and 3.4 Clearly, the contribution from the interior of X is the same as in the case of absolute boundary conditions. For the boundary contribution, we can again localize near the boundary. Thus we get directly the local contribution near V_1 from [6, §3-§5]. To get the contribution near Y_1, notice that locally the Hodge $*$ operator interchanges relative and absolute boundary conditions; and it interchanges the $\mathbb{Z}_2$-grading on $\Lambda(T^*X)$ if m is odd and preserves the $\mathbb{Z}_2$-grading on $\Lambda(T^*X)$ if m is even.

We explain first how to get the contribution near Y_1 by using the Hodge $*$ operator. We denote $D^2_{\widetilde{\mathrm{bd}}}$ the operator D^2 associated with the flat vector bundle $F^*\otimes o(TX)$ with the absolute boundary condition on Y_1 and the relative boundary condition on V_1. Then by [6, (3.3)],

$$D^2_{\widetilde{\mathrm{bd}}} = *^F D^2_{\mathrm{bd}}(*^F)^{-1}. \tag{3.27}$$

Note that if h^F is not flat, $D^2 = *D^2*^{-1}$ does not hold, cf. [4, §4g].

As the operator $*^F$ has parity $(-1)^m$, and $*^F N\,(*^F)^{-1} = m - N$, we get from (3.27) for $x\in X$ near Y_1,

$$\begin{aligned}\mathrm{Tr}_s\left[N\exp\left(-tD^2_{\mathrm{bd}}\right)(x,x)\right] &= (-1)^m\,\mathrm{Tr}_s\left[*^F N\exp\left(-tD^2_{\mathrm{bd}}\right)(x,x)(*^F)^{-1}\right]\\ &= (-1)^m\,\mathrm{Tr}_s\left[*^F N(*^F)^{-1}\exp\left(-tD^2_{\widetilde{\mathrm{bd}}}\right)(x,x)\right]\\ &= (-1)^m\,\mathrm{Tr}_s\left[(m-N)\exp\left(-tD^2_{\widetilde{\mathrm{bd}}}\right)(x,x)\right],\end{aligned} \tag{3.28}$$

and similarly

$$\begin{aligned}\operatorname{Tr}_s\left[\exp\left(-tD^2_{\mathrm{bd}}\right)(x,x)\right] &= (-1)^m \operatorname{Tr}_s\left[*^F \exp\left(-tD^2_{\mathrm{bd}}\right)(x,x)(*^F)^{-1}\right]\\ &= (-1)^m \operatorname{Tr}_s\left[\exp\left(-tD^2_{\widetilde{\mathrm{bd}}}\right)(x,x)\right].\end{aligned} \tag{3.29}$$

In the same way, for $Q_s := (*_s^F)^{-1}\frac{\partial *_s^F}{\partial s}$, with $(*_s^F)^2 = 1$ we get

$$*_s^F Q_s = -Q_s *_s^F. \tag{3.30}$$

By (3.27) and (3.30), for $x \in X$ near Y_1,

$$\begin{aligned}\operatorname{Tr}_s\left[Q_s\exp\left(-tD^2_{\mathrm{bd}}\right)(x,x)\right] &= (-1)^m \operatorname{Tr}_s\left[*_s^F Q_s \exp\left(-tD^2_{\mathrm{bd}}\right)(x,x)(*_s^F)^{-1}\right]\\ &= (-1)^m \operatorname{Tr}_s\left[*_s^F Q_s (*_s^F)^{-1}\exp\left(-tD^2_{\widetilde{\mathrm{bd}}}\right)(x,x)\right]\\ &= (-1)^{m+1} \operatorname{Tr}_s\left[Q_s\exp\left(-tD^2_{\widetilde{\mathrm{bd}}}\right)(x,x)\right].\end{aligned} \tag{3.31}$$

From (3.29), we get the factor $(-1)^m$ in (3.17) and (3.18) for the contribution from Y_1. From (3.28), (3.31), we get the factor $(-1)^{m+1}$ in c_{-1} of (3.20) and in $B(\nabla^{TX})$ of Theorem 3.4 for the contribution from Y_1.

In the rest, we prefer to write down the boundary condition corresponding to (1.12) using freely the notation in [6, §3.4, §3.5]. Let $\omega(F, h^F)$ be the 1-form on X with values in $\mathrm{End}(F)$ defined by $\omega(F, h^F) := (h^F)^{-1}(\nabla^F h^F)$. As ∇^{TX} is torsion free (cf. [6, (4.8)]),

$$(d^F)^* = -i(e_j)\nabla^{TX\otimes F}_{e_j} - i(e_j)\omega(F, h^F)(e_j). \tag{3.32}$$

For ε small enough, we identify $\partial X \times [0, \varepsilon)$ with a neighborhood $\mathscr{U}_\varepsilon$ of ∂X in X by using the exponential map $\exp_y(ue_{\mathfrak{n}})$ for $(y, u) \in \partial X \times [0, \varepsilon)$. For $x = (y, x_m) \in \mathscr{U}_\varepsilon$, let ${}^1TX_x$, ${}^2TX_x$ be obtained by parallel transport of T_yY, $\mathfrak{n}_y$ (cf. Sect. 1.2) with respect to the connection ∇^{TX} along the geodesic $[0,1] \ni u \mapsto (y, ux_m)$. Let $\nabla^{{}^jTX}$ $(j = 1, 2)$ be the connection on ${}^jTX$ induced by projection from ∇^{TX}. Let ${}^{sp}\nabla^{TX} := \nabla^{{}^1TX} \oplus \nabla^{{}^2TX}$ be the direct sum connection on $TX = {}^1TX \oplus {}^2TX$ with curvature ${}^{sp}R^{TX}$ (where "sp" refers to "split"), and set

$$A := \nabla^{TX} - {}^{sp}\nabla^{TX}. \tag{3.33}$$

Then A is a 1-form on $\mathscr{U}_{\varepsilon_0}$ taking values in the skew-adjoint endomorphisms of TX which exchange ${}^1TX$ and ${}^2TX$. Let $\{e_\alpha\}$ be an orthonormal basis of ${}^1TX$, set

$$A^{\Lambda(T^*Y)}(e_m) := -\sum_{1\le\alpha,\beta\le m-1}\left\langle e_m, A(e_\alpha)e_\beta\right\rangle w(e^\alpha)i(e_\beta). \tag{3.34}$$

We denote by $\nabla^{TX\otimes F}$ (resp. ${}^{sp}\nabla^{TX\otimes F}$) the connection on $\Lambda(T^*X)\otimes F$ induced by ∇^{TX} and ∇^F (resp. ${}^{sp}\nabla^{TX}$ and ∇^F).

By (3.33), $\nabla^{TX\otimes F}_{e_j} = {}^{sp}\nabla^{TX\otimes F}_{e_j} + \sum_{i,k=1}^{m} \left\langle e_k, A(e_j)e_i \right\rangle w(e^k) i(e_i)$ and $A(e_m) = 0$ (cf. [6, (1.9)]), we get on ∂X as in [6, (3.36)]

$$\begin{aligned} w(e^m)(d^F)^* = &- \sum_{\alpha=1}^{m-1} w(e^m) i(e_\alpha) \left({}^{sp}\nabla^{TX\otimes F}_{e_\alpha} + \omega(F, h^F)(e_\alpha) \right) \\ &- w(e^m) i(e_m) \Bigg\{ {}^{sp}\nabla^{TX\otimes F}_{e_m} - A^{\Lambda(T^*Y)}(e_m) - \sum_{\alpha=1}^{m-1} \langle A(e_\alpha)e_\alpha, e_m \rangle \\ &+ \omega(F, h^F)(e_m) \Bigg\}. \end{aligned} \tag{3.35}$$

Thus the boundary condition (1.12) near Y_1 is equivalent to the boundary condition

$$\begin{cases} w(e^m)\omega|_{Y_1} = 0, \\ i(e_m)\left({}^{sp}\nabla^{TX\otimes F}_{e_m} - A^{\Lambda(T^*Y)}(e_m) - \sum_{\alpha=1}^{m-1} \langle A(e_\alpha)e_\alpha, e_m\rangle + \omega(F, h^F)(e_m) \right)\omega|_{Y_1} = 0. \end{cases} \tag{3.36}$$

Let $B^{\Lambda(T^*Y)}$ be the 1-form defined by

$$\begin{aligned} &B^{\Lambda(T^*Y)}(e_m) := -A^{\Lambda(T^*Y)}(e_m) - \sum_{\alpha=1}^{m-1} \langle A(e_\alpha)e_\alpha, e_m\rangle + \omega(F, h^F)(e_m), \\ &B^{\Lambda(T^*Y)}(e_\alpha) := 0. \end{aligned} \tag{3.37}$$

Instead of using the connection ${}^{sp}\nabla^{TX\otimes F, A}$ as in [6, (3.38)], we use the connection

$${}^{sp}\nabla_1^{TX\otimes F, A} :={}^{sp} \nabla^{TX\otimes F} + B^{\Lambda(T^*Y)} \tag{3.38}$$

in trivializing near the boundary, as in [6, §3.5]. Then for $y_0 \in Y_1$, we get the corresponding model problem on $\mathbb{R}^m_+ := \mathbb{R}^{m-1} \times \mathbb{R}_+$ (cf. [6, Theorem 3.8]) in the form

$$\left\{ \frac{\partial}{\partial t} - \left(\frac{\partial}{\partial z_m} + \dot{S}(y_0) \right)^2 - \sum_{\alpha=1}^{m-1} \frac{\partial^2}{\partial z_\alpha^2} + \frac{1}{2} (\dot{R}^{TX}|Y)_{y_0} \right\} \omega = 0, \tag{3.39}$$

with boundary condition

$$\begin{cases} w(e^m)\omega|_{z_m=0} = 0, \\ \nabla_{e_m} i(e_m)\omega|_{z_m=0} = 0. \end{cases} \tag{3.40}$$

To conclude the final computation, we need to use the Sommerfeld formula which is the explicit solution of our model problem (3.39) and (3.40) (cf. [6, Prop. 3.21]). As in [6, (3.132)], for $y_0 \in Y_1$, let

$$Q(t, z, w) =: Q_1\, w(e^m) + Q_2\, i(e_m) + Q_3\, w(e^m) i(e_m) + Q_4\, i(e_m) w(e^m), \tag{3.41}$$

with $Q_i \in (\Lambda(T^*Y) \widehat{\otimes} \widehat{\Lambda(T^*Y)} \otimes \mathrm{End}(F))_{y_0}$, $z = (z', z_m)$, $w = (w', w_m) \in \mathbb{R}^m_+$, $t > 0$, be the fundamental solution of the following problem with respect to the Euclidean volume form $dv_{T_{y_0}X}(w)$ on $(T_{y_0}X, g^{T_{y_0}X})$,

$$\begin{cases} w(e^m)\omega|_{z_m=0} = 0, \\ i(e_m)[\nabla_{e_m} - \dot{S}(y_0)]\omega|_{z_m=0} = 0, \\ \left(\frac{\partial}{\partial t} + \Delta^{T_{y_0}X}\right)\omega = 0. \end{cases} \tag{3.42}$$

For $z', w' \in \mathbb{R}^{m-1}$, set

$$K(u, z', w') = (4\pi u)^{-\frac{m-1}{2}} \exp\Big(-\tfrac{|z'-w'|^2}{4u}\Big), \tag{3.43}$$

and define F_{Dir}, $F_{-\dot{S}}$ as in [6, (3.121), (3.122)]. By separation of variables as in [6, (3.135)], we get

$$\begin{aligned} Q_3(t, z, w) &= K(t, z', w') F_{-\dot{S}}(t, z_m, w_m), \\ Q_4(t, z, w) &= K(t, z', w') F_{Dir}(t, z_m, w_m). \end{aligned} \tag{3.44}$$

Now for $y_0 \in Y_1$, the analogue of [6, (3.128), (3.133)] holds, thus we get the contribution of Y_1 in (3.17) and (3.18) with a factor $(-1)^m$. In this way, we get a heat kernel proof of (3.17) and (3.18).

From [6, (4.23a), (4.34)] and (3.44), we get as in [6, (4.35)], the analogue of [6, Lemma 4.7], i.e., we have for $y_0 \in Y_1$

$$\lim_{t\to 0} dv_Y(y)\mathbb{I}_t(y_0) = -\frac{2}{s}\mathrm{rk}(F) \int^{B_Y} \exp\left(-\frac{1}{2}(\dot{R}_s^{TX}|Y)_{y_0}\right) \sum_{k=1}^{\infty} \frac{(-\dot{S}_s)^k}{4\Gamma(\frac{k}{2}+1)}, \tag{3.45}$$

(cf. [6, (4.27)] for the notation). Thus we get the term $B(\nabla^{TX})$ on Y_1 with a factor $(-1)^{m+1}$ in Theorem 3.4.

To establish the corresponding formula of [6, §5], we still define the operators B_s, D_1^2, $\mathscr{L}_t^{(0)}$ acting on the smooth sections of $\mathcal{F} := \Lambda(\mathbb{C}(ds \oplus d\bar{s}))\widehat{\otimes}\Lambda(T^*X) \otimes F$ on X as in [6, (5.3)]. We denote by $\mathscr{L}_{t,\mathrm{bd}}^{(0)}$ the operator associated with $\mathscr{L}_t^{(0)}$ and the following boundary condition for $\sigma \in C^\infty(X, \mathcal{F})$,

$$\begin{aligned} i(e_{\mathfrak{n}})\sigma = i(e_{\mathfrak{n}})\left(d^F\sigma - \frac{1}{2\sqrt{t}} ds \wedge *_s^{-1}\frac{\partial *_s}{\partial s}\sigma\right) &= 0 \quad \text{on } V_1, \\ w(e^m)\sigma = w(e^m)\left(d^F\sigma + \frac{1}{2\sqrt{t}} ds \wedge *_s^{-1}\frac{\partial *_s}{\partial s}\sigma\right) &= 0 \quad \text{on } Y_1. \end{aligned} \tag{3.46}$$

Then we have the analogue of [6, Theorem 5.2],

$$\frac{\partial}{\partial t}\left\{t\mathrm{Tr}_s\left[*_s^{-1}\frac{\partial *_s}{\partial s} e^{-t^2 D_{\mathrm{bd}}^2}\right]\right\} = \mathrm{Tr}_s[e^{-\mathscr{L}_{t,\mathrm{bd}}^{(0)}}]^{d\bar{s}ds}. \tag{3.47}$$

Next, from the proof of [18, Lemma 5.12, p.192–193] we have

$$\begin{aligned} &i(e_{\mathfrak{n}})d_x^F \int_0^t dt_1 \int_{y\in V_1} e^{-(t-t_1)D_{\mathrm{bd}}^2}(x,y) i(e_{\mathfrak{n}})\sigma(y) dv_Y(y) \overset{x\to x_0\in V_1}{\longrightarrow} -i(e_{\mathfrak{n}})\sigma(x_0), \\ &w(e^m)d_x^{F*} \int_0^t dt_1 \int_{y\in Y_1} e^{-(t-t_1)D_{\mathrm{bd}}^2}(x,y) w(e^m)\sigma(y) dv_Y(y) \overset{x\to x_0\in Y_1}{\longrightarrow} w(e^m)\sigma(x_0). \end{aligned} \tag{3.48}$$

Now for $y_0 \in Y_1$, we will get the limit operator $\mathcal{L}_0^{(3)}$ by replacing $-\dot{S}$ by $\dot{S}$ in $\mathcal{L}_0^{(3)}$ of [6, Theorem 5.5], but with the boundary condition (3.40). In this way, we find again the factor $(-1)^m$ for the relative Euler classes and the factor $(-1)^{m+1}$ for $B(\nabla^{TX})$ from the boundary contributions of Y_1. □

Instead of deriving Theorem 3.4 from the same arguments as used for Theorem 3.2, we will follow a different route, namely to use Theorem 2.2. Thus we will carry out two steps: first, we deform the given metrics (g_0^{TX}, h_0^F) and (g_1^{TX}, h_1^F) to product metrics, then we compare the Milnor metrics depending only on h_0^F, h_1^F, and we evaluate the left hand side of Theorem 3.4 by Theorem 2.2.

Step 1 Given a pair of metrics (g^{TX}, h^F), by using the exponential map starting from the boundary along the normal direction, there exist a neighborhood U_ε of Y, and an identification $\partial X \times [0, \varepsilon[\to U_\varepsilon$, such that for $(y, x_m) \in Y \times [0, \varepsilon[$, (0.8) holds. We trivialize F on U_ε using the parallel transport with respect to the connection ∇^F along the curve $[0, 1[\ni u \to (y, u\varepsilon)$. Then on U_ε we have (2.2), but (2.3) need not hold. Thus we fix the metric g^{TX}, and deform the metric h^F to a Hermitian metric h_0^F such that (2.3) is verified for h_0^F. Theorem 3.4 for the metric pairs (g^{TX}, h^F) and (g^{TX}, h_0^F) follows as in [6, (4.22b)].

It remains to deform the metric g^{TX} to the metric $\widetilde{g}^{TX}$ in (0.9), while fixing the metric h_0^F above satisfying (2.3). Thus we assume that the families g_s^{TX}, h_s^F $(s \in [0, 1])$ featuring in the above analysis are defined by the equation

$$g_s^{TX} = s g^{TX} + (1 - s)\widetilde{g}^{TX}, \qquad h_s^F = h_0^F. \tag{3.49}$$

The argument in [6, §4.4] shows Theorem 3.4 for the metrics (g^{TX}, h_0^F) and $(\widetilde{g}^{TX}, h_0^F)$ in (3.49).

Step 2 From Theorem 3.1 and Step 1, we reduce the proof of Theorem 3.4 to the product case, i.e., we assume that (g_1^{TX}, h_1^F) and (g_0^{TX}, h_0^F) verify (2.1) and (2.3) (which need not hold, however, in the same identification $U_\varepsilon \simeq \partial X \times [0, \varepsilon[$). We apply Theorem 2.2 by choosing a Morse function f on X induced by a $\mathbb{Z}_2$-equivariant Morse function f on $\overline{X} = X \cup_{\partial X} X$ as in the proof of Lemma 1.5, cf. Section 2.2. Set

$$h_0 = \log \left(\frac{\|\cdot\|_{\det F, 1}}{\|\cdot\|_{\det F, 0}} \right)^2. \tag{3.50}$$

By (1.32), (1.35) and (3.50),

$$\log \left(\frac{\|\cdot\|^{M, \nabla f}_{\det H^\bullet(X, Y_1, F), 1}}{\|\cdot\|^{M, \nabla f}_{\det H^\bullet(X, Y_1, F), 0}} \right)^2 = \sum_{x \in B, x \notin Y_1} h_0(x)\, (-1)^{\mathrm{ind}(x)}. \tag{3.51}$$

Note that by (3.50), $dh_0 = \theta(F, h_1^F) - \theta(F, h_0^F)$. Thus from Theorem 2.2, (0.5), (3.50) and (3.51) we get

$$\begin{aligned}
\log \left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X, Y_1, F), 1}}{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X, Y_1, F), 0}} \right)^2 &= - \int_X dh_0\, (\nabla f)^* \psi(TX, \nabla_0^{TX}) \\
&\quad - \int_X \theta(F, h_1^F)(\nabla f)^* \Big(\psi(TX, \nabla_1^{TX}) - \psi(TX, \nabla_0^{TX}) \Big) \\
&\quad - \frac{1}{2} \left(\int_{V_1} - \int_{Y_1} \right) \theta(F, h_1^F)(\nabla f)^* \Big(\psi(TY, \nabla_1^{TY}) - \psi(TY, \nabla_0^{TY}) \Big)
\end{aligned}$$

$$-\frac{1}{2}\left(\int_{V_1}-\int_{Y_1}\right)dh_0\,(\nabla f)^*\psi(TY,\nabla_0^{TY})$$
$$+\sum_{x\in B,x\notin Y_1} h_0(x)\,(-1)^{\mathrm{ind}(x)}. \tag{3.52}$$

We identify each term in (3.52). At first, by [4, (3.34), (6.1)] we know that, as a current on Y,

$$d^Y(\nabla f)^*\psi(TY,\nabla_0^{TY}) = e(TY,\nabla_0^{TY}) - \sum_{x\in B,\,x\in Y}(-1)^{\mathrm{ind}(x)}\delta_x, \tag{3.53a}$$

$$(\nabla f)^*\left(\psi(TY,\nabla_1^{TY})-\psi(TY,\nabla_0^{TY})\right)-\widetilde{e}(TY,\nabla_0^{TY},\nabla_1^{TY})\text{ is exact.} \tag{3.53b}$$

Let $g_1^{T\overline{X}}, g_0^{T\overline{X}}$ be the $\mathbb{Z}_2$-invariant metrics on the double $\overline{X}=X\cup_{\partial X}X$ induced by g_1^{TX}, g_0^{TX}. Let $\overline{F}$ be the flat vector bundle on $\overline{X}$ induced by F. We denote by $\|\cdot\|_{\det\overline{F},i}$ ($i=0,1$) the $\mathbb{Z}_2$-invariant metrics on $\det\overline{F}$ over $\overline{X}$ induced by $\|\cdot\|_{\det F,i}$.

By (3.50) and the analogue of (3.53a) for $(\nabla f)^*\psi(T\overline{X},\nabla_0^{T\overline{X}})$, we get

$$-\int_X dh_0\,(\nabla f)^*\psi(TX,\nabla_0^{TX}) = -\frac{1}{2}\int_{\overline{X}} d\log\left(\frac{\|\cdot\|_{\det\overline{F},1}}{\|\cdot\|_{\det\overline{F},0}}\right)^2(\nabla f)^*\psi(T\overline{X},\nabla_0^{T\overline{X}})$$
$$=\int_X h_0\left(e(TX,\nabla_0^{TX})-\sum_{x\in B}(-1)^{\mathrm{ind}(x)}\delta_x\right.$$
$$\left.+\frac{1}{2}\sum_{x\in B\cap\partial X}(-1)^{\mathrm{ind}(x)}\delta_x\right). \tag{3.54}$$

We verify first that the total contribution of terms in (3.52) localized in B vanishes in view of (3.53a) and (3.54). Thus

$$\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F),1}}{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F),0}}\right)^2=\int_X h_0\,e(TX,\nabla_0^{TX})$$
$$-\int_X\theta(F,h_1^F)(\nabla f)^*\left(\psi(TX,\nabla_1^{TX})-\psi(TX,\nabla_0^{TX})\right)$$
$$+\frac{1}{2}\left(\int_{V_1}-\int_{Y_1}\right)\left[h_0\,e(TY,\nabla_0^{TY})+\widetilde{e}(TY,\nabla_0^{TY},\nabla_1^{TY})\theta(F,h_1^F)\right], \tag{3.55}$$

so we still need to identify the second term in (3.55).

Let $g_s^{T\overline{X}}$ ($s\in[0,1]$) be a family of $\mathbb{Z}_2$-invariant metrics connecting $g_1^{T\overline{X}}, g_0^{T\overline{X}}$. Then

$$\int_X\theta(F,h_1^F)(\nabla f)^*\left(\psi(TX,\nabla_1^{TX})-\psi(TX,\nabla_0^{TX})\right)$$
$$=\frac{1}{2}\int_{\overline{X}}\theta(\overline{F},h_1^F)\widetilde{e}(T\overline{X},\nabla_s^{T\overline{X}})=\int_X\theta(F,h_1^F)\widetilde{e}(TX,\nabla_s^{TX}). \tag{3.56}$$

From this we see that the first two terms in (3.55) vanish if m is odd and the last two terms vanish if m is even.

As the metric $g_s^{T\overline{X}}$ is $\mathbb{Z}_2$-invariant under the action of ϕ, and ∂X is the fixed point set, ∂X is totally geodesic in $(\overline{X}, g_s^{T\overline{X}})$ and the second fundamental form is zero. By using [6, (1.4)] as in [6, p. 777, line 6, (1.25)], we see that $\dot{\widetilde{S}} = 0$. Hence, as in [6, (4.37)] we see that, with the notation of [6, (1.17), (1.45)], if m is even,

$$e_b(TX, \nabla_0^{TX}) = 0, \quad \widetilde{e_b}(Y, \nabla_s^{TX}) = 0. \tag{3.57}$$

From Theorem 2.2, (3.11) and (3.55) to (3.57), we get Theorem 3.4 in the product case.

3.3 Proof of Theorem 0.1

We establish first Theorem 0.1 when the metric g^{TX} verifies (2.1) and a general metric h^F on F.

Let h_0^F be a Hermitian metric on F such that (2.3) holds on $\partial X \times [0, \varepsilon[$. We will add a subscript 0 to indicate the objects corresponding to g^{TX}, h_0^F. Now the function h_0 in (3.50) is defined by $h_1^F := h^F$ and h_0^F.

Note that when m is even, by (3.6), $e_b(Y, \nabla^{TX}) = 0$, as $\dot{S} = 0$ if g^{TX} verifies (2.1). Thus by Theorems 2.2, 3.4, (2.1), (3.7) and (3.51),

$$\begin{aligned}
\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(X,Y_1,F)}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(X,Y_1,F)}}\right)^2 &= -\int_X \theta(F, h_0^F)(\nabla f)^*\psi(TX, \nabla^{TX}) \\
&\quad + \frac{1}{2}\int_{Y_1} \theta(F, h_0^F)(\nabla f)^*\psi(TY_1, \nabla^{TY_1}) \\
&\quad - \frac{1}{2}\int_{V_1} \theta(F, h_0^F)(\nabla f)^*\psi(TV_1, \nabla^{TV_1}) \\
&\quad - \frac{1}{2}\mathrm{rk}(F)\chi(\partial X)\log 2 + \int_X h_0\, e(TX, \nabla^{TX}) \\
&\quad + \frac{1}{2}\left(\int_{V_1} - \int_{Y_1}\right) h_0\, e(TY, \nabla^{TY}) \\
&\quad - \sum_{x\in B, x\notin Y_1} h_0(x)(-1)^{\mathrm{ind}(x)}.
\end{aligned} \tag{3.58}$$

For $T \geqslant 0$, set (cf. [4, (3.47)])

$$B_T = \frac{1}{2}\dot{R}^{TX} + \sqrt{T}\sum_{i=1}^m e^i \wedge \widehat{\nabla_{e_i}^{TX}\nabla f} + T|df|^2. \tag{3.59}$$

By [4, Def. 3.6, Remark 3.8], $(\nabla f)^*\psi(TX, \nabla^{TX})$ is a locally integrable current on X with values in $o(TX)$, which is smooth on $X \setminus B$, with B in (1.24), and

$$(\nabla f)^*\psi(TX, \nabla^{TX}) = \int_0^\infty dT \int^{B_X} \frac{\widehat{df}}{2\sqrt{T}} \exp(-B_T). \tag{3.60}$$

By (2.1), (3.59) and $\nabla f|_{\partial X} \in T\partial X$, the coefficient of $\widehat{e^m}$ in $\jmath^* B_T$ is zero on ∂X, thus

$$\jmath^* \int^{B_X} \widehat{df} \exp(-B_T) = 0, \quad \jmath^* (\nabla f)^* \psi(TX, \nabla^{TX}) = 0 \quad \text{on } \partial X = Y. \tag{3.61}$$

By the same proof of [4, (3.60)], we get in the sense of distributions

$$\lim_{T\to\infty} \int^{B_X} \exp(-B_T) = \sum_{x\in B} (-1)^{\mathrm{ind}(x)} \delta_x - \frac{1}{2} \sum_{x\in B\cap\partial X} (-1)^{\mathrm{ind}(x)} \delta_x. \tag{3.62}$$

By the equation above (3.52), (3.61) and (3.62), we get an extension of (3.54),

$$\begin{aligned}
&\int_X \Big(\theta(F, h^F) - \theta(F, h_0^F)\Big) (\nabla f)^* \psi(TX, \nabla^{TX}) \\
&\quad = \int_X dh_0 \, (\nabla f)^* \psi(TX, \nabla^{TX}) = -\int_X h_0 d(\nabla f)^* \psi(TX, \nabla^{TX}) \\
&\quad = -\int_X h_0 \Big[e(TX, \nabla^{TX}) - \sum_{x\in B} (-1)^{\mathrm{ind}(x)} \delta_x + \frac{1}{2} \sum_{x\in B\cap\partial X} (-1)^{\mathrm{ind}(x)} \delta_x \Big].
\end{aligned} \tag{3.63}$$

By (3.53a), (3.58) and (3.63), we get Theorem 0.1 if g^{TX} verifies (2.1).

For a general metric g^{TX}, we introduce $\widetilde{g}^{TX}$ as in (0.9), by combining Theorem 0.1 for $(\widetilde{g}^{TX}, h^F)$, and Theorem 3.4 for the two couples of metrics (g^{TX}, h^F) and $(\widetilde{g}^{TX}, h^F)$, and the fact that $\widetilde{g}^{TX} = g^{TX}$ on ∂X, we get Theorem 0.1 for (g^{TX}, h^F).

Remark 3.5 We denote by $\widetilde{X}$ the fibers of the fibration $X \times \mathbb{R} \to \mathbb{R}$, and denote by $g^{T\widetilde{X}}$ a metric on $T\widetilde{X}$ such that $g^{T\widetilde{X}}|_{X\times\{s\}} = g_s^{TX}$ with $g_s^{TX} = (1-s)\widetilde{g}^{TX} + s g^{TX}$ for $s \in [0, 1]$. Then the canonical connection $\nabla^{T\widetilde{X}}$ on $T\widetilde{X}$ is (cf [4, (4.50)])

$$\nabla^{T\widetilde{X}} = \nabla_s^{TX} + ds \wedge \Big(\frac{\partial}{\partial s} + \frac{1}{2} (g_s^{TX})^{-1} \frac{\partial}{\partial s} g_s^{TX} \Big). \tag{3.64}$$

By again the argument in [4, Theorem 3.18], we get on $X \times \mathbb{R}$,

$$\begin{aligned}
&d^{X\times\mathbb{R}} (\nabla f)^* \psi(T\widetilde{X}, \nabla^{T\widetilde{X}}) \\
&\quad = e(T\widetilde{X}, \nabla^{T\widetilde{X}}) - \sum_{x\in B} (-1)^{\mathrm{ind}(x)} \delta_{\{x\}\times\mathbb{R}} + \frac{1}{2} \sum_{x\in B\cap\partial X} (-1)^{\mathrm{ind}(x)} \delta_{\{x\}\times\mathbb{R}}.
\end{aligned} \tag{3.65}$$

Integrating the coefficient of ds in (3.65) on $[0, 1]$, we get

$$\begin{aligned}
&(\nabla f)^* \psi(TX, \nabla^{TX}) - d^X \int_0^1 ds \, i\Big(\frac{\partial}{\partial s}\Big) (\nabla f)^* \psi(T\widetilde{X}, \nabla^{T\widetilde{X}}) \\
&\quad = (\nabla f)^* \psi(TX, \widetilde{\nabla}^{TX}) + \widetilde{e}(TX, \nabla_s^{TX}).
\end{aligned} \tag{3.66}$$

We compute the second term in (3.66). From (3.64), the analogy of (3.59) for $T\widetilde{X}$ is

$$\widetilde{B}_T = \frac{1}{2} \dot{R}^{T\widetilde{X}} + \sqrt{T} \sum_{i=1}^m e^i \wedge \nabla_{e_i}^{T\widetilde{X}} \nabla f + T|df|^2. \tag{3.67}$$

Let $\widetilde{\jmath} : Y \times \mathbb{R} \to X \times \mathbb{R}$ be the natural embedding. By [6, Lemma 1.7 and (1.16)], for $s \in [0,1]$

$$\widetilde{\jmath}^{*} \dot{R}^{T\widetilde{X}}|_{Y\times\{s\}} = \dot{R}^{TY} + 2s^2\dot{S}^2 + s\sum_{\alpha=1}^{m-1}\Big\langle e_\alpha, \jmath^{*}R^{TX}e_m\Big\rangle \widehat{e^\alpha}\wedge\widehat{e^m} + 2ds\wedge\dot{S}\wedge\widehat{e^m}. \quad (3.68)$$

As $\nabla f|_{\partial X} \in T\partial X$, from [6, (1.14)], (3.60), (3.67) and (3.68), we get finally on Y,

$$\int_0^1 ds\, i(\frac{\partial}{\partial s})\widetilde{\jmath}^{*}(\nabla f)^{*}\psi(T\widetilde{X}, \nabla^{T\widetilde{X}}) = \pi^{-1/2}\int_0^\infty dT \int^{B_Y} \frac{\widehat{df}}{2\sqrt{T}}$$
$$\times \exp\Big(-\frac{1}{2}\dot{R}^{TY} - \sqrt{T}\sum_{\alpha=1}^{m-1} e^\alpha\wedge \widehat{\nabla^{TY}_{e_\alpha}\nabla f} - T|df|^2\Big)\sum_{k=0}^{\infty}\frac{(-1)^k\dot{S}^{2k+1}}{k!\,(2k+1)}. \quad (3.69)$$

We could not see directly whether the final formula in (3.69) is exact on ∂X. Thus we prefer to state Theorem 0.1 by using the current $(\nabla f)^{*}\psi(TX, \widetilde{\nabla}^{TX})$.

3.4 Theorem 0.3: product case

We use the notation from the Introduction. We assume first that g^{TZ} and h^F have product structure near ∂Z and V, i.e., (2.1) and (2.3) hold near ∂Z and V.

By the argument in the proof of [5, Theorem 1.10], we can choose a Morse function $f : Z \to \mathbb{R}$ in Lemma 1.5 which restricts to a Morse function on $\partial Z \cup V$ and satisfies $\nabla f|_{\partial Z\cup V} \in T(\partial Z \cup V)$, $d^2 f(x)|_{\mathfrak{n}} > 0$ for $x \in B \cap (\partial Z \cup V)$.

Let $(C^\bullet(W^u_{Z_1}/W^u_{V\cup Y_1}, F), \widetilde{\partial})$, $(C^\bullet(W^u/W^u_{Y_1\cup Y_2}, F), \widetilde{\partial})$ and $(C^\bullet(W^u_{Z_2}/W^u_{Y_2}, F), \widetilde{\partial})$ be the complexes corresponding to $(Z_1, V\cup Y_1, F)$, $(Z, Y_1\cup Y_2, F)$ and (Z_2, Y_2, F) defined by (1.32), respectively; then we have the following short exact sequence of complexes:

$$0 \to C^\bullet(W^u_{Z_1}/W^u_{V\cup Y_1}, F) \to C^\bullet(W^u/W^u_{Y_1\cup Y_2}, F) \to C^\bullet(W^u_{Z_2}/W^u_{Y_2}, F) \to 0. \quad (3.70)$$

It induces the complex (0.16), and the canonical section ϱ of $\lambda(F)$ in (0.20). Let $\|\cdot\|^{M,\nabla f}_{\lambda(F)}$ be the Milnor metric on $\lambda(F)$. As for each degree the complex (3.70) splits isometrically, by Definition 1.6, we have

$$\|\varrho\|^{M,\nabla f}_{\lambda(F)} = 1. \quad (3.71)$$

Let $\mathbf{Y} := \partial Z \cup V$. By (2.18), we get

$$\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(Z,Y_1\cup Y_2,F)}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(Z,Y_1\cup Y_2,F)}}\right)^2 = -\frac{1}{2}\chi(\partial Z)\mathrm{rk}(F)\log(2)$$
$$-\int_Z \theta(F,h^F)(\nabla f)^{*}\psi(TZ,\nabla^{TZ})$$
$$-\frac{1}{2}\int_{V_1\cup V_2}\theta(F,h^F)(\nabla f)^{*}\psi(T\mathbf{Y},\nabla^{T\mathbf{Y}})$$
$$+\frac{1}{2}\int_{Y_1\cup Y_2}\theta(F,h^F)(\nabla f)^{*}\psi(T\mathbf{Y},\nabla^{T\mathbf{Y}}), \quad (3.72)$$

$$\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(Z_1,V\cup Y_1,F)}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(Z_1,V\cup Y_1,F)}}\right)^2 = -\frac{1}{2}\chi(\partial Z_1)\mathrm{rk}(F)\log(2)$$
$$-\int_{Z_1}\theta(F,h^F)(\nabla f)^*\psi(TZ,\nabla^{TZ})$$
$$-\frac{1}{2}\int_{V_1}\theta(F,h^F)(\nabla f)^*\psi(T\mathbf{Y},\nabla^{T\mathbf{Y}})$$
$$+\frac{1}{2}\int_{V\cup Y_1}\theta(F,h^F)(\nabla f)^*\psi(T\mathbf{Y},\nabla^{T\mathbf{Y}}). \quad (3.73)$$

For the triplet (Z_2, Y_2, F), we get, again from (2.18),

$$\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(Z_2,Y_2,F)}}{\|\cdot\|^{M,\nabla f}_{\det H^\bullet(Z_2,Y_2,F)}}\right)^2 = -\frac{1}{2}\chi(\partial Z_2)\mathrm{rk}(F)\log(2)$$
$$-\int_{Z_2}\theta(F,h^F)(\nabla f)^*\psi(TZ,\nabla^{TZ})$$
$$-\frac{1}{2}\int_{V\cup V_2}\theta(F,h^F)(\nabla f)^*\psi(T\mathbf{Y},\nabla^{T\mathbf{Y}}) \quad (3.74)$$
$$+\frac{1}{2}\int_{Y_2}\theta(F,h^F)(\nabla f)^*\psi(T\mathbf{Y},\nabla^{T\mathbf{Y}}).$$

Note that $\partial Z_1 = Y_1\cup V_1\cup V$, $\partial Z_2 = Y_2\cup V_2\cup V$. From (0.20), (3.72), and (3.74) we deduce

$$\log\left(\|\varrho\|^{\mathrm{RS}}_{\lambda(F)}\right)^2 = \log\left(\|\varrho\|^{M,\nabla f}_{\lambda(F)}\right)^2 - \chi(V)\mathrm{rk}(F)\log(2), \quad (3.75)$$

and (0.21) follows from (3.71), and (3.75).

3.5 Theorem 0.3: general case

Now assume general metrics g^{TZ}, h^F. Let g_0^{TZ} and h_0^F be the metrics on TX and F such that (2.1), (2.3) hold near ∂Z and V, moreover, on $\partial Z\cup V$ we have $g_0^{TZ} = g^{TZ}$, $h_0^F = h^F$. If m is even, then by Theorem 3.4

$$\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(Z_1,V\cup Y_1,F)}}{\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet(Z_1,V\cup Y_1,F),0}}\right)^2 = \int_{(Z_1,\partial Z_1)}\log\left(\frac{\|\cdot\|_{\det F}}{\|\cdot\|_{\det F,0}}\right)^2 E(TZ_1,\nabla_0^{TZ_1})$$
$$+\int_{(Z_1,\partial Z_1)}\widetilde{E}(TZ_1,\nabla_0^{TZ_1},\nabla^{TZ_1})\theta(F,h_1^F)$$
$$+\mathrm{rk}(F)\left(\int_{V_1}-\int_{V\cup Y_1}\right)B(\nabla^{TZ_1}). \quad (3.76)$$

If m is odd, then

$$\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}(Z_1,V\cup Y_1,F)}}{\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}(Z_1,V\cup Y_1,F),0}}\right)^2 = \mathrm{rk}(F)\int_{\partial Z_1} B(\nabla^{TZ_1}). \tag{3.77}$$

We have similar equations for $\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}(Z,Y_1\cup Y_2,F)}}{\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}(Z,Y_1\cup Y_2,F),0}}\right)^2$ and $\log\left(\frac{\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}(Z_2,Y_2,F)}}{\|\cdot\|^{\mathrm{RS}}_{\det H^{\bullet}(Z_2,Y_2,F),0}}\right)^2$. Now observe that on V, computing $\dot{S}$ in Z_2 amounts to changing the inward unit normal in Z_1 such that

$$B(\nabla^{TZ_1})|_V = (-1)^{m-1}B(\nabla^{TZ_2})|_V \in \Omega^{m-1}(V, o(TV)). \tag{3.78}$$

From Theorem 0.3 for g_0^{TZ} and h_0^F, and from (3.76)–(3.78), we get Theorem 0.3 for g^{TZ}, h^F. The proof of Theorem 0.3 is complete.

3.6 Proof of Theorem 0.4

The argument for the Milnor metrics leading to (3.71) applies also to this case and gives

$$\|\widetilde{\varrho}\|^{M,\nabla f}_{\lambda(F)} = 1. \tag{3.79}$$

By Theorem 2.2 and (3.79) for (Z, Y_1, F), (Z, F) and (Y_1, F), and arguing as in (3.72) and (3.74), we obtain (0.24) in the product case, i.e., $\log(\|\widetilde{\varrho}\|^{\mathrm{RS},2}_{\lambda(F)}) = 0$.

For general metrics, we apply the anomaly formula, Theorem 3.4, and obtain as in (3.76) and (3.77),

$$\log(\|\widetilde{\varrho}\|^{\mathrm{RS},2}_{\lambda(F)}) = \mathrm{rk}(F)\left(-\int_{\partial Z} + \int_{\partial Z\setminus Y_1} +(-1)^{m+1}\int_{Y_1}\right) B(\nabla^{TZ}). \tag{3.80}$$

The proof of Theorem 0.4 is complete.

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