Stochastic Processes II Summer term 2008
(Stochastische Analysis)
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Exercises, 2nd July

12.1 (4 points) Let $(M_t)_{t\geq 0}$ be a continuous local martingale and assume that its quadratic variation is of the form

$$\langle M\rangle_t = \int_0^t \sigma^2(M_s)ds, \qquad t \geq 0,$$

with $\sigma^2 > 0$. Assume further that $\langle M\rangle_\infty = \infty$. Then M can be written as a time changed standard Brownian motion B, i.e.

$$M_t = B_{\langle M\rangle_t}, \quad B_t = M_{T_t} \quad \text{with} \quad T_t = \inf\left\{\, s \geq 0 \;\middle|\; \langle M\rangle_s > t\right\}, \quad t \geq 0,$$

due to the lecture. Use Problem 11.3 to prove the following alternative representation of the time change T:

$$T_t = \int_0^t \frac{1}{\sigma^2(B_s)}ds, \qquad t \geq 0.$$

12.2 (4 points) Let $(X_t)_{t\geq 0}$ be the Ornstein-Uhlenbeck-process from Problem 4.3:

$$X_t := e^{-at}\left(x_0 + \int_0^t e^{as}\sigma dB_s\right), \qquad t \geq 0.$$

Represent the martingale $M_t := e^{at}X_t,\ t \geq 0$, as time changed Brownian motion.

12.3 (2+2+3 points) Compute the Itô–representation, i.e. the adapted integrand $\vartheta \in L^2(B)$ with

$$H = E[H] + \int_0^T \vartheta_t dB_t$$

for the following functions H of a Brownian motion $(B_t)_{0\leq t\leq T}$:

a) B_T^2,

b) $\int_0^T B_t^2 \, dt$,

c) $\int_0^T \exp(\sigma B_t + \mu t) dt \quad (\mu, \sigma \in \mathbb{R})$.

12.4 (4 points) Let $B = (B_t)_{t \geq 0}$ be a Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ and let $(\mathcal{F}_t^B)_{t \geq 0}$ denote the filtration generated by B. Show that for every $\mathcal{F}_\infty^B$-measurable random variable $H \geq 0$ it holds

$$E[\, H \,|\, \mathcal{F}_t \,] = E[\, H \,|\, \mathcal{F}_t^B \,] \qquad P\text{-a.s.}$$

for each $t \geq 0$.

Problems 12.1 -12.4 should be solved at home and delivered at Wednesday, the 9th July, before the beginning of the tutorial.