Stochastic Processes II Summer term 2008
(Stochastische Analysis)
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Exercises, 4th June

8.1 (4 points) Let $(X_t, \mathcal{F}_t, t \geq 0)$ be a continuous martingale and H a "simple" stochastic process, i.e.

$$H_t(\omega) := \sum_{i=0}^{n-1} I_{(t_i, t_{i+1}]}(t) \cdot \widetilde{H}^i(\omega) \qquad (t \geq 0)$$

with fixed points $0 = t_0 < t_1 < \cdots < t_n < \infty$ and bounded $\mathcal{F}_{t_i}$-measurable random variables $\widetilde{H}^i$. We define the *stochastic integral* of H with respect to X as

$$\int_0^t H_s dX_s := \sum_{i=1}^{n} \widetilde{H}^{i-1}(X_{t_i \wedge t} - X_{t_{i-1} \wedge t}) \qquad (t \geq 0).$$

Prove that the stochastic integral is a continuous martingale.

8.2 (3+3 points) Let $(X_t)_{t \geq 0}$ be a local martingale with a localizing sequence of stopping times $(T_n)_{n \in \mathbb{N}}$. Show that:

a) If $\sup_{t \geq 0} |X_t| \in L^1$, then (X_t) is a martingale, and there exists an $X_\infty \in L^1$ such that

$$X_\infty = \lim_{t \to \infty} X_t \qquad P\text{-a.s. and in } L^1.$$

b) If the set $\{X_{t \wedge T_n} \mid n \in \mathbb{N}\}$ is uniformly integrable for all $t \geq 0$ (or if for some $p > 1$ $\sup_{n \in \mathbb{N}} E[\,|X_{t \wedge T_n}|^p\,] < \infty$ for all $t \geq 0$) then (X_t) is a martingale (with $X_t \in L^p$ for all $t \geq 0$) respectively.

8.3 (5 points) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and for $a > 0$ let

$$T_a := \inf\left\{\, t \geq 0 \;\middle|\; X_t \geq a \,\right\}.$$

Let further $A : [0,1] \to [0,\infty]$ be a continuous and strictly increasing process with $A(0) = 0$ and $A(1) = +\infty$. Show that the process Y defined as

$$Y_t := X_{A(t)\wedge T_a} \qquad (0 \leq t \leq 1)$$

is a local but not a "real" martingale with respect to the transformed filtration $(\mathcal{F}_{A(t)})_{t\geq 0}$.

8.4 (3+2 points)

a) Let M be a continuous square integrable martingale with independent increments. Show that $\langle M\rangle$ is deterministic, i.e. there exists a function f on $\mathbb{R}_+$ such that $\langle M\rangle_t = f(t)$ P-a.s..

b) If M is a continuous martingale and a Gaussian process, prove that $\langle M\rangle$ is deterministic.

The problems 8.1 -8.4 should be solved at home and delivered at Wednesday, the 11th June, before the beginning of the tutorial.