Stochastic Processes I Winter term 2007/2008
(Stochastik II)
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Exercises, January 9th

11.1 (3 points) Let $(X_n)_{n=0,1,\dots}$ be a Markov chain with countable state space S and transition matrix K. We say that a function $u : S \to [0,\infty]$ is *superharmonic* if

$$u(x) \geq Ku(x) := \sum_{y \in S} K(x,y)u(y) \quad \text{for all } x \in S.$$

(a) Show that $(u(X_n))_{n=0,1,\dots}$ is a P_x-supermartingale for any $x \in S$ with $u(x) < \infty$, whose Doob decomposition $u(X_n) = M_n - A_n$ is given by

$$A_n = \sum_{k=0}^{n-1} (u - Ku)(X_k), \quad n = 0, 1, \dots$$

(b) Prove that $E_x[u(X_T); T < \infty] \leq u(x)$ for any $x \in S$ and for any stopping time T.

(c) Show that any superharmonic function $u : S \to [0,\infty)$ is constant on S if $(X_n)_{n=0,1,\dots}$ is *irreducible recurrent*, i. e., for the stopping time $T_y := \min \{ n \mid X_n = y \}$, $y \in S$, it holds that

$$P_x [T_y < \infty] = 1 \quad \text{for all } x, y \in S.$$

11.2 (4 points) Consider the canonical model of a Markov chain $(X_n)_{n=0,1,\dots}$ with countable state space S and transition matrix K, and let θ denote the *shift operator* on the path space $\Omega = S^{\{0,1,\dots\}}$ defined by $(\theta\omega)(n) := \omega(n+1)$.

(a) Suppose that ϕ is a measurable bounded function on Ω satisfying $\phi = \phi \circ \theta$ ("shift invariance"). Show that h defined by

$$h(x) := E_x[\phi], \; x \in S,$$

corresponds to a bounded harmonic function on S, i. e., $Kh(x) = h(x)$ for all $x \in S$.

(b) Verify that $\lim_{n\uparrow\infty} h(X_n) = \phi$ P_x-a. s. for any $x \in S$.

(c) Does, conversely, any bounded harmonic function on S admit the representation stated in i) for an appropriate ϕ?

11.3 (2 points) Let us consider a renewal process with $p_{0j} := f_j$, $j \geq 0$, for $f_j \geq 0$, $\sum_{j\geq 0} f_j = 1$ and $p_{i,i-1} = 1, i \geq 1$. Under which condition does there exist an invariant initial distribution?

11.4 (2 points) Let $(Y_n)_{n\in\mathbb{N}}$ be an adapted sequence of nonnegative random variables such that $Y_n \leq c$, $n \in \mathbb{N}$, for some constant $c > 0$. Prove that

$$\left\{\sum_{n=1}^{\infty} Y_n = \infty\right\} = \left\{\sum_{n=1}^{\infty} E[Y_n \,|\, \mathcal{A}_{n-1}] = \infty\right\} \quad P\text{-a. s.}.$$

Hint: Use the following "dichotomy for martingales with bounded increments" (see, e. g., Shiryaev, *Probability*, 2nd edition, Chapter VII, § 5): If $(X_n)_{n=0,1,\dots}$ is a martingale satisfying $\sup_{n\in\mathbb{N}} |X_n - X_{n-1}| \in L^1(P)$ then it holds that $P[C \cup D] = 1$ where

$$\begin{aligned} C &:= \{\omega \in \Omega | \exists \lim_{n\uparrow\infty} X_n(\omega) \in \mathbb{R}\}, \\ D &:= \{\omega \in \Omega | \underline{\lim}_{n\uparrow\infty} X_n(\omega) = -\infty,\ \overline{\lim}_{n\uparrow\infty} X_n(\omega) = \infty\}. \end{aligned}$$

11.5 (4 bonus points) Consider the simple voter model where N particles ("voters") take in the next period independent of each other the state 1 ("pro") or 0 ("contra"). Let $x = (x_1, \dots, x_N) \in S := \{0,1\}^{\{1,\dots,N\}}$ denote the present configuration and

$$m(x) := \tfrac{1}{N} \sum_{i=1}^{N} x_i$$

the current "sentiment". Then the ith particle chooses the state "1" with probability

$$p_i(x) = \alpha\, x_i + (1-\alpha)\, m(x)$$

for a given parameter $\alpha \in (0,1)$.

(a) Describe this evolution as a Markov chain $(X_n)_{n=0,1,\dots}$ on the state space S by means of a transition matrix K.

(b) Show that the process is absorbed either in the state "all 1" or "all 0", namely in the state "all 1" with probabilty $m(x)$ given that the chain starts in $x \in S$.

The problems should be solved at home and delivered at Wednesday, January 16th, before the beginning of the tutorial.