Stochastic Processes I Winter term 2007/2008
(Stochastik II)
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Exercises, 23d January

13.1 (3 points) Let $(Z_n, n \geq 0)$ be a branching process with

$$P(Z_1 = 2) = p, \quad P(Z_1 = 0) = 1 - p \qquad (0 < p < 1).$$

Compute the probability π that $\lim_{n\to\infty} Z_n = 0$.
Hint: Use Problem 12.3

13.2 (4 points) The simple random walk with parameter $p \in (0, 1)$ is a Markov chain with state space $\mathbb{Z}$ and transition probabilities given by $p_{ij} = p$ for $j = i + 1$, $p_{ij} = 1 - p$ for $j = i - 1$ and $p_{ij} = 0$ otherwise. Show that for a simple random walk all states are recurrent if $p = \frac{1}{2}$, and all are transient if $p \neq \frac{1}{2}$.
Hint: Use Stirling's formula:

$$n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi} \quad \text{as} \quad n \to \infty.$$

13.3 (4 points) A woman has two suitors, Mr. Darcy and Mr. Wickham. At the start of each month she is either engaged or married to one or the other suitor. If engaged to a particular suitor at the start of the given month, there are three possibilities for the start of the following month: she may be still engaged to the current favourite, or married to him, or engaged to his rival, with probabilities $0.4, 0.3, 0.3$ respectively. Assume marriage is permanent (she's an old-fashioned girl).

a) Set this system up as a four-state Markov chain. Find the probability that the woman ultimately marries Mr. Darcy, given that she is initially engaged to him.
b) Let N be the number of completed months before she marries. Find $E[N]$.

13.4 (4 points) Consider the following two models for a diffusion of a gas. In these models the balls represent molecules and the urns represent different regions of space between which molecules may pass. Set up these models as Markov chains. In each case write down the transition probabilities and find a formula for the invariant distribution.

a) (The Ehrenfest model) A total number of N balls are places in two urns (Urn 1 and Urn 2) and X_n represents the number of balls in Urn 1 at time n. After each unit of time, one of the N balls is selected at random, removed from whichever urn it happens to be in, and placed in the other one.

b) (The Bernoulli model) N black balls and N white balls are placed in two urns so each urn contains N balls. After each unit of time, one ball is selected at random from each urn, and the two balls thus selected are interchanged. Let X_n denote the number of black balls in Urn 1 at time n.

The problems 13.1 -13.4. should be solved at home and delivered at Wednesday, the 30th January, before the beginning of the tutorial.