Stochastic Processes I Winter term 2007/2008
(Stochastik II)
Prof. Dr. Uwe Küchler
Dipl. Math. Irina Penner

Exercises, 31st October

3.1 (4 points) Assume that X_1 and X_2 are independent random variables, both having a Poisson distribution with parameter $\lambda > 0$. Let further $Y := X_1 + X_2$. Compute $P[\, X_1 = i \,|\, Y\,]$ for $i = 0, 1, \ldots$.

3.2 (5 points) Let $n \in \mathbb{N}$ and let $\mathfrak{Z}_n$ be the partition

$$\left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \mid k = 0, 1, \ldots, 2^n - 1 \right\}$$

of $\Omega := [0, 1)$. We denote by $\mathfrak{B}_{[0,1)}$ the σ-algebra of Borel subsets of Ω and by λ the Lebesgue measure on $\mathfrak{B}_{[0,1)}$. Consider a random variable X on $(\Omega, \mathfrak{B}_{[0,1)}, \lambda)$ defined by $X(\omega) = \omega$, $\omega \in \Omega$.

a) Calculate $X_n := E[X|\mathfrak{B}_n]$, where $\mathfrak{B}_n := \sigma(\mathfrak{Z}_n)$, $n = 1, 2, \ldots$.

b) Show that

$$E[X_{n+1}|\mathfrak{B}_n] = X_n \qquad P\text{-a.s.}$$

for all $n = 1, 2, \ldots$.

3.3 (4 points) For a random variable $X \in L^2(\Omega, \mathcal{A}, P)$ and a σ-algebra $\mathcal{A}_0 \subseteq \mathcal{A}$ we define the conditional variance of X w.r.t. $\mathcal{A}_0$ as

$$\operatorname{var}(X|\mathcal{A}_0) := E\left[\, (X - E[X|\mathcal{A}_0])^2 \,|\, \mathcal{A}_0 \,\right].$$

Show that

$$\operatorname{var}(X|\mathcal{A}_0) = E[X^2|\mathcal{A}_0] - (E[X|\mathcal{A}_0])^2$$

and

$$\operatorname{var}(X) = E[\operatorname{var}(X|\mathcal{A}_0)] + \operatorname{var}\left(E[X|\mathcal{A}_0]\right).$$

3.4 (3 points) Assume that X_i, $i = 1, 2, 3, \dots$ are independent identically distributed random variables with $P[X_1 = 1] = \frac{1}{2} + \alpha$ and $P[X_1 = -1] = \frac{1}{2} - \alpha$ for some $\alpha \in [0, \frac{1}{2}]$. Let further $\mathcal{A}_n$ be the σ-algebra generated by $X_1, \dots, X_n$ and $S_n := X_1 + \dots + X_n$, $n = 1, 2, \dots$. Show that

$$E[S_n | \mathcal{A}_k] = S_k + 2\alpha(n - k)$$

for $k = 1, \dots, n$.

The problems 3.1 -3.4. should be solved at home and delivered at Wednesday, the 7th November, before the beginning of the tutorial.