Stochastic Processes I Winter term 2007/2008
(Stochastik II)
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Exercises, 14th November

5.1 (3 points) Assume that $(Y_k, k \geq 1)$ is a sequence of i.i.d. random variables with

$$P(Y_1 = i) = p^i(1-p)^{1-i},\ i \in \{0,1\},$$

for some $p \in (0,1)$. Let N be a Poisson distributed random variable (parameter $\lambda > 0$), independent from $(Y_k, k \geq 1)$. Put

$$S := \sum_{k=1}^{N} Y_k \quad \text{with} \quad \sum_{k=1}^{0} Y_k := 0.$$

Determine $E(S|N)$ and $E(N|S)$.

5.2 (4 points) Let $(X_1, X_2, \ldots, X_n, X_{n+1})^T$ be an $N(0, \sum_{n+1})$ - distributed random vector. The elements of the matrix $\sum_{n+1}$ are denoted by $\sigma_{ij}, i, j = 1, 2, \ldots, n+1$.
Assume the matrix $\sum_n := (\sigma_{ij})_{i,j=1,\ldots,n}$ is regular. Verify the identity

$$E(X_{n+1}|X_1, \ldots, X_n) = (\sigma_{1n+1}, \ldots, \sigma_{n,n+1}) \sum_n^{-1} (X_1, \ldots, X_n)^T \quad \text{P-a.s.}$$

Conclude that for every two centered random variables X, Y with a common Gaussian distribution the equation

$$E(X|X+Y) = \frac{Cov(X,Y) + Var(X)}{Var(X) + Var(Y) + 2Cov(X,Y)}\,(X+Y)$$

is valid. What does change if X and Y are not centered?

5.3 Let X be a random variable with $E|X| < \infty$. Prove that the following assertions hold:

a) (2 points) If X is discrete with $P(X = k) = p_k > 0, k \in Z =$ set of all integers, then

$$E(X \mid |X|) = \frac{p_{|X|} - p_{-|X|}}{p_{|X|} + p_{-|X|}} |X|$$

b) (4 bonus points) If X has a density f with $f(x) > 0, x \in R_1$, then

$$E(X \mid |X|) = \frac{f(|X|) - f(-|X|)}{f(|X|) + f(-|X|)} |X| \quad \text{P-a.s.}$$

5.4 (4 points) Assume that X and Y are nonnegative i.i.d. random variables with $E|X| = E|Y| < \infty$. Then we have

$$E(X|X+Y) = \frac{1}{2}(X+Y) \quad \text{P-a.s.,} \tag{$*$}$$

due to Exercise 4.3.

a) Does it follow from a similar symmetry argument as for $(*)$ that

$$E(X|XY) = (XY)^{\frac{1}{2}} \quad \text{P-a.s.?}$$

b) Calculate $E(X|XY)$ explicitly given that X and Y are uniformly distributed on $(0, 1]$.

The exercises should be solved at home and delivered at Wednesday, November 21st, before the beginning of the tutorial.