§3 Universality

From now on assume:

A1 Either no $\tau < \Theta$ is Woodin in an inner model or else V_Θ is closed under #.

A2 Let $M \in V_\Theta$ be a 1-small premouse and $\mathcal{I}$ a normal iteration of M of length Θ. Then $\mathcal{I}$ has a cofinal branch

A3 Θ is a Mahlo cardinal.

Note A1 subsumes our earlier A0. A1 is known to imply that K^c is iterable below Θ.

Note A2 holds if $V_\Theta^\#$ exists or if Θ is not Woodin in an inner model.

Note A3 is assumed in order to insure a large supply of inaccessible $\kappa < \Theta$ which can serve as critical points for the extenders in the K^c construction.

We prove:

<u>Thm 5</u> K^c is universal wrt. 1-small premice in V_Θ – i.e. if $N \in V_\Theta$ is a 1-small premouse, then any coiteratio of N, K^c will terminate below Θ.

<u>Note</u> A failure of well foundedness is considered a termination. Hence the lemma would hold even without A1, which is used to ensure an iteration strategy for K^c. The interesting case occurs when both sides have a normal iteration strategy, thus preventing a failure of well foundedness.

We now prove Thm 5. Suppose not. Let $Q \in V_\Theta$ be a 1-small premouse and let $\langle \mathcal{T}^Q, \mathcal{T}^K \rangle$ be a coiteration of Q, K^c of length Θ. Let:

$$\mathcal{T}^Q = \langle \langle Q_i \rangle, \langle \nu_i \rangle, \langle \eta_i^Q \rangle, \langle \pi_{ij}^Q \rangle, T^Q \rangle$$

$$\mathcal{T}^K = \langle \langle K_i \rangle, \langle \nu_i \rangle, \langle \eta_i^K \rangle, \langle \pi_{ij}^K \rangle, T^K \rangle.$$

Making use of this we shall prove:

<u>Lemma 5.1</u> There are a stationary $S \subset \Theta$ and a commutative system $\langle \pi_{\alpha\beta} \mid \alpha \leq \beta \text{ in } S \rangle$ s.t.

(i) α is inaccessible for $\alpha \in S$

(ii) Set $\tilde{K}_\alpha = (J^E_{\alpha^+})^{K^C}$ ($\alpha \in S$). Then

$$\pi_{\alpha\beta} : \tilde{K}_\alpha \xrightarrow[\Sigma_0]{} \tilde{K}_\beta \text{ cofinally} ; \ \pi_{\alpha\beta} \upharpoonright \alpha = \mathrm{id} ;$$
$$\pi_{\alpha\beta}(\alpha) = \beta \quad \text{for } \alpha \leq \beta \text{ in } S.$$

proof.

Let b be a cofinal branch in $\mathcal{T}^Q$. This has a transitive limit model Q_b and maps π_i^Q to Q_b for $i \in b$. There is $i_0 \in b$ s.t. no truncation occurs on b above i_0. Hence π_i^Q is total on Q_i for $i_0 \leq i \in b$. For $i \in b$ set:

$\kappa_i = \kappa_i^Q = \mathrm{crit}(\pi_{ib})$; $\tau_i = \tau_i^Q = (\kappa_i^+)^{Q_i}$.

A standard proof shows that there is a club $C \subset b \setminus i_0$ s.t. if $\alpha \in C$, then $\alpha = \kappa_\alpha$ and $\pi_{\alpha\beta}(\alpha) = \beta$ for $\alpha, \beta \in C$, $\alpha \leq \beta$. It follows that $\pi_\alpha(\alpha) = \Theta$; $\alpha = \mathrm{crit}(\pi_\alpha)$ for $\alpha \in C$.

Now let $\kappa \in C$ be inaccessible. Then $\kappa \in Q_\kappa \cap K_\kappa$, since otherwise K_κ would be a segment of Q_κ and the coiteration would terminate.

(1) $\kappa \neq \pi^K_{i\kappa}(\bar{\kappa})$ for any $\bar{\kappa} < \kappa$.

proof. Otherwise we could repeat the proof that coiterations terminate (§4 of [NFS]) to show that the coiteration of Q, K must have terminated below κ. QED(1)

Hence:

(2) There is no truncation on the branch $\{i \mid i \leq_{TK} \kappa\}$. (Hence $\pi^K_{i\kappa}$ is total for $i \leq_{TK} \kappa$.)

(Otherwise $\kappa = \pi^K_{i\kappa}(\bar{\kappa})$ where $\bar{\bar{K}}_i < \kappa$)

It follows easily that:

(3) $\pi^K_{i\kappa}(\xi) < \kappa$ for $\xi < \kappa$,

hence:

(4) $\pi^K_{i\kappa}(\kappa) = \kappa$.

Let $\tau = \kappa^{+K^c}$. Then

(5) $\pi^K_{i\kappa}(\tau) = \tau$.

Now set: $\tilde{K}_\kappa = J_\tau^{E^{K^c}}$; $K_\kappa^* = \pi_{0\kappa}^K(\tilde{K}_\kappa)$.

It follows easily that:

(1) $K_\kappa^* = J_\tau^{E^{K_\kappa}} = J_\tau^{E^{Q_\kappa}}$.

(Note that $\tau = \kappa^{+Q_\kappa}$).

Set: $\tilde{\pi}_\kappa = \pi_{0\kappa}^K \upharpoonright \tilde{K}_\kappa$. Then

(2) $\tilde{\pi}_\kappa : \tilde{K}_\kappa \xrightarrow[\Sigma_0]{} K_\kappa^*$ cofinally.

But by (1):

(3) $\pi_{\kappa\kappa'}^Q : K_\kappa^* \xrightarrow[\Sigma_0]{} K_{\kappa'}^*$ cofinally

for $\kappa, \kappa' \in C'$, $\kappa \leq \kappa'$,

where C' = the of inaccessibles in C

We wish to define a stationary $S \subset C'$ s.t. $\pi_{\kappa\kappa'}^Q \tilde{\pi}_\kappa \subset \tilde{\pi}_{\kappa'}$ for $\kappa, \kappa' \in S$, $\kappa \leq \kappa'$. Recalling that $\tau_\kappa = \mathrm{ht}(\tilde{K}_\kappa)$ for $\kappa \in C'$, we set $\tau_\kappa^* = \mathrm{ht}(K_\kappa^*)$ and note that:

(4) $\mathrm{cf}(\tau_\kappa) = \mathrm{cf}(\tau_\kappa^*) = \mathrm{cf}(\tau_{\kappa'}^*)$

for $\kappa, \kappa' \in C'$, $\kappa \leq \kappa'$,

by (2), (3).

Let $cf(\tau_\kappa) = \gamma$ for all $\kappa \in C'$. Pick $\langle \xi_i^\kappa \mid i < \gamma \rangle$ monotone and cofinal in τ_κ for $\kappa \in C'$. Assume w.l.o.g. that $\kappa > \gamma$ for $\kappa \in C'$. Then there is $\delta(\kappa) < \kappa$ s.t.

$\{\tilde{\pi}_\kappa(\xi_i^\kappa) \mid i < \gamma\} \subset \text{rng}(\pi^Q_{\delta\kappa})$

for $\delta \geq \delta(\kappa)$. Hence by Fodor there is a stationary $S' \subset C'$ s.t. $\delta(\kappa) = \delta$ is constant for $\kappa \in S'$. Set: $h(\kappa) = \langle \pi^Q_{\delta\kappa} \tilde{\pi}_\kappa(\xi_i^\kappa) \mid i < \gamma \rangle$. Then $h(\kappa) \in {}^\gamma Q_\delta$. Hence there is stationary $S \subset S'$ s.t. h is constant on S. But then:

(5) $\pi^Q_{\kappa\kappa'} \tilde{\pi}_\kappa(\xi_i^\kappa) = \tilde{\pi}_{\kappa'}(\xi_i^{\kappa'})$ for $i < \gamma$; $\kappa, \kappa' \in S$, $\kappa \leq \kappa'$.

(6) $\tilde{\pi}_\kappa \upharpoonright \kappa = \tilde{\pi}_{\kappa'} \upharpoonright \kappa$ for $\kappa, \kappa' \in S$, $\kappa \leq \kappa$

proof
$\tilde{\pi}_\kappa \upharpoonright \kappa = \pi_{0\kappa} \upharpoonright \kappa = \pi_{\kappa\kappa'} \pi_{0\kappa} \upharpoonright \kappa = \pi_{0\kappa'} \upharpoonright \kappa = \tilde{\pi}_{\kappa'} \upharpoonright$

Hence:

(7) $\mathrm{rng}(\pi_{\kappa\kappa'}\tilde{\pi}_\kappa) \subset \mathrm{rng}(\tilde{\pi}_{\kappa'})$ for $\kappa, \kappa' \in S$, $\kappa \leq \kappa'$.

prf.

Let $x \in \tilde{K}_\kappa$. Then $x \in J^E_{\xi_i \kappa}$ for some $i < \gamma$ where $E = E^{K^i}$. Let $f = f^\kappa_i =_{\mathrm{df}}$ the $\tilde{K}_\kappa$-least $f : \kappa \xrightarrow{\text{onto}} J^E_{\xi^\kappa_i}$. Then $x = f(\nu)$ for a $\nu < \kappa$. We have:

$\tilde{\pi}_\kappa(f) =$ the K^*_κ-least $f : \kappa \xrightarrow{\text{onto}} J^{E^{K^*_\kappa}}_{\tilde{\pi}_\kappa(\xi^\kappa_i)}$

$\pi^Q_{\kappa\kappa'}\tilde{\pi}(f) =$ the $K^*_{\kappa'}$-least $f : \kappa' \xrightarrow{\text{onto}} J^{E^{K^*_{\kappa'}}}_{\tilde{\pi}_{\kappa'}(\xi^{\kappa'}_i)}$

$= \tilde{\pi}_{\kappa'}(f^{\kappa'}_i)$.

Hence: $\pi^Q_{\kappa\kappa'}\tilde{\pi}_\kappa(x) = \pi^Q_{\kappa\kappa'}\tilde{\pi}_\kappa(f(\nu)) =$

$= \pi^Q_{\kappa\kappa'}(\tilde{\pi}_\kappa(f)(\tilde{\pi}_\kappa(\nu))) = \tilde{\pi}_{\kappa'}(f^{\kappa'}_i)(\tilde{\pi}_{\kappa'}(\nu)) =$

$= \tilde{\pi}_{\kappa'}(f^{\kappa'}_i(\nu)) \in \mathrm{rng}(\tilde{\pi}_{\kappa'})$.

(Note $\tilde{\pi}_\kappa(\nu) = \tilde{\pi}_{\kappa'}(\nu)$ by (6)). QED (7)

Now set: $\pi_{\kappa\kappa'} = \tilde{\pi}^{-1}_{\kappa'}\pi^Q_{\kappa\kappa'}\tilde{\pi}_\kappa$. Then $\pi_{\kappa\kappa'} : \tilde{K}_\kappa \xrightarrow[\Sigma_0]{} \tilde{K}_{\kappa'}$ cofinally. Moreover $\pi_{\kappa\kappa'} \upharpoonright \kappa = \mathrm{id}$ by (6) and $\pi_{\kappa\kappa'}(\kappa) = \kappa'$

QED (Lemma 5.1)

Now let $\tilde{K}, \langle \pi_\alpha \mid \alpha \in S \rangle = \lim_{\alpha \le \beta \text{ in } S} (\tilde{K}_\alpha, \pi_{\alpha\beta})$

Then $\tilde{K}$ extends K^c. Clearly $\mathrm{cf}(\alpha + K^c) = \gamma$ for all $\alpha \in S$ where w.l.o.g. $\gamma < \min S$. Hence $\mathrm{cf}(\mathrm{ht}(\tilde{K})) = \gamma < \theta$. Hence $\tilde{K} \in H_{\theta^+}$. Let $\theta \in H \prec H_{\theta^+}$ s.t. $\overline{\overline{H}} = \theta$, H is transitive, $\tilde{K}, \langle \tilde{\pi}_\alpha \mid \alpha \in S \rangle \in H$. Let $f : \theta \leftrightarrow H$ and set: $X_\alpha = f''\alpha$ for $\alpha \le \theta$. Set: $C =$ the set of $\alpha \le \theta$ s.t. $\tilde{K}, \langle \tilde{\pi}_\alpha \mid \alpha \in S \rangle \in X_\alpha \prec H$ and $V_\alpha = V_\theta \cap X_\alpha$. Then C is cub in θ

Pick $\kappa \in C \cap S$. Let $\sigma : N \overset{\sim}{\leftrightarrow} X_\kappa$. Set: $F^* = \sigma \upharpoonright \mathbb{P}(\kappa)$. Then F^* is an extender on N and $\sigma : N \longrightarrow_{F^*} H$. (Hence $V_\theta \subset H = \mathrm{Ult}(N, F^*)$.) Clearly $N \models ZFC^-$ and $V_\kappa \in N$.

It is easily seen that:

(8) $\sigma(\langle \pi_{\alpha\beta} \mid \alpha \le \beta < \kappa \text{ in } S \rangle) =$

$= \langle \pi_{\alpha\beta} \mid \alpha \le \beta \text{ in } S \rangle$.

Hence since $\tilde{K}_\kappa, \langle \pi_{\alpha\kappa} \mid \alpha \in S \cap \kappa \rangle =$

$=: \lim_{\alpha \le \beta < \kappa \text{ in } S} (\tilde{K}_\alpha, \pi_{\alpha\beta})$, we have:

(9) $\sigma(\pi_{\alpha\kappa}) = \pi_\alpha \quad (\alpha \in \kappa \cap S)$. Hence:

(10) $\sigma \upharpoonright \tilde{K}_\kappa = \pi_\kappa$, since $\sigma \pi_{\alpha\kappa}(x) = \pi_\alpha(x) = \pi_\kappa \pi_{\alpha\kappa}(x)$

Set: $F' = F^* \upharpoonright \tilde{K}_\kappa = \pi_\kappa \upharpoonright \tilde{K}_\kappa$. Then $\langle \tilde{K}, F' \rangle$ satisfies all premouse conditions except the initial segment condition. We shorten F' so as to satisfy this condition: Let $\lambda < \theta$ be least s.t. $\sigma(f)(\alpha) < \lambda$ whenever $f : \kappa \to \kappa$, $f \in \tilde{K}_\kappa$ and $\alpha < \lambda$. Set: $F = F' | \lambda$.

Let $\bar{\pi} : \tilde{K}_\kappa \to_F \bar{K}$. $\langle \bar{K}, F \rangle$ is easily seen to be a 1-small premouse

(11) $\bar{K}$ is an initial segment of $\tilde{K}$ (hence of K^c).

proof.

~~Let~~ $k : \bar{K} \longrightarrow \tilde{K}_\kappa$ be defined by $k(\bar{\pi}(f)(\alpha)) = \pi_\kappa(f)(\alpha)$. Then k is a cofinal Σ_0 preserving map and $k \upharpoonright \lambda = id$, $k(\lambda) = k\bar{\pi}(\kappa) = \pi_\kappa(\kappa) = \theta$.

Let $\lambda < \xi < ht(\bar{K})$ s.t. $\omega\rho^\omega_{J_\xi^{E^{\bar{K}}}} = \lambda$. (Such ξ are cofinal in $ht(\bar{K})$).

Set: $\bar{Q} = J_\xi^{E^{\bar{K}}}$, $Q = \pi_\kappa(\bar{Q}) = J^{E^{\tilde{K}}}_{\pi_\kappa(\xi)}$; $k' = k \upharpoonright \bar{Q}$. Then $\bar{Q}$, Q are sound

and $k' : \overline{Q} \longrightarrow_{\Sigma^*} Q$ where $\omega\rho^{\omega}_{\overline{Q}} = \mathrm{crit}(k')$.
We apply §8 Lemma 4 of [NFS].
(a) is impossible since $k' \neq \mathrm{id}$ + Q is sound
(c) is impossible since $\omega\rho^{\omega}_{\overline{Q}} \geq \mathrm{crit}(k')$.
Hence (b) holds – i.e. $\overline{Q}$ is a segment of Q, hence of $\tilde{K}$. QED(11)

We note that λ is a limit cardinal in $\tilde{K}$ by the definition of λ. Hence:

(12) $J_{\lambda}^{E^{K^c}} = J_{\lambda}^{E^{\tilde{K}}}$, where λ is a limit cardinal in K^c.

(13) $\langle N, F^* \rangle$ is a certificate for $\langle \tilde{K}, F$
Moreover, if $f : \xi \to \mathrm{dom}(F)$, $\xi < \kappa$, then $f \in N$.

proof.
$\langle N, F^* \rangle$ is trivially a certificate.
Since $\tilde{K}_{\kappa} \in N$, there is $g \in N$ s.t. $g : \kappa \longleftrightarrow \tilde{K}_{\kappa}$. Set $\overline{f} = g^{-1} f$.
Then $\overline{f} \in V_{\kappa} \subset N$, since κ is regular.
Hence $f = g\overline{f} \in N$. QED(13)

By §11 Lemma 2.2, 2.4 it follows from (13) that for $\delta = \delta(\lambda) = \sup\{\xi \mid \tilde{\mu}_\xi < \lambda\}$ we have:

(14) $M_\delta = N_\delta = \langle J_\lambda^{E^{\bar{K}}}, \emptyset \rangle$ and $\tilde{\kappa}_\delta = \lambda$.

Now let $\bar{K} = J_\alpha^{E^{\bar{K}}} = J_\alpha^{E^{K^c}}$. Since λ is the largest cardinal in $\bar{K}$, α is cardinally absolute in K^c. By §11 Lemma 2.2 we conclude that for $\delta' = \delta(\alpha)$ we have:

(15) $M_{\delta'} = N_{\delta'} = \langle \bar{K}, \emptyset \rangle$.

Hence by (13) $N_{\delta'+1} = \langle \bar{K}, F \rangle$. Hence $\omega\rho^\omega_{N_{\delta'+1}} < \lambda$. This is a contradiction, since $\lambda = \tilde{\kappa}_\delta = \min\{\omega\rho^\omega_{N_\xi} \mid \xi \geq \delta\}$. Contr!

QED (Lemma 5)

<u>Corollary 6.1</u> Let $Q \in V_\theta$ be a normally iterable 1-small premouse in V_θ. Then Q is iterable in V_θ.

proof of Corollary 6.1

Coiterate Q, K^c to Q', K'. Since K^c is preSolid, the usual proofs tell us that one side of the iteration must be simple and that the simple side must be a segment of the nonsimple side, if this occurs. K' cannot be a proper segment of Q', since the K-side would then have to be non-simple and K' would be unsound. Thus Q' is a segment of K'. Hence Q' is a mouse. Hence Q is a mouse, since $\pi : Q \longrightarrow_{\Sigma^*} Q'$, where π is the iteration map. QED(6.1)

Corollary 6.2 If Q is a $\overset{1\text{-small}}{\text{MS}}$ mouse in V_θ, then Q is a mouse in V_θ.

We now prove a refinement of Corollary 6.1.

<u>Thm 7</u> Let Q be a countable 1-small premouse which is countably normally iterable. Then Q is iterable in V_Θ.

proof. Suppose not.

Coiterate Q, K^c, getting $\langle \mathcal{T}^Q, \mathcal{T}^K \rangle$ of length λ. On the K-side we employ a strategy definable in V_Θ. Hence there must be a failure of well foundedness on the Q-side. (Since otherwise Q coiterates out to a segment of a mouse and is therefore a mouse in K.) Let:

$$\mathcal{T}^Q = \langle \langle Q_i \rangle, \langle \nu_i \rangle, \langle \eta_i^Q \rangle, \langle \pi_{ij}^Q \rangle, T^Q \rangle$$

$$\mathcal{T}^K = \langle \langle K_i \rangle, \langle \nu_i \rangle, \langle \eta_i^K \rangle, \langle \pi_{ij}^K \rangle, T^K \rangle.$$

Since we have a strategy on the K-side, we suppose $\mathcal{T}^K$ has been continued to length $\lambda + 1$ if $\mathrm{Lim}(\lambda)$. On the Q-side we employ the "economical" strategy: At $\mathrm{Lim}(\gamma)$, choose $\{i \mid i\, T^Q \gamma\}$ if possible to be "economical" – i.e. s.t. there is no $\nu \geq \sup_{i<\gamma} \lambda_i$ with $E_\nu^{M_\gamma} \neq \emptyset$. This means, in particular, that if $\{i \mid i\, T^Q \gamma\}$ is non economical, then it is the unique cofinal well founded branch. Now let:

$\langle y^Q, y^R \rangle \in X \prec V_\theta$, $\overline{\overline{X}} = \omega$. Let $\sigma : \overline{V} \overset{\sim}{\longleftrightarrow} X$, where $\overline{V}$ is transitive. Let $\sigma(\langle y^{\overline{Q}}, y^{\overline{R}} \rangle) = \langle y^Q, y^R \rangle$. Let:

$y^{\overline{Q}} = \langle \langle \overline{Q}_i \rangle, \langle \overline{\nu}_i \rangle, \ldots, T^{\overline{Q}} \rangle$.

$y^{\overline{R}} = \langle \langle \overline{R}_i \rangle, \langle \overline{\nu}_i \rangle, \ldots, T^{\overline{R}} \rangle$. Let $\sigma(\overline{\lambda}) = \lambda$.

Note that $\overline{Q}_0 = Q$

$\langle y^{\overline{Q}}, y^{\overline{R}} \rangle$ is then a countable coiteration. If $\gamma < \overline{\lambda}$ is a limit ordinal, then we know that $\{i \mid i T^{\overline{Q}} \gamma\}$ is the unique cofinal well founded branch through $T^{\overline{Q}} | \gamma$ in the sense of $\overline{V}$, but not necessarily in V. Set: $\tilde{\lambda}$ = the least $\tilde{\lambda} \leq \overline{\lambda}$ s.t. for all $\gamma < \tilde{\lambda}$, $\{i \mid i T^{\overline{Q}} \gamma\}$ is the unique cofinal well founded branch in $T^{\overline{Q}} | \gamma$. Then

(1) $\mathrm{Lim}(\tilde{\lambda})$

pf. Suppose not. Let $\tilde{\lambda} = h+1$. Then $\tilde{\lambda} = \overline{\lambda}$ and $\overline{V}$ thinks that the coiteration cannot be continued. Since $\overline{\lambda}$ is countable and we have followed the unique iteration strategy for Q, and Q is countably normally iterable, then it can be continued. Hence $\overline{V}$ thinks so by an easy absoluteness argument. QED (1)

$T^{\bar{Q}}|\tilde{\lambda}$ must have a well founded cofinal branch **b**, since Q is countably normally iterable. Choose b s.t. the limit model Q_b has minimal height.

<u>Case 1</u> $ht(Q_b) \in \bar{V}$.

<u>Case 1.1</u> b is the unique well fnd cofinal branch. Then $\tilde{\lambda} = \bar{\lambda}$ and $\mathcal{J}^{Q}$ has no founded cofinal branch. Hence $\bar{V}$ thinks that $\mathcal{J}^{\bar{Q}}_{\bar{\lambda}}$ has no such branch. We derive a contradiction. Let $\bar{\lambda} < \delta \in \bar{V}$ s.t. $L_\delta[\mathcal{J}^{\bar{Q}}]$ is admissible and $ht(Q_b) < \delta$. Let $\xi = ht(Q_b)$. Let $\mathcal{L} = \mathcal{L}_{\delta,\xi} = \mathcal{L}_{\delta,\xi,\mathcal{J}^{\bar{Q}}}$ be the following infinitary language on $L_\delta[\mathcal{J}^{\bar{Q}}]$: Predicates: $\in, =$; Constants: $\underline{x}$ $(x \in L_\delta[\mathcal{J}^{\bar{Q}}])$, $\mathring{b}$.

<u>Axioms</u>: ZF^-; $\bigwedge v (v \in \underline{x} \leftrightarrow \bigvee_{z \in x} v = \underline{z})$ $(x \in L_\delta[\mathcal{J}^{\bar{Q}}])$; $\mathring{b}$ is a cofinal branch in $\mathcal{J}^{\bar{Q}}$; $\mathring{b}$ has a transitive limit model $Q_{\mathring{b}}$ s.t. $ht(Q_{\mathring{b}}) = \underline{\xi}$.

Clearly $\mathcal{L}$ is consistent. Every model $\mathfrak{A}$ of $\mathcal{L}$ is isomorphic to one which is good in the sense that its well founded core is a transitive $\in$-structure. Hence we may work only with good models. We then have $\underline{x}^{\mathfrak{A}} = x$ for $x \in L_\delta[\mathcal{J}^{\bar{Q}}]$. But then $\mathring{b}^{\mathfrak{A}}$ is really a well founded cofinal branch in $\mathcal{J}^{\bar{Q}}$; hence $\mathring{b}^{\mathfrak{A}} = b$. By the completeness theorem for countable admissibles it follows that:

$\nu \in b \leftrightarrow \mathcal{L} \models \underline{\nu} \in \mathring{b}$. Hence $b \in \bar{V}$. Contr! QED (Case 1.1)

<u>Case 1.2</u> $\mathcal{J}^{\bar{Q}} | \tilde{\lambda}$ has another well founded cofinal branch.

By §6 and the minimality of $ht(Q_b)$, Q_b has the form $J_\alpha^{E^N}$, where $N = J_\delta^E$, $\delta = \sup_{i<\tilde{\lambda}} \nu_i^{Q_i}$ and $N = \bigcup_{i<\tilde{\lambda}} J_{\lambda_i}^{E^{Q_i}}$. Moreover δ is Woodin in Q_b if $\alpha > \delta$.

<u>Claim 1</u> $\bar{K}_{\tilde{\lambda}}$ is not a proper segment of Q_b

Suppose not. Then there is no truncation on the main branch to $\bar{K}_{\tilde{\lambda}}$, since otherwise $\bar{K}_{\tilde\lambda}$ would be unsound. Hence $ht(\bar{K}_{\tilde\lambda}) = On \cap \bar{V} > ht(Q_b)$. Contr!

<u>Claim 2</u> Q_b is a segment of $\bar{K}_{\tilde\lambda}$.

Suppose not. There is ν s.t. $\delta < \nu \le \alpha$ and $E_\nu^{\bar{K}_{\tilde\lambda}} \neq \emptyset$, where δ is sound in $J_\alpha^{E^{\bar{K}_{\tilde\lambda}}}$. It follows easily that $\bar{K}_{\tilde\lambda}$ is not 1-small. Contr! QED (Claim 2).

But then Q_b is a mouse in $\bar{V}_\sigma$! Q_b is a simple iterate of Q, since otherwise Q_b could not be a proper segment of $\bar{K}_{\tilde\lambda}$. Hence $\bar{K}_{\tilde\lambda}$ is not a simple iterate of $\bar{K}$. Contradiction! Hence $\pi^Q_{cb} : Q \xrightarrow{\Sigma^*} Q_b$ and Q is a mouse in $\bar{V}$. Contr!

QED (Case 1.2)

<u>Case 2</u> Case 1 fails. (Hence $\tilde\lambda = \bar\lambda$)

We repeat an argument of Woodin. Let Γ = the set of $\gamma \in \bar{V}$ which are admissible in $y^{\bar{Q}}$. For $\gamma \in \Gamma$ let $\mathcal{L}_\gamma = \mathcal{L}_{\gamma, y^{\bar{Q}}}$ be the following theory in the infinitary language of $L_\gamma[y^{\bar{Q}}]$:

Predicates: $\in, =$. Constants $\underline{x}$ ($x \in L_\gamma[\mathcal{T}^{\bar{Q}}]$) and $\mathring{b}$. Axioms ZF^-, $\bigwedge v (v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$ ($x \in L_\gamma[\mathcal{T}^{\bar{Q}}]$), $\mathring{b}$ is a cofinal well founded branch through $\mathcal{T}^{\bar{Q}}$, $\underline{\xi} \in Q_{\mathring{b}}$ ($\xi < \gamma$).

Clearly $\mathcal{L}_\gamma$ is consistent. By Villé's Lemma $\mathcal{L}_\gamma$ has a good model $\mathfrak{A}$ whose well founded core has rank exactly γ. (As above, good means that the well founded core is a transitive $\in$-model.) Let $b_\gamma = \mathring{b}^{\mathfrak{A}}$. Then b_γ is a cofinal branch through $\mathcal{T}^{\bar{Q}}$ and $\tilde{Q}_\gamma = (Q_{\mathring{b}})^{\mathfrak{A}}$ is a good limit model whose well founded core has rank exactly γ. But then $b_\gamma \neq b_{\gamma'}$ for $\gamma, \gamma' \in \Gamma$, $\gamma < \gamma'$. Repeating some arguments from §6 we see that if $\delta = \sup_{i < \bar{\lambda}} \bar{\nu}_i$ and $N = J_\delta^E = \bigcup_{i < \bar{\lambda}} J_{\bar{\nu}_i}^{E^{\bar{Q}_i}}$, then δ is Woodin in J_γ^E. But this holds for arbitrarily large $\gamma \in \Gamma$. Hence δ is Woodin in $L_{\bar{\theta}}^E$, where $\bar{\theta} = On \cap \bar{V}$.

Let $\sigma(E) = E'$, $\sigma(\delta) = \delta'$. Then δ' is Woodin in $L^{E'}_{\theta}$, hence in $L^{E'}$, since E' is inaccessible. Hence by A1, V_θ is closed under #. Hence $\bar{V}$ is closed under #.

Now let $ht(Q_b) = \xi$ and let $\gamma > \xi$ be admissible in $y\bar{Q}$. Let $\mathcal{L} = \mathcal{L}_{\gamma, \xi, y\bar{Q}}$ be as in Case 1.1. Then $\mathcal{L}$ is consistent. If τ is the first indiscernable for $L[y\bar{Q}]$ given by $(y\bar{Q})^{\#}$, then $L_\tau[y\bar{Q}] \prec L[y\bar{Q}]$. Hence there are $\bar{\gamma}, \bar{\xi} < \tau$ s.t. $\bar{\mathcal{L}} = \mathcal{L}_{\bar{\gamma}, \bar{\xi}, y\bar{Q}}$ is consistent. Let $\mathfrak{A}$ be a good model of $\bar{\mathcal{L}}$ and let $\bar{b} = b^{\mathfrak{A}}$. Then $\bar{b}$ is a cofinal well founded branch through $y\bar{Q}$ and $ht(Q_{\bar{b}}) = \bar{\xi} < \bar{\theta}$. Contradiction!

QED (Theorem 7)

We recall that M was called a <u>weak mouse</u> iff whenever $\sigma : Q \xrightarrow[\Sigma^*]{} M$ and Q is countable, then Q is countably iterable. Call M a <u>very weak mouse</u> iff every such Q is countably normally iterable.

Then:

<u>Corollary 7.1</u> Every 1-small very weak mouse is a weak mouse in V_θ.

Hence:

<u>Corollary 7.2</u> Every 1-small weak MS-mouse is a weak mouse.

We believe that these facts should be provable without A1–A3, but don't know how.

The condition A3 was imposed only to ensure enough inaccessible cardinals to verify the background conditions for the construction of K^c. As mentioned in §2 of this addendum, we can get by with a weaker background condition if we are willing to work with MS-mice. Thus it would seem that we can prove Thm 5 for the K^c of MS-mice without use of A3. However, we also made use of A3 in proving Lemma 5.1. It would be natural to prove Lemma 5.1 replacing (i) by (i') $\alpha = \overline{\overline{V_\alpha}}$ and $cf(\alpha) > \omega$. The rest of the proof would then be essentially as before. However I don't see how to prove even this version of Lemma 5.1 without adopting some new assumption in place of A3. The following will do

A4 A2 holds for $M \subset V_\theta$ s.t. M is a 1-small premouse.

(In particular, this will hold for $M = K^c$)

The details are left to the reader.
We then get:

<u>Thm 8</u> Assume A1, A2, A4. Let K^c be the K^c of MS-mice. Then K^c is universal wrt. 1-small premice in V_θ.

We would expect this to yield a version of Corollary 6.1 for MS-mice. We must, however, be careful in formulating. If $Q \in V_\theta$ is a normally iterable 1-small premouse in V_θ, then Thm 8 shows that Q normally iterates up to a normal iterate Q' of a sound MS-mouse $\tilde{Q}$. Let $\pi : Q \longrightarrow Q'$ be the iteration map. Q' itself may not be MS-iterable and hence the existence of π does not entitle us to assert that Q is MS-iterable. It can be shown, however, that core(Q') is an MS-mouse. Suppose now that Q' is a solid core mouse. Then $\pi(p_Q) = p_{Q'}$ and π induces a map

$\bar{\pi} : Q \xrightarrow[\Sigma^*]{} \mathrm{core}(Q')$. Hence Q is MS-iterable. Thus we get:

<u>Cor 8.1</u> Assume A1, A2, A4. Let $Q \in V_\theta$ be a normally iterable 1-small solid core mouse in V_θ. Then Q is an MS-mouse in V_θ.

Similarly, modifying the proof of Thm 7

<u>Thm 9</u> Assume A1, A2, A4. Let Q be a countable 1-small solid core mouse which is countably normally iterable. Then Q is an MS-mouse in V_θ.

Hence:

<u>Thm 10</u> Assume A1, A2, A4. Let Q be a very weak 1-small solid core mouse. Then Q is a weak MS-mouse.