Addendum to Robust Extenders

We present a simplified version of the methods needed for a proof of [RE]* §1 Theorem 1, which says that Steel's main realizability lemma for arrays with "background certification" also holds for arrays in which the extenders satisfy the weaker condition of robustness. As in [RE] we prove the theorem in ZFC, whereas Steel worked in V_Ω for an inaccessible Ω. Our new proof is technically simpler. In particular, we dispense with Steel's functions $c(i,j)$.

§1 Basic definitions and results

We recall the definition of robustness:

Def Let $N = \langle J_\nu^E, F\rangle$ be an active premouse. F is robust wrt. N iff whenever $U \subset \lambda = \lambda_\nu$ and $W \subset \mathbb{P}(\kappa) \cap N$ $(\kappa = \kappa_\nu)$ are countable, then there is $g: U \to \kappa$ s.t.

(a) $\langle g(\vec{\alpha})\rangle \in X \longleftrightarrow \langle\vec{\alpha}\rangle \in F(X)$ whenever $\alpha_1, \ldots, \alpha_n \in U$ and $X \in W$. (Here $\langle\ \rangle$ is Gödel's tuple function on ordinals.)

(b) Let $v_1, \ldots, v_n \subset U$, $\tau = \operatorname{lub} U$, $\bar{\tau} = \operatorname{lub} g''U$. Let φ be a Σ_1 fmla. Then

$C^E_{\bar{\tau},\kappa} \models \varphi[g''v_1, \ldots, g''v_n] \longleftrightarrow C^E_{\tau,\infty} \models \varphi[v_1, \ldots, v_n]$.

(Here $C^E_{\tau,\alpha}$ is the Chang hierarchy up to α over $\langle L_\tau[E], E \cap L_\tau[E]\rangle$. The precise definition is given in [RE] §1.)

* [RE] = "Robust Extenders"

(We recall the condensation principle;
Let $C \prec_{\Sigma_1} C^E_{\gamma,\eta}$ s.t. $\gamma \subset C$ and $[C]^\omega \subset C$.
Then C is isomorphic to a $C^E_{\gamma,\bar\eta}$ for an $\bar\eta \leq \eta$.
From this it follows that if $\alpha^\omega < \kappa$ for all $\alpha < \kappa$, then $C^E_{\gamma,\kappa} \prec_{\Sigma_1} C^E_{\gamma,\infty}$ for all $\gamma < \kappa$. This holds in particular for $\kappa = (2^\beta)^+$.)

As in [RE] § 2 we also define:

Def Let $N = \langle J^E_\nu, F\rangle$ be as above. Let $\kappa = \kappa_\nu$, $\lambda = \lambda_\nu$ be as above. Let $\kappa \leq \gamma \leq \lambda$. F is robust up to γ in N iff for every pair of countable sets $U \subset \lambda$, $W \subset \mathbb{P}(\kappa) \cap N$ there is a $g : U \to \kappa$ s.t. (a) holds as above and:

(b') Let $c = \operatorname{lub}(U \cap \gamma)$, $\bar c = \operatorname{lub} g''(U \cap \gamma)$. Then for all $v_1, \ldots, v_n \subset U \cap \gamma$ and all Σ_1 fmlae φ:
$$C^E_{\bar c,\kappa} \models \varphi[g''v_1, \ldots, g''v_n] \longleftrightarrow C^E_{c,\infty} \models \varphi[v_1, \ldots, v_n].$$

Def A premouse N is robust iff whenever $N \| \nu = \langle J^E_\nu, F\rangle$ is active and $\gamma \in (\kappa_F, \lambda_F]$ is a cardinal in N, then F is robust up to γ in $N\|\nu$.

As we remarked in [RE], if $\langle N_i \mid i \leq \theta\rangle$ $(\theta \leq \infty)$ is an array in which all extenders added are robust, then each N_i is a robust premouse.

We again prove only a special case of the theorem, which reduces to:

<u>Main Claim</u> Let N be a mouselike, robust premouse satisfying ZFC^-. Let $\mathcal{J} = \langle\langle P_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a countable normal putative*) iteration of a countable P_0 without truncations, where $\sigma : P_0 \prec N$. Then one of the following holds:

(a) $lh(\mathcal{J}) = h+1$ and there is $\sigma' : P_i \prec N$ s.t. $\sigma'\pi_{0h} = \sigma$.

(b) $\mathcal{J}$ has a maximal branch b, which is of limit length, and there is $\sigma' : P_b \prec N$ s.t. $\sigma'\pi_b = \sigma$.

. . . .

Before beginning the proof we reintroduce the notions of "world" and "enlargement". For technical reasons, some definitions will differ inessentially from the earlier ones. The normal iteration $\mathcal{J}$ is fixed from now on. Let ZFC^* be the theory

$ZFC^- + \bigwedge\alpha\ [\alpha]^\omega \in V$.

As before we define:

<u>Def</u> A <u>world</u> is a transitive structure W s.t. $W \models ZFC^*$ and $[\tau]^\omega \cap W = [\tau]^\omega \cap V$ for $\tau = On \cap W$.

*) "putative" means that if $lh(\mathcal{J}) = h+1$, then P_h does not need to be well founded.

(Note Since we shall often find our worlds in generic extensions of V, the clause $[\tau]^\omega \wedge W \subset V$ is not vacuous.]

Fix $A \subset On$ s.t. whenever $\beta > \omega$, $\beta = \overline{\overline{V_\beta}}$, then $V_\beta = L_\beta[A]$ and $L_{\beta^+}[A] \models (\beta$ is the largest cardinal)

Clearly $L_{\beta^+}[A]$ is a world whenever $\beta = \overline{\overline{V_\beta}}$ and $cf(\beta) > \omega$.

Def By a standard world we mean a $W = \langle L_{\beta^+}[A], \in, A \cap \beta^+, N, P_1, \ldots, P_m \rangle$ s.t. $P_1, \ldots, P_m \in W$ and $N \in L_\beta[A]$ is a robust mouselike premouse, and $cf(\beta) > \omega$.

(Hence $(C^e_{\tau,\infty})_W = C^e_{\tau,\beta^+} \prec_{\Sigma_1} C^e_{\tau,\infty}$ for $e, \tau \in W$.)

Clearly $\mathcal{Y} \subset H_{\omega_1} \subset W$ for any world W. For standard worlds we shall assume, in addition, that $\mathcal{Y}$ is W-definable (e.g. $\mathcal{Y} = P_1$).

[Note We generally write: $W = \langle L_{\beta^+}[A], N, \vec{P} \rangle$ or $W = \langle |W|, N, \vec{P} \rangle$.]

If $W = \langle L_{\beta^+}[A], N, \vec{P} \rangle$ is a standard world, we define for each $\mu < \infty$ the μ-th enhancement:

$W^{(\mu)} = \langle L_{\beta_\mu^+}[A], N, \vec{p}, \beta \rangle$, where

β_i = the i-th $\beta' \geq \beta$ s.t. $\beta = \overline{\overline{V}}_\beta$ and $cf(\beta) > \omega$.

(Hence $W^{(0)} = \langle W, \beta \rangle$.)

Note that $\langle \beta_i \mid i \leq \mu \rangle$ is uniformly $W^{(\mu)}$ - definable.

We then define a good world to be an arbitrary world which has the salient features of a standard world:

Def $\tilde{W} = \langle L_\alpha[\tilde{A}], \in, \tilde{A}, \tilde{N}, p_1 \ldots p_n \rangle$ is a good world iff $\tilde{W}$ is a world in which the following hold:

(a) If $\beta = \overline{\overline{V}}_\beta$, then $V_\beta = L_\beta[\tilde{A}]$. Moreover, if β^+ exists, then β is the largest cardinal in $L_{\beta^+}[\tilde{A}]$

(b) There is a largest cardinal β. Moreover, $\beta = \overline{\overline{V}}_\beta$ and $cf(\beta) > \omega$.

(c) $\tilde{N} \in V_\beta$ is a robust, mouse like premouse.

If $\tilde{W}$ is a good world, we denote its largest cardinal by $\beta_{\tilde{W}}$.

(Note If we were doing the full proof for arrays, then an array would take the place of N.)

Def Let $\tilde{W} = \langle L_\alpha[\tilde{A}], \tilde{N}, \vec{P} \rangle$ be a good world. By an <u>enhancement</u> of $\tilde{W}$ we mean a good world
$$W' = \langle L_{\alpha'}, [A'], \tilde{N}, \vec{P}, \beta \rangle$$
s.t. $\beta = \beta_{\tilde{W}}$, $\tilde{A} = A' \cap \alpha$, $\alpha = \beta + W'$ and $\nabla_\beta^{\tilde{W}} = \nabla_\beta^{W'}$. If W' is an enhancement of $\tilde{W}$, then in W' we can define a sequence $\langle \beta_i \mid i \leq \mu \rangle$ by:

$\beta_i \simeq$ the least $\beta' \geq \beta$ s.t. $\beta' > \beta_h$ for $h < i$, $\beta' = \bar{\nabla}_{\beta'}$, and $\mathrm{cf}(\beta') > \omega$.

Then $\beta_\mu^{W'} = \beta_{W'}$. By $\tilde{W}^{(i)}$ $(i \leq \mu)$ we denote $\langle L_{\beta_i^+}[A'], \tilde{N}, \vec{P}, \beta \rangle$, where β_i^+ denotes $(\beta_i^+)^{W'}$.
We denote β_i by $\beta_i^{W'}$ and μ by $\mu^{W'}$.
Clearly each $\tilde{W}^{(i)}$ is a good world.

We now define the notion of <u>enlargement</u>. We shall demand less of enlargements than we did in [RE].

Def Let $\eta \le lh(\mathcal{I})$. $\mathbb{E} = \langle\langle W_i, \sigma_i\rangle \mid i<\eta\rangle$ is an enlargement of $\mathcal{I}|\eta$ iff

(a) $W_i = \langle L_{\alpha_i}[A_i], N, \vec{P}\rangle$ is a good world

(b) $\sigma_i \in V$ s.t. $\sigma : P_i \prec N_i$

(Hence $\sigma \in W_i$ since $\sigma \subset V \cap W_i$ is countable.)

(c) Set: γ_i = the largest $\gamma \le \lambda_i$ which is a cardinal in P_i (hence $\gamma_i \le \lambda_i \le \gamma_j$ for $i<j<\eta$). Set: $c_i = \sup \sigma_i''\gamma_i$, $E_i = E^{N_i}$. Then

$\sigma_i \upharpoonright \gamma_h = \sigma_h \upharpoonright \gamma_h$, $J^{E_i}_{c_h} = J^{E_h}_{c_h}$ for $h<i$.

(d) Set: $\delta_i = \sigma_i \upharpoonright \gamma_i$. Then $\delta = \langle \delta_i \mid i<\eta\rangle \in V$.

(Hence $\langle \delta_h \mid h \le i\rangle \in W_i$.)

(e) Set $t_i = th(\langle W_i, \sigma_i, \langle t_h \mid h<i\rangle\rangle)$

(where $th(\mathfrak{A})$ is the complete theory of $\mathfrak{A}$). Then $t = \langle t_i \mid i<\eta\rangle \in V$. (Hence $t \in W_i$ since $t \subset H_{\omega_1}$ is countable.)

Note The same definition gives the notion "enlargement of a phalanx", since we have not used the fact that $\mathcal{I}$ is an iteration but merely that $P_i|\lambda_h = P_h|\lambda_h$ and λ_h is a cardinal in P_i for $i<h$.

Def $\langle t, \delta\rangle$ is called the trace of $\mathbb{E}$. We refer to t as the first trace and δ as the second trace.

We write $\langle t, \delta\rangle = tr(\mathbb{E})$.

Def $s = \langle t, \delta \rangle$ is a trace for $\gamma | \eta$ iff there is a s.t. $\mathrm{coll}(\omega, \alpha) \Vdash \bigvee \mathbb{E}\ s = \mathrm{tr}(\mathbb{E})$.

Note that the trace of an enlargement $\mathbb{E}$ is always in V, even if $\mathbb{E}$ is not.

Def Let $N = \langle J^E_\nu, F \rangle$ be a premouse. $s = \langle t, \delta \rangle$ is an N-conforming trace for $\gamma | \eta$ iff there is an s.t.
$\mathrm{coll}(\omega, \alpha) \Vdash \bigvee \mathbb{E} (s = \mathrm{tr}(\mathbb{E})\ \wedge$
$\wedge\ J^{E_i}_{c_i} = J^{E}_{c_i}$ for $i < \eta)$.

Def Let N be as above and let $\sigma : P_\eta \prec N$. $s = \langle t, \delta \rangle$ is an N, σ-conforming trace iff s is an N-conforming trace and $\delta_i = \sigma \upharpoonright \gamma_i$ for $i < \eta$.

(Note This definition makes sense for an arbitrary $\sigma : \sup_{i<\eta} \gamma_i \to On$.)

A very basic lemma is the following:

Lemma 1 There is a Σ_1 fmla φ s.t. whenever $N = \langle J_\alpha^E, F \rangle$ is a premouse, then s is an N-conforming trace for $\mathcal{Y}|\eta$ iff the following hold:

(a) $s = \langle t, \delta \rangle$ where $\delta = \langle \delta_i \mid i < \eta \rangle$ and
$\delta_i : \gamma_i \to On_N$ for $i < \eta$

(b) $C^E_{c,\infty} \models \varphi[s, \mathcal{Y}|\eta]$, where
$c = \sup_{i<\eta} c_i$, $c_i = \sup \delta_i{}''\gamma_i$.

proof.
(a) is obviously a necessary condition. Now let (a) hold. For appropriate φ we show:

Claim s is an N-conforming trace $\leftrightarrow$ (b).

The condition φ says:

There exist β, α s.t.

(a) $c < \beta < \alpha$, $s = \langle t, \delta \rangle \in C_{c,\beta}$, $C_{c,\alpha}$ is admissible

(b) T is consistent, where $T = T_{\beta,\alpha}(s)$ is the theory in $\mathcal{L}_{C_{c,\alpha}}$ with:

Predicate: $\dot{\in}$

Constants: $\underline{x}$ $(x \in C_{c,\alpha})$, $\dot{\mathbb{E}}, \dot{\beta}, \dot{A}, \dot{N}, \dot{\sigma}, \dot{W}, \dot{p}$

Axioms

(1) ZFC^-, $\bigwedge v (v \in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v = \underline{z})$ for $x \in C^E_{c,\alpha}$

(2) $\dot{\mathbb{E}} = \langle \langle \dot{W}_i, \dot{\sigma}_i \rangle \mid i < \eta \rangle$ where the following hold:

(i) Each of $\dot{W}, \dot{\beta}, \dot{A}, \dot{\sigma}, \dot{p}, \dot{N}$ maps $\underline{\eta}$ into

$\underline{C^{E}_{c,\beta}}$

(ii) $\dot{\beta}_{\underline{i}} < \underline{\beta}$, $\dot{A}_{\underline{i}} \subset \dot{\beta}_i$, $\dot{\sigma}_{\underline{i}} : \underline{\dot{p}_i} \prec \dot{N}_i$,

$\dot{W}_{\underline{i}} = \langle L_{\dot{\beta}_{\underline{i}}}[\dot{A}_i], \in, \dot{A}_{\underline{i}}, \dot{N}_{\underline{i}}, \dot{p}_{\underline{i}} \rangle$ $(i < \eta)$

(iii) $\dot{W}_{\underline{i}} \cap [\dot{\beta}_{\underline{i}}]^{\omega} = \underline{[\beta]^{\omega}} \cap \mathcal{P}(\dot{\beta}_i)$ $(i < \eta)$

(Note $[\beta]^{\omega} = \{x \mid C^{E}_{c,\beta+1} \models (x \subset \beta \wedge \bar{\bar{x}} \leq \omega)\}$; hence $\langle [\beta]^{\omega} \mid \beta < \alpha \rangle$ is $\Delta_1(C^{E}_{c,\alpha})$.)

(iv) $\dot{W}_{\underline{i}} \models (ZFC^{*} \wedge (a) \wedge (b) \wedge (c))$, where (a), (b), (c) are as in the definition of "good world". $(i < \eta)$

(v) $\underline{t_i} = th(\langle \dot{W}_i, \dot{\sigma}_i, \underline{t \restriction i} \rangle)$ $(i < \eta)$

(vi) $\underline{\dot{\sigma}_{i}} \restriction \underline{\gamma_h} = \underline{\delta_h}$; $J^{E \dot{N}_i}_{\underline{c_h}} = \underline{J^{E N}_{c_h}}$

$(h \leq i < \eta)$.

proof of Claim:

$(\rightarrow)$ Let $\mathbb{E} = \langle \mathbb{E}_i \mid i < \eta \rangle$ be an $\underbrace{\text{(N-conforming)}}$ enlargement of $\mathcal{I} \restriction \eta$ with traces s, where $\mathbb{E} = V[G]$ and G is coll(ω, γ)-generic for some γ. Let $\mathbb{E} \in V_{\beta}[G]$ and let $\alpha > \beta$ s.t. $C^{E}_{c,\alpha}$ is admissible. Let $\alpha \leq \mu$ s.t. $V_{\mu}[G] \models ZFC^{-}$. Then $\langle V_{\mu}[G], \mathbb{E} \rangle$ models $T_{\beta,\alpha}$. Hence $T_{\beta,\alpha}$ is consistent.

(⟸) Let $\mathfrak{A}$ model $T_{\beta,\alpha}$ (in $V[G]$, where G is coll(ω,α)-generic). We can assume w.l.o.g. that $\mathfrak{A}$ is solid (i.e. wfcore$(\mathfrak{A})$ is transitive and $\in_{\mathfrak{A}} \upharpoonright$ wfcore$(\mathfrak{A}) = \in \upharpoonright$ wfcore$(\mathfrak{A})$).
Then $\underline{x}^{\mathfrak{A}} = x$ for $x \in C_{c,\alpha}$. It follows easily that $E = \dot{E}^{\mathfrak{A}}$ is an enlargement with trace s. QED (Lemma 1)

An easy modification of the proof yields:

Cor 1.1 "s is a trace" is uniformly $\Sigma_1(C_{0,\infty})$-definable.

The proof of Lemma 1 can be carried out in any good world. Hence:

Cor 1.2 Let $W = \langle L_\alpha[A], N, p\rangle$ be a good world. Let $s \in W$. Then $W \models$ "s is an N-conforming trace" holds iff (a) holds and
(b) $(C^E_{c,\infty})_W \models \varphi[s, y|\eta]$,
where $c = \sup_{i<\eta} \delta_i{}''\gamma_i$, $s = \langle t, \delta\rangle$.

Hence:

Cor 1.3 "s is an N-conforming trace" is absolute in standard worlds W, since $(C^E_{c,\infty})_W \prec C^E_{c,\infty}$ for $c \in W$.

As a corollary of the proof we also get:

<u>Cor 1.4</u>. Let $N^* = \langle J_\gamma^{E^*}, F^* \rangle$ be a premouse. Let $\mathbb{E}$ be an N^*-conforming enlargement of $\mathcal{Y}|\eta$. Let $W = \langle |W|, N, p \rangle$ be a good world. Let $J_c^{E^N} = J_c^{E^*}$, where $c = \sup_{i<\eta} c_i$, $c_i = \sup \delta_i{}'' \gamma_i$. Suppose moreover that $rn(\mathbb{E}) < On \cap W$. Let $s = tr(\mathbb{E})$. Then $W \models s$ is an N-conforming trace.

prf.

Let $rn(\mathbb{E}) = \beta$. Let $\alpha > \beta$ be least s.t. $C_{c,\alpha}^{E^N}$ is admissible. Then $\alpha \in W$. But then the theory $T_{\alpha,\beta}(s)$ in the proof of Lemma 1 has a model. Hence $T_{\alpha,\beta}(s)$ is consistent.

QED(1.4)

We shall confine ourselves largely to traces having the following property:

<u>Def</u> Let $\mathbb{E}$ be an enlargement of $\mathcal{Y}|\eta$. $\mathbb{E}$ is <u>neat</u> (or <u>self justifying</u>) iff, letting $s = tr(\mathbb{E})$, we have

$W_i \models s \upharpoonright i$ is an N_i-conforming trace

for $i < \eta$. (Here $s \upharpoonright i$ denotes $\langle t \upharpoonright i, \delta \upharpoonright i \rangle$, where $s = \langle t, \delta \rangle$.)

(<u>Note</u> It follows that $W_i \models s \upharpoonright i$ is an N_i, σ_i-conforming trace.)

Def a <u>neat trace</u> is the trace of a neat enlargement.

Note Let $s = \langle t, \delta \rangle = tr(\mathbb{E})$. Then $\mathbb{E}$ is neat iff t satisfies a syntactical condition of the form: $x_i \in t_i$ for $i < \eta$.

Hence:

Lemma 1.5 Let s be a trace for $\mathcal{Y}|\eta$.

(a) s is neat iff it satisfies the above syntactical condition

(b) If s is neat and $\mathbb{E}$ is any enlargement with trace s, then $\mathbb{E}$ is neat.

We call a good world <u>reflective</u> if it countenances the existence of smaller imitations of itself in the following sense:

Def Let W be a good world. W is <u>reflective</u> iff whenever, $e, \tau \in W$ and $t = th(\langle W, e\rangle)$, then $t \in W$ and the following holds in W:
For sufficiently large γ it is forced by $coll(\omega, \gamma)$ that there is a good world $\bar{W}$ s.t. $\tau \in \bar{W}$ and for some $\bar{e} \in \bar{W}$:
$t = th(\langle \bar{W}, \bar{e}\rangle)$ and $J_\tau^{\bar{e}} = J_\tau^{e}$.

The method of proof used in Lemma 1 yields:

<u>Lemma 2</u> There is a Σ_1 fmla ψ s.t. for all good worlds W:
W is reflective iff whenever $e, \tau \in W$ and $t = tr(\langle W, e\rangle)$, then $t \in W$ and $(C^e_{\tau,\infty})_W \models \psi[t]$.

proof.
ψ says that there exist β, α s.t. $\tau < \beta < \alpha$, $C^e_{\tau,\alpha}$ is admissible, and $T^* = T^*_{e,\beta,\alpha}(t)$ is consistent, where T^* is the following theory in the infinitary language of $C^e_{\tau,\alpha}$:

Predicate: $\in$

Constants $\underline{x}$ $(x \in C_{\tau,\alpha})$; $\dot{W}, \dot{e}, \dot{A}, \dot{N}, \dot{p}$

Axioms

(1) ZFC^-, $\bigwedge v (v \in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v = \underline{z})$ for $x \in C_{\tau,\alpha}$

(2) $\dot{W} = \langle L_\beta[\dot{A}], \in, \dot{A}, \dot{N}, \dot{p} \rangle$, where the following hold:

(i) $\dot{A} \subset \underline{\beta}$ and $\dot{W} \models (ZFC^* + (a) \wedge (b) \wedge (c))$, where (a), (b), (c) as in the definition of "good world"

(ii) $\dot{e} \in \dot{W} \wedge \underline{t} = th(\langle W, \dot{e} \rangle)$

(iii) $J_{\underline{\tau}}^{\dot{e}} = \underline{J_\tau^e}$

(iv) $\dot{W} \cap [\underline{\beta}]^\omega = \underline{[\beta]^\omega}$

(where $[\beta]^\omega = \{x \mid C^e_{\tau,\beta+1} \models (x \subset \beta \wedge \bar{\bar{x}} \leq \omega)\}$ is $C^e_{\tau,\alpha}$ - definable.)

If W is reflective and $e, t \in W$, then clearly there is β s.t., letting α = the smallest $\alpha > \beta$ s.t. $C^e_{\tau,\alpha}$ is admissible, then T^* has a model in $W[G]$ for a $coll(\omega, \gamma)$ - generic G. Hence T^* is consistent. Hence $C^e_{\tau,\infty} \models \Psi[t]$ in W. Conversely, if T^* is consistent and G is $coll(\omega, \alpha)$ - - generic over W, then T^* has a solid model of σ in $W[G]$. Set $\bar{w} = \dot{w}^{\sigma}$, $\bar{e} = \dot{e}^{\sigma}$. Then $\bar{w}, \bar{e}$ have the desired properties. QED (Lemma 2)

<u>Def</u> A good world W is <u>enhanceable</u> iff it has a proper enhancement (i.e. there is a good world W' of the form $W^{(1)}$).

<u>Note</u> If $W = \bar{W}^{(\mu)}$, then $W^{(i)}$ is enhanceable for all $i < \mu$.

As a corollary of the above proof:

<u>Cor 2.1</u> If W is enhanceable, then it is reflective.

prf.

Let $W' = W^{(1)}$. Let $e, \tau \in W$. Let $\beta = On \cap W$ and let α = the smallest $\alpha > \beta$ s.t. $C^e_{\tau,\alpha}$ is admissible. Then $\alpha \in W'$. $T^* = T_{e,\beta,\alpha}(t)$ has the model $\langle W', W, e \rangle$, where W interprets $\dot{W}$ and e interprets $\dot{e}$ (where $t = \mathrm{th}(\langle W, e \rangle)$). Thus $(C^e_{\tau,\infty})_{W'} \models \varphi(t)$. But $(C^e_{\tau,\infty})_W \prec_{\Sigma_1} (C^e_{\tau,\infty})_{W'}$. Hence $(C^e_{\tau,\infty})_W \models \varphi(t)$. QED (2.1)

<u>Lemma 2.2</u> Let $\mathbb{E} = \langle\langle W_i, \sigma_i\rangle \mid i < \eta\rangle$ be a neat enlargement of $\mathcal{T}|\eta$. Let W_i be reflective. Let $s = tr(\mathbb{E})$. Then $W_i \models (s \restriction i+1)$ is an $N_i; \sigma_i$ - conforming trace.

Prf.

Let $s = \langle t, \delta\rangle$. We must show that, if G is $coll(\omega, \gamma)$ - generic over W_i for a sufficient γ, then $W_i[G]$ contains an N_i, σ_i - conforming enlargement of $\mathcal{T}|i+1$ with trace $s \restriction i+1$. Since s is neat, we can assume γ large enough that there is $\bar{\mathbb{E}} \in W_i[G]$ enlarging $\mathcal{T}|i$ with trace $s \restriction i$. Writing:

$(a, b, c) = a \times \{0\} \cup b \times \{1\} \cup c \times \{2\}$,

set: $e = (\sigma_i, E^{N_i}, t \restriction i)$. Clearly $t_i = th(\langle W_i, \sigma_i, t \restriction i\rangle)$ is recursive in $t^* = th(\langle W_i, e\rangle)$. (In fact, t_i is <u>uniformly</u> recursive in t^* in the sense that $t_i = F(t^*)$ and for <u>any</u> good world W and any $a, b, c \in W$ s.t. $b = N^W$, we have $th(\langle W, a, c\rangle) = F(th(\langle W, (a, b, c)\rangle))$.)

If we have chosen γ large enough, it follows by reflectivity that there is a good world $\bar{w} \in W_i[G]$ and a $\bar{e} \in \bar{w}$ s.t. $J_c^{\bar{e}} = J_c^{e}$ and $t^* = th(\langle \bar{w}, \bar{e} \rangle)$. But then, letting $\bar{N} = N^{\bar{w}}$, $\bar{E} = E^{\bar{N}}$, we have: $\bar{e} = (\bar{\sigma}, \bar{E}, t \restriction i)$, where $\bar{\sigma} : P_i \prec \bar{N}$, $J_c^{\bar{E}} = J_c^{E_i}$, and $\bar{\sigma} \restriction \gamma_i = \sigma_i \restriction \gamma_i$. Moreover $t_i = th(\langle \bar{w}, \bar{\sigma}, t \restriction i \rangle) = F(t^*)$. But then $\bar{\mathbb{E}} {}^\frown \langle \bar{w}, \bar{\sigma} \rangle \in W[G]$ is then an enlargement of $\mathbb{Y}|i+1$ with trace $s \restriction i+1$.

QED (Lemma 2.2)

Before proceeding further we adopt the following linguistic conventions: We often make a statement of the form: "There is an enlargement with property A". This means that there is such an enlargement in $V[G]$ if G is $coll(\omega, \gamma)$-generic over V for a sufficient γ. If we say: "Let $\mathbb{E}$ be an enlargement with property A. Then there is an enlargement $\mathbb{F}$ with property B", it is understood that we are working in such a $V[G]$ and are asserting that an $\mathbb{F}$ with property B exists in $V[G][G']$, where G' is coll(ω, γ) generic over $V[G]$ for a sufficiently large γ. (In fact, statements of this sort will generally turn out to be equivalent to a Σ_1 statement in $C^{e}_{\bar{\tau},\infty}$ for appropriate $e, \bar{\tau}$.) We recall that the ground model V is fixed throughout the discussion and was used in defining the notions "world" and "enlargement".

. . . .

A major tool in proving iterability will be the following "superlemma". We first define:

<u>Def</u> An enlargement $\mathbb{E}$ of $\mathbb{J}|i+1$ is <u>powerful</u> iff $\mathbb{E}$ is neat and W_i is reflective. (where $\mathbb{E} = \langle\langle W_h, \sigma_h\rangle \mid h \le i\rangle$).

<u>Def</u> W' is a <u>segment</u> of W iff W is a good world satisfying: $W = \bar{W}^{(\mu)}$, where $\bar{W}$ is a good world. and $W' = \bar{W}^{(i)}$ in W for some $i \le \mu$.

<u>Lemma 3</u> (Superlemma)
Let $\mathbb{E} = \langle\langle W_i, \sigma_i\rangle \mid i \leq \eta\rangle$ be a powerful enlargement of $\mathcal{T}|\eta+1$ with trace $s = \langle t, \delta\rangle$. Let $\xi = T(\eta+1)$ in $\mathcal{T}$ and let $\overline{W}$ be a reflective segment of W_ξ.

(A) $\mathbb{E}\upharpoonright\xi$ extends to a powerful enlargement $\overline{\mathbb{E}} = \langle\langle \overline{W}_i, \overline{\sigma}_i\rangle \mid i \leq \eta+1\rangle$ of $\mathcal{T}|\eta+2$ with trace $\overline{s} = \langle \overline{t}, \overline{\delta}\rangle$ s.t.
(i) $\overline{\mathbb{E}}\upharpoonright\xi = \mathbb{E}\upharpoonright\xi$, (ii) $t = \overline{t}\upharpoonright\eta+1$,
(iii) $\overline{W}_{\eta+1} = \overline{W}$, (iv) $\overline{\sigma}_{\eta+1}\pi_{\xi,\eta+1} = \sigma_\xi$

(B) If $\overline{W}$ is a proper segment of W_ξ, then $\mathbb{E}$ extends to a powerful enlargement $\mathbb{E}' = \langle\langle W_i, \sigma_i\rangle \mid i \leq \eta+1\rangle$ of $\mathcal{T}|\eta+2$ with trace $s' = \langle t', \delta'\rangle$ s.t.
(i) $\mathbb{E} = \mathbb{E}'\upharpoonright\eta+1$, (ii) $On_{W'_{\eta+1}} < On_{W_\eta}$;
(iv) $t'_{\eta+1} = \overline{t}_{\eta+1}$, where $\overline{\mathbb{E}}, \overline{s} = \langle\overline{t}, \overline{\delta}\rangle$ are as in (A).

prf. of Superlemma.

We first prove (A)

Since N_η is robust in W_η, there is a $g : \lambda_\eta \longrightarrow \sigma_\eta(\kappa_\eta)$ s.t.

(*) $\langle \vec{\alpha} \rangle \in E^{P_\eta}_{\nu_\eta}(X) \longleftrightarrow \langle g(\vec{\alpha}) \rangle \in \sigma_\eta(X)$

for $\alpha_1, \ldots, \alpha_n < \lambda_\eta$, $X \in \mathcal{P}(\kappa_\eta) \cap P_\eta$

(**) Let $v_1, \ldots, v_n < \delta_\eta$, $c_\eta = \sup \sigma_\eta{}''\delta_\eta$, $c = \sup g''\delta_\eta$. (Hence $c < \sigma_\eta(\kappa_\eta)$.)

Then in W_η we have:

$C^{E_\eta}_{c_\eta, \infty} \models \varphi[\sigma_\eta{}''\vec{v}] \longleftrightarrow C^{E_\eta}_{c, \delta_\eta(\kappa_\eta)} \models \varphi[g''\vec{v}]$

for all Σ_1 fmlae φ.

Then:

(1) $g \upharpoonright \kappa_\eta = \sigma_\eta \upharpoonright \kappa_\eta$

(Apply (*) with $X = \{\xi\}$ for $\xi < \kappa_\eta$)

By (*) there is $\sigma : P_{\eta+1} \prec N_\xi$ defined by:

(2) $\sigma(\pi_{\xi,\eta+1}(f)(\alpha)) = \sigma_\xi(f)(g(\alpha))$.

Hence $\sigma \upharpoonright \lambda_\eta = g$, $\sigma\pi_{\xi,\eta+1} = \sigma_\xi$.

Let $s = \langle t, \delta \rangle = \mathrm{tr}(\mathbb{F})$. Since W_η is reflective we know by Lemma 2.2:

(3) $W_\eta \models s$ is an N_η, σ_η - conforming trace.

Let $f : \omega \xrightarrow{\text{onto}} \eta + 1$. Set:

$t^* = \{\langle i, k\rangle \mid i < \omega \wedge k \in t_{f(i)}\}$, where $\langle\, ,\rangle$ is Gödel's pairing function on ordinals. Then $t^* \subset \omega$. (We assume $t_h \subset \omega$ for $h \leq \eta$)

We also set:

$u = \{\langle i, \mu\rangle \mid i < \omega \wedge \mu < \gamma_{f(i)}\}$.

Then $u \subset \delta_\eta$. Moreover $\sigma_\eta(\langle i, \mu\rangle) =$
$= \langle i, \sigma_\eta(\mu)\rangle$ for $\mu < \gamma_{f(i)}$, $i \leq \eta$.

Since $\delta_i = \sigma_\eta \restriction \gamma_i$ for $i \leq \eta$, we see that $s = \langle t, \delta\rangle$ is straightforwardly coded by $t^*, \sigma_\eta{}''u$, and e, where is any $e \subset \omega$ coding $\mathcal{J}|\eta+1$ and f.

Clearly $\sigma_\eta \restriction \omega = \mathrm{id}$. By Lemma 2 it follows that (3) can be expressed in W_η by:

(4) $C^{E_\eta}_{c_\eta, \infty} \models \Psi[t^*, \sigma_\eta{}''u, e]$,

where Ψ is Σ_1. Hence by $(**)$:

(5) $C^{E_\eta}_{c, \sigma_\eta(\kappa_\eta)} \models \Psi[t^*, g''u, e]$.

But $t^*, g''u, e$ code $\bar{s} = \langle t, \bar{\delta}\rangle$, where $\bar{\delta}_i = g \restriction \gamma_i = \sigma \restriction \gamma_i$ for $i \leq \eta$.

But $J_c^{E_\eta} = J_c^{E_\xi}$, since $c < \sigma_\eta(\kappa_\eta) = \sigma_\xi(\kappa_\eta) < c_\xi$. Thus

(6) $C^{E_\xi}_{c,\sigma_\xi(\kappa_\xi)} \models \Psi[t^*, g''u, e]$.

But this tells us (since $\sigma_\xi(\kappa_\xi), \sigma \in \bar{W}$):

(7) $\bar{s}$ is an N_ξ, σ - conforming trace for $\eta+1$ in $\bar{W}$.

Note that $\bar{s}\upharpoonright\xi = s\upharpoonright\xi$, since $\sigma\upharpoonright\delta_h = \sigma_\xi\upharpoonright\delta_h$ for $h < \xi$ by (1). By (7), if G is $\text{coll}(\omega,\delta)$ - generic over $\bar{W}$ for a sufficient δ, there is $\bar{\mathbb{E}}'' \in \bar{W}[G]$ which is an enlargement of $\mathcal{Y}|\eta+1$ with trace $\bar{s}$. Set $\bar{\mathbb{E}}' = \mathbb{E}\upharpoonright\xi \cup \bar{\mathbb{E}}''\upharpoonright[\xi, \eta+1]$. Then $\bar{\mathbb{E}}'$ is also an enlargement of $\mathcal{Y}|\eta+1$ with trace $\bar{s}$. Finally set: $\bar{\mathbb{E}} = \bar{\mathbb{E}}'{}^\frown\langle\bar{W}, \sigma\rangle$. $\bar{\mathbb{E}}$ is easily seen to have the desired properties.

We now prove (B). Let $t' = \bar{t}_{\eta+1} = th(\langle\bar{W}, \sigma, t\rangle)$. For $c < \beta < \alpha$ with $C^{E_\xi}_{c,\alpha}$ admissible consider the theory $T_{\beta\alpha}(\bar{s})$ in the language of $C^{E_\xi}_{c,\alpha}$ with:

<u>Predicate</u> G

<u>Constants</u> $\underline{x}$ $(x \in C^{E_\xi}_{c,\alpha})$, $\dot{W}, \dot{\sigma}, \dot{A}, \dot{N}, \dot{p}$

Axiome

(A) ZFC^-, $\bigwedge v (v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$ for $x \in C_{c,\alpha}$

(B) $\dot{W} = \langle L_{\underline{\beta}}[\dot{A}], \in, \dot{A}, \dot{N}, \dot{p} \rangle$ with:

(i) $\dot{A} \subset \underline{\beta}$, $W_i \models ZFC^{*} + (a) \wedge (b) \wedge (c)$, where (a), (b), (c) as in the def. of "good world"

(ii). $\underline{s} \in \dot{W} \wedge \underline{t}' = Th(\langle \dot{W}, \dot{\sigma}, \underline{t} \rangle) \wedge \dot{\sigma} \in \dot{W} \wedge \bigwedge_{i \leq \eta} \underline{\bar{\sigma}_i} = \dot{\sigma} \upharpoonright \underline{\gamma_i} \wedge \dot{\sigma} : \mathbb{P}_{\eta+1} \prec \dot{N}$

(iii) $\dot{W} \cap [\underline{\beta}]^{\omega} = \underline{[\beta]^{\omega}}$

(Recall that $\{[\beta]^{\omega}\}$ is $C^E_{c,\alpha}$ - definable in β)

(iv) $\dot{W}$ is reflective.

Then $T = T_{\beta\alpha}(\bar{s})$ is consistent, since $\langle W_{\eta+1}, \bar{W}, \sigma, A^{\bar{W}}, N^{\bar{W}}, p^{\bar{W}} \rangle$ is a model.

The statement: "There exist β, α s.t. $c < \beta < \alpha$, $C^E_{c,\alpha}$ is admissible, and $T_{\beta,\alpha}(\bar{s})$ is consistent" has the form:

(7) $C^E_{c,\infty} \models \chi'[s, t']$

Hence it has the form:

(8) $C^E_{c,\infty} \models \chi[t^*, g''u, e, t']$, where χ is Σ_1.

Since (7) holds in W_3 and $(C^E_{c,\infty})_{\bar{W}} \prec_{\Sigma_1} (C^E_{c,\infty})_{W_3}$, we conclude that (7) holds in $\bar{W}$. But it follows just as in [RE] by the mouselikeness and robustness of N_ξ in $\bar{W}$ that:

(9) $C^E_{c,\kappa_\eta} \prec_{\Sigma_1} (C^E_{c,\infty})_{\bar{W}}$.

Hence:

(10) $C^E_{c,\kappa_\eta} \models \chi[t^*, g''u, e, t']$

By (**) we conclude:

(11) $(C^{E_\eta}_{c_\xi,\infty})_{W_\eta} \models \chi[t^*, \sigma_\eta''u, e, t']$ or

in other words:

(12) $(C^{E_\eta}_{c_\xi,\infty})_{W_\eta} \models \chi'[s, t']$

(where $s = tr(\mathbb{E})$).

But this means that there are $\beta, \alpha \in W_\eta$ s.t. $c_\eta < \beta < \alpha$, and $C^{E_\eta}_{c_\eta,\alpha}$ is admissible, and $T_{\beta,\alpha}(s)$ consistent. Now let G be $\mathrm{coll}(\omega,\alpha)$-generic over W_η. Then $T_{\beta,\alpha}(s)$ has a solid model $\mathfrak{A}$ in $W_\eta[G]$. Set: $W' = \dot{W}^{\mathfrak{A}}$, $\sigma' = \dot{\sigma}^{\mathfrak{A}}$. Then:

(13) W' is a reflective good world;

(14) $s \in W'$, $\sigma' \in W'$ and $\sigma' : P_{\eta+1} \prec N' = N^{W'}$ with $\sigma_i = \sigma' \upharpoonright \delta_i$ $(i \le \eta)$.

Moreover

(15) $t' = th(\langle W', \sigma', s \rangle)$, hence:

(16) $W' \models s$ is an N', σ'-conforming trace.

If we set $\mathbb{E}' = \mathbb{E}^\frown \langle W', \sigma' \rangle$, then $\mathbb{E}'$ has the desired properties.

QED (Superlemma)

§2 Iterability

We now give the proof of the Main Claim in §1. N is a mouselike, robust premouse satisfying ZFC$^-$. We can place it inside a standard world $W = \langle L_\alpha[A], N, p\rangle$, where $\mathcal{T}$ is W-definable. Hence there is a W-definable bijection $m^*: lh(\mathcal{T}) \longleftrightarrow \omega$.

Following Steel we define:

Def $m(i) = \min\{m^*(j) \mid i \leq_T j \text{ in } \mathcal{T}\}$ $(i < lh(\mathcal{T}))$

Def i survives at j (i surv j) iff
$i \leq j \wedge m(i) = m(j) \wedge m(l) \geq m(i)$ for all $l \in (i, j)$.

Steel establishes:

Lemma 1

(a) i surv $j \rightarrow i \leq_T j$

(b) $(h \text{ surv } i \text{ surv } j) \rightarrow h$ surv j

(c) $(h \text{ surv } j \wedge h \leq_T i \leq_T j) \rightarrow h$ surv i surv j

(d) Let b be a branch of limit length in $\mathcal{T}$. b is maximal in $\mathcal{T}$ iff
$\bigwedge i \in b \bigvee j \in b\, (i < j \wedge \neg\, i \text{ surv } j)$

(Hence if $b = \{h \mid h <_T \lambda\}$, λ Lim(λ), then
$\bigvee i \in b \bigwedge j \in b\, (i < j \rightarrow i \text{ surv } j)$)

We now define:

Def $i <_* j \longleftrightarrow_{Df} (i \leq_T j \wedge \neg\, i \text{ surv } j)$

Def i dies at $j \longleftrightarrow_{Df} (j < lh(\mathcal{T}) \wedge$
$\wedge\, i < j \wedge$ whenever $h \geq j$, $T(h) = i$, then
$T(h) <_* h$.

Finally:

<u>Def</u> i is a <u>break point</u> at $\gamma \le lh(\mathcal{T})$ iff
iff $i < \gamma$ and whenever $i < h < \gamma$ and $T(h) \le i$, then $T(h) <_* h$. (In other words: In $\mathcal{T}|\gamma$ every $l \le i$ dies at $i+1$.)

. . . .

We now turn to the proof of the <u>main claim</u>. We are given $\sigma : P \prec N$, where P is a putative iteration of P, and wish to show that one of the following holds:

(a) $lh(\mathcal{T}) = h+1$ and there is $\sigma' : P_h \prec N$ s.t. $\sigma' \pi_{0,h} = \sigma$.

(b) There is a maximal branch b of limit length in $\mathcal{T}$ and a $\sigma' : P_b \prec N$ s.t. $\sigma' \pi_b = \sigma$.

We shall assume (b) to be false and prove (a). We begin by reformulating (b):

<u>Def</u> $\mathbb{R} = \mathbb{R}_N$ is defined by:

$$\Gamma = \Gamma_N = \{\sigma \mid \bigvee i \ \sigma : P_i \prec N\}$$

$$\mathbb{R} = \{\langle \sigma', \sigma \rangle \mid \bigvee i, j \ (i <_* j \wedge \sigma : P_i \prec N \wedge \sigma' : P_j \prec N \wedge \sigma' \pi_{ij} = \sigma)\}$$

If $\sigma \in \Gamma$ we also set:

$$\mathbb{R}^\sigma = \mathbb{R}^\sigma_N = \{\langle \sigma', \sigma'' \rangle \mid \sigma'' \mathbb{R} \sigma' \wedge (\sigma' \mathbb{R} \sigma \vee \sigma' = \sigma)\};$$

Then (b) is equivalent to:

(b') R^{σ} is ill founded

for our given $\sigma : \bar{P} \prec N$. Thus we are assuming R^{σ} to be well founded.

In the following we shall always deal with good world $W' = \langle |W'|, N, p' \rangle$ in which J, m^* are W'-definable by the same definition as in our standard world W. Hence there is a relation $R_{W'} = R_{N'}$ definable in W' as R was defined in W. With this convention we define:

<u>Def</u> An enlargement $\mathbb{E} = \langle\langle W_h, \sigma_h \rangle \mid h \leq i \rangle$ of $J \mid i+1$ is <u>proud</u> iff

(a) $\mathbb{E}$ is neat

(b) $R_h = R_{W_h}^{\sigma_h}$ is well founded for $h \leq i$

(c) W_h has the form $\overline{W}_h^{(\mu_h)}$, where $\mu_h \geq$ the rank of σ_h in R_h $(h \leq i)$

(d) W_i is reflective

(e) If $h < i$ does not die at $i+1$, then W_h is reflective.

<u>Def</u> $\mathbb{E}$ is <u>semi-proud</u> iff (a)-(c) hold.

<u>Def</u> s is a (semi) proud trace iff s is the trace of a (semi) proud enlargement.

Note Semiproudness is equivalent to a syntactic condition of the form: $x \in t_h$ for $h \leq i$. Hence any enlargement of a semiproud trace is semiproud.

We prove:

Main Lemma Let $j+1 \leq lh(\mathcal{I})$. Let $i < j$ and let $\mathbb{E}$ be a proud enlargement of $\mathcal{I}|i+1$.

(a) If i is a breakpoint at $j+1$, then $\mathbb{E}$ extends to a proud enlargement $\mathbb{F}$ of $\mathcal{I}|j+1$ s.t. $On_{W_l} < On_{W_i}$ for $i < l \leq j$.

(b) If i survives at j, then $\mathbb{E} \upharpoonright i$ extends to a proud enlargement $\mathbb{F}$ of $\mathcal{I}|j+1$ s.t. $On_{W_l^{\mathbb{F}}} < On_{W_i^{\mathbb{E}}}$ for $i \leq l < j$ and $W_j^{\mathbb{F}} = W_i^{\mathbb{E}}$, $\sigma_j^{\mathbb{F}} \pi_{ij} = \sigma_i^{\mathbb{E}}$.

Before proving this, we show that it implies the main claim. Note that if $\mu =$ the rank of σ in $\mathbb{R}$, then $\mathbb{E} = \langle\langle W^{(\mu)}, \sigma \rangle\rangle$ is a proud enlargement of $\mathcal{I}|1$. Suppose, first, that $lh(\mathcal{I}) = j+1$. W.l.o.g. we may suppose: $n^*(j) = 0$.

Then 0 survives at j. Hence $\mathbb{E} \upharpoonright 0 = \emptyset$ extends to a proud enlargement $\mathbb{F}$ of $\mathcal{Y}$ with $W_j^{\mathbb{F}} = W$ and $\sigma_j^{\mathbb{F}} \pi_{0j} = \sigma$.

Now suppose that $\mathcal{Y}$ has limit length. We derive a contradiction. We suppose w.l.o.g. that $m^*(0) = 0$. Hence 0 is a breakpoint at $lh(\mathcal{Y})$.

Define ξ_i $(i < \omega)$ by: $\xi_0 = 0$;

$\xi_{i+1} =$ that ξ s.t. $m^*(\xi) =$
$= \min \{ m^*(h) \mid h > \xi_i \}$.

An easy induction shows that each ξ_i is a breakpoint at $lh(\mathcal{Y})$, hence at $\xi_i + 1$. By successive applications of (a) we get enlargements $\mathbb{E}_i$ of $\mathcal{Y} | \xi_i + 1$ s.t. $W_{\xi_{i+1}} \in W_{\xi_i}$ for $i < \omega$. Contr!

We now prove the main lemma by induction on j. Let it hold below j. We first prove (a), assuming (b) to hold:

Case 1 There is an $h < j$ which survives at j. Then $h > i$, since i is a break point at $j+1$. By the induction hypothesis $\mathbb{E}$ extends to a proud enlargement $\mathbb{E}'$ of $\mathbb{J}|h+1$ with $On_{W_l} < On_{W_i}$ for $i < l \le h$. We then apply (b) to $\mathbb{E}'$, getting a proud $\mathbb{F}$ extending $\mathbb{E}'|h$ s.t. $On_{W_l^{\mathbb{F}}} < On_{W_h^{\mathbb{E}'}} < On_{W_i^{\mathbb{E}}}$ for $h \le l < j$ and $W_j^{\mathbb{F}} = W_h^{\mathbb{E}'}$ (hence $On_{W_j^{\mathbb{F}}} < On_{W_i^{\mathbb{E}}}$). Then $\mathbb{F}$ extends $\mathbb{E}$ and has the right properties.

Case 2 Case 1 fails. Then j is a successor ordinal $h+1$ and $T(j) <_* j$. By the ind. hypothesis $\mathbb{E}$ extends to a proud $\mathbb{E}'$ enlarging $\mathbb{J}|j$ s.t. $On_{W_l} < On_{W_i}$ for $i < l < j$. We then apply the Superlemma. By Superlemma (A) there is $\sigma \in W_\zeta$ ($\zeta = T(j)$) s.t. $\sigma : P_{j+1} \prec N_\zeta$ and

$\bar\sigma \pi_{\zeta, j+1} = \sigma_\zeta$. Hence $\bar\sigma R_\zeta \sigma_\zeta$ and hence $\bar\mu < \mu_\zeta$, where $\bar\mu$ = the rank of $\bar\sigma$ in R_ζ. We then apply Superlemma (B) to $\bar W_\zeta^{(\bar\mu)}$, getting an enlargement $\mathbb{E}''$ of $\mathcal{Y}|j+1$ extending $\mathbb{E}'$ s.t. $On_{W_j} < On_{W_h}$ and W_j is reflective. Moreover $\mathbb{E}'$ is neat, $R_j^{\sigma_j}$ is well founded and $W_j = \bar W_j^{(\mu_j)}$ for μ_j = the rank of σ_j in R_j, since

$$\langle \bar W_\zeta^{(\bar\mu)}, \bar\sigma \rangle \equiv \langle W_j, \sigma_j \rangle.$$

Hence $\mathbb{E}''$ is proud. QED((b)→(a))

We now prove (b). We first note:

<u>Fact</u> If i survives at j, then every $h \in [i, j)$ dies at $j+1$.

prf. Suppose not.

Let $\eta \geq j+1$, $\zeta = T(\eta)$ s.t. ζ survives at η and $\zeta \in [i, j)$. Then $n(\eta) = n(\zeta) \geq n(i) = n(j)$ since $\zeta \in [i, j)$. But $n(\eta) \neq n(j)$, since $\eta \not\leq_T j$ and $j \not\leq_T \eta$. Hence $n(\zeta) > n(i)$.

Hence ζ does not survive at η, since $n(\zeta) > n(j)$, $j \in (\zeta, \eta)$. Cont!

<u>Case 1</u> Lim(j)

Let $\langle i_n \mid n<\omega\rangle$ be monotone s.t. $i_0 = i$, $\sup_n i_n = j$. (Hence i_n succ i_m for $n<m$.) We first apply the induction hypothesis to get successive enlargements $\mathbb{E}_n$ $(n<\omega)$ s.t. $\mathbb{E}_0 = \mathbb{E}$ and $\mathbb{E}_{n+1}$ extends $\mathbb{E}_n \restriction i_n$ with:

$\mathbb{E}_{n+1}$ is proud; $W_{n+1}^{\mathbb{E}_{n+1}} = W_n^{\mathbb{E}_n} = W_i$,

$\sigma_{n+1}\pi_{i_n, i_{n+1}} = \sigma_n$ for $n<\omega$.

Let $\tilde{s}_n = \langle t_n, \delta_n\rangle = tr(\mathbb{E}_n)$. Then $\tilde{s}_n$ is a semi proud trace in $W_i =$ $= W_n^{\mathbb{E}_n}$, since $\mathbb{E}_n$ is semi proud and W_i is reflective. (hence neat) and

Hence $\langle\langle \tilde{s}_n, \sigma_n\rangle \mid n<\omega\rangle$ forms a descending chain in the following relation S, which is defined in W_i:

<u>Def</u> $D =$ the set of $\langle s, \sigma\rangle$ s.t. for some n $\sigma : P_{i_n} \prec N^{\nu}$, s is an N_i, σ-conforming, semi proud trace for $\mathcal{Y}|i_n+1$, and $s\restriction i = \tilde{s}_0 \restriction i$. ($\tilde{s}_0 = tr(\mathbb{E})$)

$\langle\langle s', \sigma'\rangle, \langle s, \sigma\rangle\rangle \in$

$S =$ the set of $\langle\langle s', \sigma'\rangle, \langle s, \sigma\rangle\rangle$ $\in D^2$ s.t. for some $n<\omega$:

$\sigma : P_{i_n} \prec W_i$, $\sigma' : P_{i_{n+1}} \prec N_i$, $\sigma'\pi_{i_n, i_{n+1}} = \sigma$,

$s' \restriction i_n = s \restriction i_n$.

Thus S is ill founded. Let $\langle\langle s_n, \sigma_n\rangle \mid n<\omega\rangle \in$ $\in W_i$ be a chain through S. Define $\sigma : P_i \prec N_i$ by $\sigma \pi_{i_n i} = \sigma_n$. Set:
$s = \bigcup_{n<\omega} s_n \upharpoonright i_n$.

<u>Claim</u> s is an N, σ conforming trace in W_i

proof.
s is obviously σ-conforming, since $\sigma \upharpoonright \gamma_h = \sigma_n \upharpoonright \gamma_h = \delta_{n,h}$ for $h < i_n$. Now let μ_n be least s.t.

$\mathrm{coll}(\omega, \mu_n) \Vdash$ (There is an N_i-conforming enlargement $\mathbb{F}$ of $\mathcal{I} | i_n$ s.t. $s_n \upharpoonright i_n = \mathrm{tr}(\mathbb{F})$)

Then $\langle \mu_n \mid n<\omega\rangle$ is W_i-definable from $\langle s_n \mid n<\omega\rangle$. Set:
$\mu = \mathrm{lub}_{n<\omega}\, \mu_n$. Let G be $\mathrm{coll}(\omega, \mu)$-generic over W_i. Then in $W_i[G]$ we find $\langle \mathbb{F}_n \mid n<\omega\rangle$ s.t. $\mathbb{F}_n$ is an N_i-conforming enlargement of $\mathcal{I} | i_n$.
Define $\mathbb{F}$ by: $\mathbb{F} \upharpoonright i_0 = \mathbb{F}_0$, $\mathbb{F} \upharpoonright [i_n, i_{n+1}) = \mathbb{F}_{n+1} \upharpoonright [i_n, i_{n+1})$. Then $\mathbb{F}$ is an N_i-conforming enlargement of $\mathcal{I} | j$.

QED (Claim)

Now let $\mathbb{E}'$ be an N_i-~~conforming~~ enlargement of $\mathcal{I} | j$. Define

$\mathbb{E}''$ by $\mathbb{E}''\restriction i = \mathbb{E}\restriction i$, $\mathbb{E}''\restriction[i,j) = \mathbb{E}'\restriction[i,j)$. Then $\mathbb{E}''$ is an N_i-conforming enlargement of $\mathcal{Y}|j$. Note that s is a neat enlargement, since each s_m is neat. Set $\mathbb{E}^* = \mathbb{E}''{}^\frown\langle W_i, \sigma\rangle$. Then $\mathbb{E}^*$ is a neat enlargement of $\mathcal{Y}|j+1$ by the above Claim. Moreover $\mathbb{E}^*\restriction i = \mathbb{E}\restriction i$, and $\sigma\pi_{ij} = \sigma_i^{\mathbb{E}}$. We note that $\mathbb{E}^*$ is semi proud since each s_m is semi proud (hence $W_h \models R_h^{\sigma_h}$ is well founded and $W_h = \overline{W}_h^{(\mu_h)}$ for a $\mu_h \geq$ rank of σ_h in R_h), and $W_i \models R_i^{\sigma_i}$ is well founded and $W_i = \overline{W}_i^{(\mu_i)}$ for a $\mu_i \geq$ rank of σ_i in R_i). Thus $\mu_i \geq$ rank of σ in R_i, since whenever $\sigma R_i \sigma'$ (i.e. $\sigma' : P_l < N_i$ for an $l_* > j$ and $\sigma'\pi_{jl} = \sigma$), then $\sigma_i R_i \sigma'$, since $\sigma'\pi_{il} = \sigma_i$. Thus $\mathbb{E}^*$ is semi proud. But $W_i^{\mathbb{E}^*} = W_i$ is reflective and $W_h^{\mathbb{E}^*} = W_h$ is reflective for all $h < i$ which does not die at j (since it then does not die at $i+1$). By the above Fact it then follows that $\mathbb{E}^*$ is proud. QED (Case 1)

Case 2 $j = h+1$.

Let $\xi = T(j)$. Then ξ is a break point at j, since if $k \in (\xi, j)$ and $T(k) \leq \xi$, then $T(k) <_* k$. (Otherwise $n(T(k)) = n(k) \geq n(\xi)$, since $k \in (\xi, j)$. But $n(k) \neq n(j) = n(\xi)$, since $k \not<_T j$ and $j \not<_T k$. Hence $n(k) > n(\xi)$, where $\xi \in [T(k), k)$. Hence $\neg T(k)$ surv k.) Clearly $\xi \geq i$ and i survives at ξ. By the induction hypothesis $\mathbb{E}\restriction i$ extends to a proud $\mathbb{E}'$ enlarging $\mathcal{Y}|\xi+1$, s.t. $On_{W_l^{\mathbb{E}'}} < On_{W_i^{\mathbb{E}}}$ for $i \leq l < \xi$, $W_\xi^{\mathbb{E}'} = W_i^{\mathbb{E}}$, and $\sigma_\xi^{\mathbb{E}'} \pi_{i,\xi} = \sigma_i^{\mathbb{E}}$. Hence we can assume w.l.o.g. that $\xi = i$. By the induction hypothesis we can apply (a) to $\mathbb{E}$, getting a proud $\mathbb{E}'$ extending $\mathbb{E}$ and enlarging $\mathcal{Y}|j$ s.t. $On_{W_l} < On_{W_\xi}$ in $\mathbb{E}'$ for $\xi < l < j$. Applying Superlemma (A) to $\mathbb{E}'$ then gives the desired result: We obtain $\overline{\mathbb{E}}$ extending $\mathbb{E}\restriction i$ s.t., letting $\bar{s} = \langle \bar{t}, \bar{\sigma} \rangle = tr(\overline{\mathbb{E}})$, $s = \langle t, \sigma \rangle = tr(\mathbb{E})$, we have $t = \bar{t}\restriction i$, $\bar{\sigma}_j \pi_{ij} = \sigma_j$ $(\bar{\sigma}_h = \sigma_h^{\overline{\mathbb{E}}})$, and $\overline{W}_j = W_i$ $(\overline{W}_h = W_h^{\overline{\mathbb{E}}})$.

Then $\bar{\mathbb{E}}$ is semi proud, since s is a semi proud trace and (as in Case 1) $W_i \models (R_i^{\bar{\sigma}_i})$ is well founded and $V = \bar{W}^{(\mu)}$ for a $\mu \geq$ rank of $\bar{\sigma}_i$ in R_i).

But W_i is reflective and $\bar{W}_h = W_h$ is reflective for all $h < i$ which do not die at $i+1$ (hence for $h < i$ which do not die at $j+1$). By the above Fact it follows that $\bar{\mathbb{E}}$ is proud.

QED (Case 2).

This proves the Theorem.