# Almost Subcomplete Forcing

## §0 Preface

In this paper we develop the notion of "almost complete forcing". We first noticed this concept a few years ago, when we generalized Shelah's notions of '$\aleph_1$-proper' and 'dee-complete' to '$\aleph_1$-subproper' and 'dee-subcomplete' in [DSP] (Here a word of apology is necessary: I inadvertently turned 'dee-complete' into 'dee-proper', then compounded the violation of standard terminology by coining the term 'dee-subproper'. We shall here endeavor to stick to accepted terminology. I am grateful to Menachem Magidor for pointing this out to me.)

Almost subcomplete forcing is in one sense narrower and in another sense broader than dee-subproper forcing. A potentially interesting feature is that it could be applied to forcings which are semi subproper but not subproper.

(We can construct such an example but have not found any "real" applications.) The property of almost subcompleteness is preserved under RSC iterations, subject to some standard minimal conditions. The proof is simpler than the iterability proof in [DSF] and does not involve the auxiliary notion of $\aleph_1^S$ - subproperness. However, we had no application which could not be done by other methods. Our interest in the matter revived when we (mistakenly) appeared to find such an application: In [DSF] we had shown that a variant $\mathbb{N}'$ of Namba forcing is dee - subcomplete, assuming CH, and can therefore be iterated. However, the proof did not go through for another Namba variant $\mathbb{N}^*$. After pondering this for some time, we were able to show that, assuming CH + $2^{\omega_1} = \omega_2$, both $\mathbb{N}'$ and $\mathbb{N}^*$ are almost subcomplete. This led us to resuscitate our very sketchy notes on ASC forcing and write the fuller version given here. We then realized, however, that we had proven more than intended: $\mathbb{N}'$ and $\mathbb{N}^*$ are, in fact, fully subcomplete, assuming $CH + 2^{\omega_1} = \omega_2$. Thus, alas, we still have no real application of ASC forcing. Theories of iteration are, however, by their very nature a work for the future. We offer these notes in that sense. In a separate set of notes we shall give the proof that $\mathbb{N}'$ and $\mathbb{N}^*$ are subcomplete.

We now define the class of almost complete forcings. We first define:

**Def** Let $N = L_\alpha^A = \langle L_\alpha[A], \in, A \cap L_\alpha[A] \rangle$ be a ZFC$^-$ model. Let $\mathbb{B} \in N$ be a complete Boolean algebra in $N$.

$G \subset \mathbb{B}$ is <u>weakly</u> $\mathbb{B}$ - <u>generic</u> <u>over</u> $N$ iff the following hold:

(a) $G$ is an ultrafilter on $\mathbb{B} \setminus \{0\}$

(b) Whenever $\Delta \in N$ is dense in $\mathbb{B} \setminus \{0\}$ and $\mathrm{card}(\Delta) \leq \omega_1$ in $N$, then $\Delta \cap G \neq \emptyset$.

(<u>Note</u> (b) can be equivalently replaced by: Whenever $\Vdash t \in \check{\omega}_1$ in $N$, then $\bigvee b \in G \ \bigvee \alpha < \omega_1^N \ \ b \Vdash t = \check{\alpha}$.

We also recall the definitions:

Def Let $N$, $\mathbb{B}$ be as above.

$\delta(\mathbb{B}) = \delta_N(\mathbb{B}) =:$ the minimal cardinality (in the sense of $N$) of a dense subset of $\mathbb{B} \setminus \{0\}$.

Def Let $N$ be as above. Let $\delta$ be a cardinal in $N$. Let $X \subset N$.

$C_\delta(X) = C_\delta^N(X) =:$ the smallest $Y \prec N$ s.t. $X \cup \delta \subset Y$.

Def Let $N$ be as above. $N$ is almost full iff there is a $ZFC^-$ model $\mathfrak{A}$ s.t. $N \in \mathfrak{A}$ and $\mathfrak{A} = \langle |\mathfrak{A}|, \in_{\mathfrak{A}} \rangle$

(a) $\mathfrak{A}$ is grounded - i.e. the well founded core $\mathrm{wfc}(\mathfrak{A})$ of $\mathfrak{A}$ is transitive and:

$$\in_{\mathfrak{A}} \cap \mathrm{wfc}(\mathfrak{A})^2 = \in \cap \mathrm{wfc}(\mathfrak{A})^2$$

(b) $N \in \mathrm{wfc}(\mathfrak{A})$.

We are now ready to define:

Def Let $\mathbb{B}$ be a complete BA. $\mathbb{B}$ is an almost subcomplete (ASC) forcing as witnessed by $\langle \theta, \mathbb{S} \rangle$ iff the following hold:

(I) There is a $\Sigma_1$ formula $\varphi$ s.t.

$\mathbb{S} = \{ \langle \bar{N}, \bar{\theta}, \bar{\mathbb{B}}, \bar{G} \rangle \mid H_{\omega_1} \models \varphi[\bar{N}, \bar{\theta}, \bar{\mathbb{B}}, \bar{G}] \}$

(II) $\mathbb{B} \in H_\theta$ where $\theta > \omega$ is a cardinal. Let $N = L_\tau^A$ be a ZFC$^-$ model such that $H_\theta \subset N$, $\theta < \tau$. Let $\pi : \bar{N} \prec N$ where $\bar{N}$ is countable and almost full. Let $\pi(\bar{\theta}, \bar{\mathbb{B}}) = \theta, \mathbb{B}$. Then:

(a) $S = S(\bar{N}, \bar{\theta}, \bar{\mathbb{B}}) =: \{ \bar{G} \mid \langle \bar{N}, \bar{\theta}, \bar{\mathbb{B}}, \bar{G} \rangle \in \mathbb{S} \}$ is a set of $\bar{G} \subset \bar{\mathbb{B}}$ which are weakly generic over $\bar{N}$.

(b) Ak $a \in \bar{\mathbb{B}} \setminus \{0\}$, there is $\bar{G} \in S$ with $a \in \bar{G}$.

(c) Let $\bar{G} \in S$, $\bar{s} \in \bar{N}$, $\pi(\bar{s}) = s$. Then there is $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic, then there is $\sigma \in V[G]$ with:

(i) $\sigma : \bar{N} \prec N$

(ii) $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}) = \theta, \mathbb{B}, s$

(iii) $C_\delta^N(\operatorname{rng} \sigma) = C_\delta^N(\operatorname{rng} \pi)$ where $\delta = \delta(\mathbb{B})$ in $N$

(iv) $\bar{G} = \sigma^{-1}{}'' G$.

Def $\mathbb{B}$ is *almost subcomplete* (ASC) iff it is ASC as witnessed by some $\langle \theta, \delta \rangle$.

(*Note* Our notion of dee-subcompleteness developed in [DSF] does not generalize Shelah's full notion of dee-completeness, but only that of *simple* dee-completeness. (I) is the corresponding simplicity condition for ASC forcing.)

(*Note* We can considerably weaken (iii) and still obtain the iterability theorem. We can require only that if $\bar{G}, \bar{\pi}, \pi$ are as above and $\tau_1, \ldots, \tau_n \geq \delta$ are regular cardinals in $N$, then there there is $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic, there is $\sigma \in V[G]$ satisfying (i), (ii), (iv) and:

(iii') $\sup \sigma''\tau_i = \sup \pi''\tau_i$ for $i = 1, \ldots, n$.

We adopted the more rigorous condition (iii) because we find it easier to work with and because in all cases hitherto, the verification of the weaker condition has turned upon the verification of (iii))

If we can replace "weakly generic" by "fully generic" in (a), then the forcing $\mathbb{B}$ is subproper. We say that it is strictly almost subcomplete (SASC). Clearly, every SASC forcing is dee-subcomplete, since our definition implies that it has a completeness system with exactly one element. However, we do not know whether all SASC forcings are $\omega_1$-subproper. Our iteration theorem will hold, however, for all ASC forcings even those which are not subproper or not $\omega_1$-subproper. (This suggests that there may be an expanded notion of dee-subcomplete forcing which applies to some forcings which are not subproper. An iteration theorem for these would presumably have to involve a substitute for the notion "$\omega_1$-subproper" — perhaps "$\omega_1$-semi-subproper"? We have not pursued this possibility, however, since we have no candidate for a concrete application.)

(Note Both of the forcings $\mathbb{N}'$ and $\mathbb{N}^*$ are $\omega_1$-subproper, so the iteration results for these forcings could have been obtained by the methods of [DSP].)

In §1 we develop the theory of ASC forcings. In the ensuing sections we develop the theory of $\mathbb{N}'$ and $\mathbb{N}^*$ and prove that they are SASC, assuming $CH + 2^{\omega_1} = \omega_2$. We deal primarily with $\mathbb{N}^*$, indicating the slight changes necessary to get the theorems for $\mathbb{N}'$.

# Bibliography


[PIF] Proper and Improper Forcing

[LF] $\mathcal{L}$ - Forcing

[SPSC] Subproper and Subcomplete Forcing

[Sing] Singapore Notes

[FCH] Forcing Axioms Compatible with CH

[ITSC] Iteration Theorems for Subcomplete and Related Forcings

[EN] The Extended Namba Problem

[DSP] ~~Dee~~ Subproper Forcing