§4 Namba'-forcing

$\mathbb{N}'$ is a variation of the Namba-conditions $\mathbb{N}$ which is discussed at length in [PIF] and by us in [LF] §6 and [SPSC]. We define $\mathbb{N}$ as the set of all subtrees T of $(\omega_2)^{<\omega}$ = the set of monotone $f: n \to \omega_2$ $(n<\omega)$ s.t. whenever $s \in T$, then $\{t \mid s \leq t \text{ in } T\}$ has size ω_2.

$\mathbb{N}'$ is the set of $T \in \mathbb{N}$ s.t. for some $s \in T$, $T = \{t \mid t \leq_T s \vee s \leq_T t\}$ and if $t \geqq_T s$ in T, then t has ω_2 many immediate successors.

s is called the stem of T ($s = \mathrm{stm}(T)$).

In both cases the partial ordering of the conditions is inclusion.

In [PIF] it was shown that, assuming CH, we have:

- $\mathbb{N}'$ adds no reals
- $\mathbb{N}'$ is different from $\mathbb{N}$ in the

sense that no $\mathbb{N}'$ - generic contains an $\mathbb{N}$ - generic sequence and conversely.

(Note We say that $c = \langle \delta_i \mid i < \omega \rangle$ is $\mathbb{N}$-generic iff $c = c_G = \bigcup \bigcap G$, where G is $\mathbb{N}$-generic. G is then recoverable from c as the set G_c of $T \in \mathbb{N}$ in which c is a branch. Similarly for $\mathbb{N}'$.)

In [SPSC] we showed that, assuming $CH + 2^{\omega_1} = \omega_2$, $\mathbb{N}'$ is subproper, (hence can be iterated without collapsing ω_1).

Here we assume only CH and show that $\mathbb{N}'$ is ω_1-subproper and Dee-subproper. Hence it can be iterated without adding reals.

<u>From now on assume CH.</u>

$\mathbb{N}'$ satisfies a <u>weak</u> <u>amalgamation</u> <u>lemma</u>:

<u>Def</u> $stm(T)$ = the <u>stem</u> of T = the maximal $s \in T$ s.t. $\bigwedge t \in T\ (s \le t \vee t \le s)$.

<u>Lemma 1</u> Let $T \in \mathbb{N}'$, $s = stm(T)$. Let $\langle T_u \mid u \in (\omega_2)^{<\omega} \rangle$ be s.t.

- $T_u \in \mathbb{N}'$ and $T_\emptyset = T$
- $|s_u| = |s| + |u|$ where $s_u = stm(T_u)$
- $s_{u^\frown\langle\nu\rangle} \neq s_{u^\frown\langle\xi\rangle}$ for $\nu < \xi < \omega_2$

Then $T' \in \mathbb{N}'$, where

$$T' = \bigcap_{n<\omega} \bigcup_{|u|=n} T_u = \bigcup_{f:\omega\to\omega_2} \bigcap_{n<\omega} T_{f\restriction n}$$

(Note: $stm(T') = stm(T)$.)

The <u>refinement</u> <u>lemma</u> reads:

<u>Lemma 2</u> Let $T \in \mathbb{N}'$, $f : T \to \omega_1$. There exist $T' \le T$, $g : \omega \to \omega_1$ s.t.

- $stm(T') = stm(T)$
- $f(t) = g(|t|)$ for $t \in T'$.

This is proven with a game theoretic argument due to Shelah. A proof is given in [LF] §6. Using these two lemmas one can show:

Lemma 3 $\mathbb{N}'$ adds no reals.

A proof of this is also given in [LF] §6.

We now prove:

Thm 4 $\mathbb{N}'$ is Dee-subproper.

The proof stretches over many sublemmas. We first define the completeness system which will verify Dee-subproperness. Let $\bar{N} = L^A_\tau$ be a countable ZFC$^-$ model and let $\bar{\mathbb{N}}' \in \bar{N}$ satisfy the definition of $\mathbb{N}'$ in $\bar{N}$. Let $\bar{c} = \langle \bar{\delta}_i \mid i < \omega \rangle$ be monotone and cofinal in $\bar{\omega}_2 = \omega_2^{\bar{N}}$.

Set:

$A_{\bar{c}}$ = the set of G s.t.

- G is $\bar{\mathbb{N}}'$-generic over $\bar{N}$
- Let $c = \langle \delta_i \mid i < \omega \rangle = c_G = \bigcup \cap G$. Then there is $n < \omega$ s.t. for all $i \geq n$ there is $j < \omega$ s.t.

If $u \in H_{\omega_1}$ is not such a $\bar{c}$, we simply set:

A_u = the set of G s.t. G is $\bar{\mathbb{N}}'$-generic over $\bar{N}$.

Set:

$\mathbb{D}(\bar{N}, \bar{\mathbb{N}}') = \{A_u \mid u \in H_{\omega_1}\}$.

<u>Lemma 4.1</u> $\mathbb{D}$ is a completeness system.

prf.
We verify (a) – (d) in the definition of "completeness system". (a), (d) are trivial. We verify (b), (c).

(b) Let $T \in \bar{\mathbb{N}}'$, $s = \mathrm{stm}(T)$, $n = |s|$.

<u>Claim</u> There is $G \ni \dot{T}$ s.t. $G \in \mathbb{D}(\bar{N}, \bar{\mathbb{N}}')$.

prf.
Let $\bar{c} = \langle \bar{\gamma}_i \mid i < \omega \rangle$ be monotone and cofinal in $\bar{\omega}_2 = \omega_2^{\bar{N}}$. We show:

<u>Claim 1</u> There is $G \ni T$ s.t.

- G is $\bar{\mathbb{N}}'$-generic over $\bar{N}$
- $\bigwedge i \geq n \bigvee j\ (\gamma_i \leq \bar{\gamma}_j \leq \bar{\gamma}_{j+1} \leq \gamma_{i+1})$

where $c = \langle \gamma_i \mid i < \omega \rangle = c_G$.

This follows easily by:

Claim 2 Let $T \in \bar{\mathbb{N}}'$, $s = stm(T)$. Let $\Delta \in \bar{\mathbb{N}}$ be strongly dense in $\bar{\mathbb{N}}'$. There is $T' \leq T$ in $\bar{\mathbb{N}}'$ s.t. $T' \in \Delta$ and, letting $s' = stm(T')$:

(*) If $|s| \leq i < i+1 \leq |s'|$, then
$$V_i \; s'_i \leq \bar{\gamma}_i < \bar{\gamma}_{i+1} \leq s'_{i+1}.$$

proof

For $t \in T$ set:
$$n(t) = \begin{cases} 1 & \text{if there is } T' \leq T \text{ s.t.} \\ & stm(T') = t \wedge T' \in \Delta \\ 0 & \text{if not} \end{cases}$$

By the refinement lemma there is $T^* \leq T$ s.t. $stm(T^*) = stm(T)$ and $n(t) = g(|t|)$ for $t \in T^*$. But then $g(n) = 1$ for some n, since there is $T' \leq T^*$ s.t. $T' \in \Delta$. Let $g(n) = 1$. Clearly $|s| \leq n$ where $s = stm(T)$. But then there is $s' \in T^*$ s.t. $|s'| = n$ and (*) holds for s'.

QED (Claim 2)

This proves (b). We prove (c).

Let $\bar{c}_i = \langle \gamma^i_j \mid j < \omega \rangle$ be monotone and cofinal in $\bar{\omega}_2$ for $i < \omega$. Set $\gamma^*_i = \sup\{\gamma^h_j \mid h, j \leq i\}$ for $i < \omega$. Then $c^* = \langle \gamma^*_i \mid i < \omega \rangle$ is monotone and cofinal in $\bar{\omega}_2$. Clearly, however, $A_{c^*} \subset \bigcap_i A_{\bar{c}_i}$.

QED (Lemma 4.1)

Now let $\theta > 2^{2^{\omega_2}}$. We show:

<u>Main Claim</u> $\langle \theta, \mathbb{D} \rangle$ witnesses the Dee-subproperness of $\mathbb{N}$:

Let $N = L^A_\tau$ be a ZFC^- model s.t. $H_\theta \subset N$ and $\theta < \tau$. Let $\pi : \bar{N} \prec N$ s.t. $\bar{N}$ is countable and full, where $\pi \in N$. Let $\pi(\bar{\theta}, \bar{\mathbb{N}}) = \theta, \mathbb{N}$. We claim that there is a $\bar{c} = \langle \gamma_i \mid i < \omega \rangle$ monotone and cofinal in $\bar{\omega}_1 = \omega_1^{\bar{N}}$ s.t. whenever $\bar{G} \in A_{\bar{c}}$, $\bar{s} \in \bar{N}$; $s = \pi(\bar{s})$, Then there is $T \in \mathbb{N}'$ forcing that whenever $G \ni T$ is $\mathbb{N}'$-generic, then there is $\sigma \in V[G]$ s.t.

(a) $\sigma : \bar{N} \prec N$

(b) $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}) = \theta, \mathbb{B}, s$

(c) $C^N_{\omega_2}(\mathrm{rng}\,\sigma) = C^N_{\omega_2}(\mathrm{rng}\,\pi)$

(d) $\sigma''\bar{G} \subset G$.

In order to find such a $\bar{c}$ we work in $V[G^*]$ where G^* is $\mathbb{N}'$-generic. Note that $H^V_{\omega_1} = H^{V[G^*]}_{\omega_1}$.

<u>Def</u> Let $c = \langle \gamma_i \mid i < \omega \rangle$ be monotone and cofinal in ω_2. c is <u>good</u> iff whenever $F \in V$, $F : \omega_2 \to \omega_2$, then

$$\bigvee n \bigwedge i \geq n \;\; \gamma_i > \sup_{h<i} F(\gamma_h).$$

<u>Lemma 4.2</u> There is a good $c \in V[G^*]$

proof.

Let $c = c_{G^*} = \bigcap \bigcup G^*$. Then c is good. To see this, note that if $F \in V$, $F : \omega_2 \to \omega_2$ and

$T \in \mathbb{N}'$, then $T' \in \mathbb{N}'$ where T' is the set of $t \in T$ s.t., letting $n = \mathrm{stm}(T)$,

$\bigwedge i \in [n, lt1)\ t_i > \sup_{h<i} F(t_h)$.

Then $T' \leq T$ in $\mathbb{N}'$ and

$T' \Vdash \bigwedge i \geq \check{n}\ \dot{\gamma}_i > \sup_{h<i} \check{F}(\dot{\gamma}_h)$.

QED (Lemma 4.2)

(<u>Note</u> This shows that <u>every</u> $\mathbb{N}'$-generic sequence is good. However, no $\mathbb{N}$-generic sequence is good.)

Until further notice let c be a fixed good sequence in $V[G^*]$. We note that if $c' = \langle \gamma'_i \mid i < \omega \rangle$ is monotone and cofinal in ω_1 and $\bigvee n \bigwedge i \geq n \bigvee j\ \gamma'_i \leq \gamma_j < \gamma_{j+1} \leq \gamma'_{i+1}$, then c' is trivially good.

<u>Def</u> $N' =_{\mathrm{Df}} \langle N, \theta, \mathbb{N}', s, \pi \rangle$

$\sigma^* : \tilde{N}' \overset{\sim}{\longleftrightarrow} Z$, where Z = the smallest $Z \prec N'$ s.t. $c \in Z$.

Let $\tilde{N}' = \langle \tilde{N}, \tilde{\theta}, \tilde{\mathbb{N}}', \tilde{s}, \tilde{\pi} \rangle$.

Then $\sigma^* : \tilde{N}' \prec N'$.

<u>Def</u> $D =$ the set of $\tilde{N}'$-definable $X \subset \omega_2$

$\tilde{D} = (\sigma^*)^{-1}{}'' D$.

$F = \sigma^* \restriction \tilde{D}$.

Then F bijects $\tilde{D}$ onto D.

<u>Lemma 4.3</u> $F \in V$

prf.

Clearly $D \in V$ is countable. Let $\langle X_i \mid i < \omega \rangle$ enumerate D. Set $\tilde{X}_i = (\sigma^*)^{-1}(X_i)$.
Then $\langle \tilde{X}_i \mid i < \omega \rangle \in H_{\omega_1} \subset V$. Hence
$F = \{ \langle X_i, \tilde{X}_i \rangle \mid i < \omega \} \in V$. QED (4.3)

<u>Def</u> Let $f : \tilde{\omega}_2 \to \omega_2$ cofinally, where $\tilde{\omega}_2 =_{df} \omega_2^{\tilde{N}}$. f is <u>smooth</u> iff for each $X \in \tilde{D}$ and all $\nu_1, \ldots, \nu_n < \tilde{\omega}_2$:
$\langle \nu_1, \ldots, \nu_n \rangle \in X \leftrightarrow \langle f(\nu_1), \ldots, f(\nu_n) \rangle \in F(X)$,
($\langle\,\rangle$ being Gödel's tupel function on ordinals).

<u>Lemma 4.4</u> Let f be smooth. There is a unique extension $f^* : \tilde{N}' \prec N'$.

prf. of Lemma 4.4

(1) $\tilde{N}' \models \varphi[\vec{\nu}] \longleftrightarrow N' \models \varphi[f(\vec{\nu})]$
for $\nu_1, \ldots, \nu_n < \tilde{\omega}_2$.

prf.
There is $Y \in D$ s.t.
$\langle \vec{\nu} \rangle \in Y \longleftrightarrow N' \models \varphi[\vec{\nu}]$ for $\nu_1, \ldots, \nu_n < \omega_2$.

Set $\tilde{Y} = (\sigma^*)^{-1}(Y)$. Then
$\langle \vec{\nu} \rangle \in \tilde{Y} \longleftrightarrow \tilde{N}' \models \varphi[\vec{\nu}]$ for $\nu_1, \ldots, \nu_n < \tilde{\omega}_2$.
The conclusion is immediate. QED(1)

But each $x \in \tilde{N}'$ is $\tilde{N}'$-definable from ordinals $\nu_1, \ldots, \nu_n < \tilde{\omega}_2$. At $x = \tilde{g}(\vec{\nu})$, $\tilde{g}$ being a definable function, set:

$$f^*(x) = g(f(\vec{\nu})),$$

where g is N'-definable by the same formula. It is apparent from (1) that this definition is independent of the choice of $\vec{\nu}$ and the defining formula of $\tilde{g}$. Now let $x_1, \ldots, x_n \in \tilde{N}$, with $x_i = \tilde{g}_i(\vec{\nu})$ for $i = 1, \ldots, n$. By (1) we have:

$\tilde{N}' \models \varphi[\vec{x}] \longleftrightarrow \tilde{N}' \models \varphi[\tilde{g}_1(\vec{\nu}), \ldots, \tilde{g}_m(\vec{\nu})] \longleftrightarrow$
$\longleftrightarrow N' \models \varphi[g_1(f(\vec{\nu})), \ldots, g_m(f(\vec{\nu}))] \longleftrightarrow$
$\longleftrightarrow N' \models \varphi[f^*(x_1), \ldots, f^*(x_n)]$.

This proves existence. To see uniqueness, let f' have the same property. Then

$$f'(\tilde{g}(\vec{v})) = g(f(\vec{v})) = f^*(\tilde{g}(\vec{v})),$$

QED (Lemma 4.4)

<u>Lemma 4.5</u> Let f^* be as above. Set: $C = C^N_{\omega_2}(\text{rng}\,\pi)$, $\tilde{C} = C^{\tilde{N}}_{\omega_2}(\text{rng}\,\tilde{\pi})$. Then $\tilde{C} \in \tilde{N}$ and $f^*(\tilde{C}) = C$.

proof.

Let $\pi : \bar{N} \prec N^*$ cofinally. Then $N^* = N|\lambda \in N$ and $N^* \prec N$, where $\lambda = \text{On} \cap N^*$. Hence $C = C^{N^*}_{\omega_2}(\text{rng}\,\pi)$. Similarly $\tilde{C} = C^{\tilde{N}^*}_{\omega_2}(\text{rng}\,\tilde{\pi})$, where $\tilde{\pi} : \bar{N} \prec \tilde{N}^*$ cofinally. But $f^*(\tilde{N}^*) = N^*$, since $f^*(\tilde{\pi}) = \pi$. QED (4.5)

<u>Lemma 4.6</u> There is $\tilde{\sigma} : \bar{N} \prec \tilde{N}$ s.t.

- $\tilde{\sigma}$ takes $\bar{\omega}_2 = \omega_2^{\bar{N}}$ cofinally to $\tilde{\omega}_2$
- $C^{\tilde{N}}_{\tilde{\omega}_2}(\text{rng}\,\tilde{\sigma}) = C^{\tilde{N}}_{\tilde{\omega}_2}(\text{rng}\,\tilde{\pi})$.
- $\tilde{\sigma}(\bar{\theta}, \bar{\mathbb{N}}', \bar{s}) = \tilde{\theta}, \tilde{\mathbb{N}}', \tilde{s}$

prf. of Lemma 4.6

Let $k : \tilde N_1 \stackrel{\sim}{\longleftrightarrow} C^{\tilde N}_{\tilde\omega_2}(\mathrm{rng}(\tilde\pi))$. Then $k, \tilde N_1 \in \tilde N$

Let $\tilde\pi_1 = k^{-1}\tilde\pi : \bar N \prec \tilde N_1$.

Let $\langle \tilde N_0, \tilde\pi_0 \rangle =$ the liftup of $\langle \bar N, \tilde\pi \restriction \bar\omega_1 \rangle$.

$k_0 : \tilde N_0 \prec \tilde N_1$ s.t. $k_0 \tilde\pi_0 = \tilde\pi_1$ and $k_0 \restriction \tilde\omega_2^{\tilde N_0} = \mathrm{id}$.

The maps $\tilde\pi_0, \tilde\pi_1$ are cofinal; hence so is k_0. $\tilde N_0, \tilde N$ are almost full, since $\bar N$ is full. Let $\mathcal{L}_1$ be the following language on $L_{\delta_1}(\tilde N_1)$ $(\delta_1 = \delta_{\tilde N_1})$:

Predicate $\in$; Constants $\dot\sigma$, $\underline{x}$ $(x \in L_{\delta_1}(\tilde N_1))$

Axioms ZFC^-, $H_{\omega_1} = \underline{H_{\omega_1}}$, $\bigwedge v\,(v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$,

$\dot\sigma : \underline{\bar N} \prec \underline{\tilde N_1}$ $\bar\omega_1$-cofinally,

$\dot\sigma(\underline{\bar\theta}, \underline{\bar N}, \underline{\bar 1}) = \underline{k^{-1}(\tilde\theta)}, \underline{k^{-1}(\tilde N')}, \underline{k^{-1}(\tilde\pi)}$

Then $\mathcal{L}_1$ is consistent since the corresponding language $\mathcal{L}_0$ on $L_{\delta_0}(\tilde N_0)$ $(\delta_0 = \delta_{\tilde N_0})$ is modeled by:

$\langle H_{\omega_2}, \in, \tilde N_0, \tilde\pi_0 \rangle$.

Let $\mathfrak{A}$ be a solid model of $\mathcal{L}_1$.
Then $\tilde\sigma = k_0 \circ \dot\sigma^{\mathfrak{A}}$ has the desired properties. QED (Lemma 4.6)

(Note N is called full iff N is a second order ZFC$^-$ model in $L_\mu[N]$, where $L_\mu(N)$ satisfies ZFC$^-$. δ_N = the smallest δ s.t. $L_\delta(N)$ is admissible. The concepts "liftup" and "almost full" are elucidated in [SPSC] §3.)

From now on let $\tilde{\sigma}$ be a fixed map $\tilde{\sigma} : \bar{N} \prec \tilde{N}$ satisfying Lemma 4.6.

Lemma 4.7 Let f be smooth and set $\sigma = f^* \tilde{\sigma}$. Then $C^N_{\omega_2}(\mathrm{rng}\,\sigma) = C^N_{\omega_2}(\mathrm{rng}\,\pi)$.

prf.

$(\subset)$ $\mathrm{rng}\,\sigma = f^{*}{}''\mathrm{rng}\,\tilde{\sigma} \subset f^*(C^{\tilde{N}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\sigma})) =$
$= f^*(C^{\tilde{N}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\pi})) = C^N_{\omega_2}(\mathrm{rng}\,\pi)$ by Lemma 4.5

Hence $C^N_{\omega_2}(\mathrm{rng}\,\sigma) \subset C^N_{\omega_2}(\mathrm{rng}\,\pi)$.

$(\supset)$ Let $x \in C^N_{\omega_2}(\mathrm{rng}\,\pi)$. Then $x = \pi(g)(\xi)$ where $\xi < \omega_2$ and $g \in \bar{N}$. Hence $\tilde{\pi}(g) \in C^{\tilde{N}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\pi}) = C^{\tilde{N}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\sigma})$. Let $\tilde{\pi}(g) = \tilde{\sigma}(h)(\zeta)$, $\zeta < \tilde{\omega}_2$, $h \in \bar{N}$.

Then $x = (\sigma(h)(f(\zeta)))(\xi)$ where $f(\zeta), \xi < \omega_2$. Hence $x \in C^N_{\omega_2}(\mathrm{rng}\,\sigma)$.

QED (4.7)

Remark $F = \sigma^* \upharpoonright \tilde{D}$ is smooth with $F^* = \sigma^*$. Hence $\sigma' = \sigma^* \circ \tilde{\sigma}$ satisfies Lemma 4.7. We shall see that there are many other smooth functions in $V[G^*]$.

Lemma 4.8 Let σ be as above. There is $c' \in V[G^*]$ s.t. c' is good and $c' \subset \mathrm{rng}\,\sigma$.

pf.
Let $c = \langle \gamma_i \mid i < \omega \rangle \in V[G^*]$ be good. Define $c' = \langle \gamma_i' \mid i < \omega \rangle$ by: $\gamma_0' = 0$;
$\gamma_{i+1}' =$ the least $\gamma \in \mathrm{rng}\,\sigma$ s.t. $\bigvee_j \gamma_i' \leq \gamma_j < \gamma_{j+1} \leq \gamma$,
QED(4.8)

Now let $\sigma' = \sigma^* \circ \tilde{\sigma}$, assume that $c \in V[G^*]$ is not only good but that $c \subset \mathrm{rng}\,\sigma'$. Set: $\bar{c} = (\sigma')^{-1} {}'' c$.

We show:
Claim Let $\bar{G} \in A_{\bar{c}}$. There is $J \in N'$ satisfying the conclusion of the Main Claim.

Let $\bar{c}' = \langle \bar{\gamma}_i' \mid i < \omega \rangle = c_{\bar{G}} =_{H} \cap \bigcup \bar{G}$.
Set $\tilde{c}' = \sigma' {}'' \bar{c}'$. Let $\mathcal{L}$ be the following language over the admissible structure $\langle H_\theta, < \rangle$, where $<$ well orders H_θ.

<u>Predicate</u>: $\in$, <u>Constants</u> $\mathring{f}$, $\underline{x}$ $(x \in H_\theta)$

<u>Axioms</u> ZFC^-, $H_{\omega_1} = \underline{H}_{\omega_1}$, $\bigwedge v (v \in \underline{x} \leftrightarrow \bigvee_{z \in x} v = \underline{z})$,

$\mathring{f}: \tilde{\omega}_2 \to \omega_2$ cofinally, "$\mathring{f}$ is smooth" - ...,

$\bigwedge_{\nu_1 \dots \nu_n < \tilde{\omega}_2} \bigwedge_{X \in \underline{D}} (\langle \vec{\nu} \rangle \in X \leftrightarrow \langle \mathring{f}(\vec{\nu}) \rangle \in \underline{F}(X))$

for $n < \omega$.

Clearly this is consistent, since it is modeled by $\langle H_\theta, <, F \rangle$.

We now define a subtree of $(\omega_2)^{<\omega}$ by:

<u>Def</u> $t \in T$ iff $\mathcal{L} + \bigwedge_{i < |t|} \underline{t}_i = \mathring{f}(\underline{\tilde{\gamma}}_i')$ is consistent, where $\tilde{C}' = \langle \tilde{\gamma}_i' \mid i < \omega \rangle = \tilde{\sigma}''\bar{C}'$.

Thus $C' = \langle \gamma_i' \mid i < \omega \rangle$ is a branch through T.

<u>Lemma 4.9</u> There is $T' \subset T$ s.t.

- $T' \in N'$
- $s = \langle \gamma_i' \mid i < |s| \rangle$ where $s = \mathrm{stm}(T')$

proof.

For $n < \omega$ consider the following game G_n: In the i-th move player I picks an η_i s.t. $\eta_i > \eta_j$ for $j < i$.

Player II then picks - if possible - a ν_i s.t. $\langle \nu_h \mid h \leq i \rangle \in T$.

If $i < n$ he must pick $\nu_i = \gamma_i'$.

If $i \geq n$ " " " $\nu_i > \eta_i$.

If at some i II has no move, then I wins. Otherwise II wins.

<u>Claim</u> For arbitrarily large n, II has a winning strategy.

proof.

Suppose not. Then I has a winning strategy for all $n \geq n_0$. Let $D \subset \omega_2$ be the set of δ s.t. for all $n \geq n_0$ and all plays $\langle \nu_0, \ldots, \nu_m \rangle$ by II s.t. $\nu_m < \delta$, we have $S_n(\nu_0, \ldots, \nu_j) < \delta$ for $j \leq m$, where S_n is I's winning strategy. Then D is club in ω_2. Set:

$F(\xi) =$ the least $\delta \in D$ s.t. $\xi < \delta$.

Since C' is good, there is $n \geq n_0$ s.t. $\bigwedge i \geq n \ \gamma_i' > \sup_{h<i} F(\gamma_h')$. But then $\langle \gamma_i' \mid i < \omega \rangle$ defeats the strategy S_n. Contr! QED (Claim).

Now let S be II's winning strategy for G_n. Let T' be the set of all $S(\eta_0, \ldots, \eta_i)$ where $\langle \eta_0, \ldots, \eta_i \rangle$ is any possible play by I. Then

T' has the desired properties.

QED (4.9)

Let T' be as in Lemma 4.9. We show that T' satisfies the conclusion of the Main Claim. Let $G \ni T'$ be $\mathbb{N}'$-generic. We claim that there is $\sigma \in V[G]$ satisfying (a)–(d) of The Main Claim. Let $c'' = \langle \gamma_i'' \mid i < \omega \rangle = c_G = \wedge \cup G$. For $i < \omega$, let:

f_i'' = the N-least $f : \omega_1 \xrightarrow{\text{onto}} (\gamma_i'' + 1)$

$\tilde{f}_i'$ = " $\tilde{N}$- " $f : \tilde{\omega}_1 \xrightarrow{\text{onto}} (\tilde{\gamma}_i' + 1)$

Set $f_i = f_i'' \circ (\tilde{f}_i'^{-1})$. Then $f_i : \tilde{\gamma}_i' + 1 \to \gamma_i'' + 1$. But if $\mathfrak{A}$ is a solid model of $\mathcal{L} + \dot{f}(\underline{\tilde{\gamma}_i'}) = \underline{\gamma_i''}$, and $f = \dot{f}^{\mathfrak{A}}$, then f, being smooth, extends to $f^* : \tilde{N}' \prec N'$. Hence $f^*(\tilde{f}_i') = f_i''$ and $f_i = f \upharpoonright (\tilde{\gamma}_i' + 1)$. Hence $f_i \subset f_j$ and $f_j(\tilde{\gamma}_i') = \gamma_i''$ for $i \leq j < \omega$. Set: $f = \bigcup_i f_i$

Then:

(1) $f: \tilde{\omega}_1 \to \omega_1$ cofinally and $f(\tilde{\gamma}_i') = \gamma_i''$.

Moreover:

(2) f is smooth

proof.

Let $\nu_1, \ldots, \nu_n < \tilde{\omega}_1$. Then $\vec{\nu} < \tilde{\gamma}_i'$ for some i

Let $\mathfrak{A}$ be a solid model of $\mathcal{L} + \dot{f}(\underline{\tilde{\gamma}_i'}) = \underline{\gamma_i''}$.

Then $\dot{f}^{\mathfrak{A}}$ is smooth and $\dot{f}^{\mathfrak{A}} \restriction \tilde{\gamma}_i' = f \restriction \tilde{\gamma}_i'$ by the above argument. Hence, for $X \in \tilde{A}$, we have:

$$\langle \vec{\nu} \rangle \in X \longleftrightarrow \langle \dot{f}^{\mathfrak{A}}(\vec{\nu}) \rangle = \langle f(\vec{\nu}) \rangle \in F(X)$$

QED (2)

But then f extends to:

(3) $f^*: \tilde{N}' \prec N'$,

and we set: $\sigma = f^* \circ \tilde{\sigma}$. We show that (a)–(d) of the ain Claim hold for this σ. (a), (b) are immediate by (3) and $\tilde{\sigma}(\bar{\theta}, \bar{N}', \bar{\tau}) = \tilde{\theta}, \tilde{N}', \tilde{\tau}$. (c) follows by Lemma 4.7. Finally, (d) follows, since $C'' = C_G$, $\bar{C}' = C_{\bar{G}}$, and $f``\bar{C}' = C''$.

QED (Theorem 4)

It remains only to show that $\mathbb{N}'$ is ω_1-subproper. Before doing so, we attempt to distill some more information from the proofs just given. From now on let θ be $(2^{2^{\omega_2}})^+$. It is enough, however, to take θ as having this meaning in the sense of an N which is <u>viable</u> in the following sense:

<u>Def</u> N is <u>viable</u> iff

- $N = L_\tau^A$ is a model of $ZFC^- + CH$
- There are $\theta, H \in N$ s.t.

$$L_\delta(N) \models (\theta = (2^{2^{\omega_2}})^+ \wedge H = H_\theta),$$

where $\delta = \delta_N$ = the least δ s.t. $L_\delta(N)$ is admissible.

When dealing with a fixed viable N we shall write $\omega_1, \omega_2, \theta, H_\theta, \mathbb{N}'$ etc. to denote the relativization of the concept to N. All that we did in the proofs just given goes through for an arbitrary viable N.

Assuming $N = L^A_\tau$ to model $ZFC^- + CH$, N will be viable in either of the cases:

- $2^{2^{\omega_2}} = \theta$ in V and $H^V_\theta \subset N$
- N is almost full.

For transitive M, $c \subset M$, set:

$$M[c] =_{\text{df}} \bigcup_{x \in M} L_{On_M}(TC(x \cup \{c\}))$$

We define:

<u>Def</u> Let N be viable and $c = \langle \gamma_i \mid i < \omega \rangle$ monotone and cofinal in ω_2.

c is <u>tame</u> iff the following hold:

- $\delta_N = \delta_{N[c]}$
- $H^N_{\omega_1} = H^{L_\delta(N[c])}_{\omega_1}$ $\quad (\delta = \delta_N)$
- $\theta = (2^{2^{\omega_2}})^+$ in $L_\delta(N[c])$
- $H_\theta[c] = H^{L_\delta(N[c])}_\theta$.

c is <u>good</u> iff for all $F : \omega_2 \to \omega_2$ in N we have: $\bigvee n \bigwedge_{j \geq n} \gamma_j > \sup_{h < j} F(\gamma_h)$.

Clearly, any $\mathbb{N}'$-generic sequence over N is both good and tame.

Lemma 5.1 Let c be tame. Let $c' \in N[c]$ s.t. $c' = \langle \gamma_i' \mid i < \omega \rangle$ is monotone and cofinal in ω_2. Then $H_\theta[c] = H_\theta[c']$. (Hence c' is tame.)

prf.

$c' \in H_\theta[c]$, since c is tame.

Claim $c \in H_\theta[c']$

Let f map ω_1 onto ω_2, $f \in H_\theta[c']$. Set $\bar{c} = f^{-1} \circ c$. Then $\bar{c} \in H_{\omega_1}$ and $c = f \circ \bar{c} \in H_\theta[c']$. QED (5.1)

Now let $\pi : \bar{N} \prec N$ where $\pi \in N$ and $\bar{N} = L_{\bar{\tau}}^{\bar{A}}$ is countable and full in N. Let $\pi(\bar{\theta}, \bar{\mathbb{N}}', \bar{s}) = \theta, \mathbb{N}', s$ ($\bar{s} \in \bar{N}$ being arbitrary. We can literally repeat our previous proof: Setting $N' = \langle N, \theta, \mathbb{N}', s, \pi \rangle$, let:

Y = the smallest $Y \prec N'$ s.t. $c \subset Y$, where c is tame for N'. Then $Y \in L_\delta(N[c])$ where $\delta = \delta_N$. Hence $\tilde{N}', \sigma^* \in L_\delta(N[c])$ where

$\sigma^*: \mathbb{N}' \overset{\sim}{\longleftrightarrow} Y$ and $\tilde{N}' = \langle \tilde{N}, \tilde{\theta}, \tilde{\mathbb{N}}', \tilde{\imath}, \tilde{\pi} \rangle$ is transitive. Hence $\tilde{N}' \in H_{\omega_1}$. Define $D, \tilde{D}, F = \sigma^* \upharpoonright \tilde{D}$ exactly as before. As before we have $D, \tilde{D}, F \in H = H_{\omega_2}$. Define the notion of a <u>smooth</u> $f: \tilde{\omega}_2 \to \omega_2$ exactly as before. A smooth f then again extends uniquely to $f^*: \tilde{N}' \prec N'$ s.t. $f^* \upharpoonright \tilde{\omega}_2 = f$. As before we get the existence of $\tilde{\sigma} \in H_{\omega_1}$ s.t. $\tilde{\sigma}: \bar{N} \prec \tilde{N}$, $\tilde{\sigma}(\bar{\theta}, \bar{\mathbb{N}}', \bar{\imath}) = \tilde{\theta}, \tilde{\mathbb{N}}, \tilde{\imath}$, and $C^{\tilde{N}}_{\tilde{\omega}_2}(\text{rng}\, \tilde{\sigma}) = C^{\tilde{N}}_{\tilde{\omega}_2}(\text{rng}\, \tilde{\pi})$. Thus setting $\sigma = f^* \circ \tilde{\sigma}$, where $f \in N[c]$ is smooth, we get:

$$C^N_{\omega_2}(\text{rng}\, \sigma) = C^N_{\omega_2}(\text{rng}\, \pi),$$

But then:

<u>Lemma 5.2</u> $\sigma \in N[c]$.

proof.
Clearly $\sigma \in L_\delta[N[c]]$. Let $k: N^* \overset{\sim}{\longleftrightarrow} C^N_{\omega_2}(\text{rng}\, \pi)$. Since $\text{rng}\, \sigma \subset C^N_{\omega_2}(\text{rng}\, \pi)$, we can set $u = k^{-1}{}''\text{rng}\, \sigma$. Then $u \subset N^*$ is countable, where $N^* \in H^N_\theta$. Hence

$u \in H_\theta[c] = H_\theta^{N[c]}$, Hence

$\mathrm{rng}\,\sigma = k''u \in N[c]$. QED (5.2)

The main thing we now distill from this is:

<u>Lemma 5.3</u> Let c be tame. There is $\sigma \in N[c]$ s.t.

- $\sigma : \bar{N} \prec N$
- $\sigma(\bar{\theta}, \bar{N}', \bar{s}) = \theta, N', s$
- $C^N_{\omega_2}(\mathrm{rng}\,\sigma) = C^N_{\omega_2}(\mathrm{rng}\,\pi)$.

But by the proof of Lemma 4.8:

<u>Lemma 5.4</u> Let σ, c be as above, where c is both tame and good. There is $c' \in N[c]$ s.t. $c' \subset \mathrm{rng}\,\sigma$ and c' is tame and good (hence $N[c] = N[c']$).

<u>Note</u> It might appear that our formulation of Lemma 5.3 throws out valuable information since, in order to get σ, we first defined $\sigma^*, \tilde{N}, \tilde{\sigma}$.

Using Lemma 5.4, however, we can quickly restore the missing information:

Def $\langle c, \sigma \rangle$ is excellent for N, π iff

- c is tame and good
- $\sigma \in N[c]$
- $\sigma : \bar{N} \prec N$
- $\sigma(\bar{\theta}, \bar{\mathbb{N}}') = \theta, \mathbb{N}'$
- $C^N_{\omega_2}(\mathrm{rng}\,\sigma) = C^N_{\omega_2}(\mathrm{rng}\,\pi)$
- $c \subset \mathrm{rng}\,\sigma$.

Lemmas 5.3, 5.4 then say:

Cor 5.5 Let c' be tame and good, Let $\pi(\bar{s}) = s$. There is an excellent $\langle c, \sigma \rangle$ s.t. $\sigma(\bar{s}) = s$ and $N[c] = N[c']$.

From now on suppose that $\langle c, \sigma \rangle$ is excellent, $\bar{s} \in \bar{N}$, and $\sigma(\bar{s}) = s$. ($\bar{s}$ is not necessarily the same as our previous $\bar{s}$ with $\pi(\bar{s}) = s$.) Set:

$N' = \langle N, \theta, \mathbb{N}', \pi, s \rangle$. Since $\mathrm{rng}\,\sigma \subset C^N_{\omega_2}(\mathrm{rng}\,\pi)$, each $x \in \mathrm{rng}\,\sigma$ is N'-definable from elements of $\omega_1 \cup c$. Since σ is countable in $L_\delta(N[c])$, there is $\alpha < \omega_1$ s.t. each $x \in \mathrm{rng}\,\sigma$ is N-definable in parameters from $\alpha \cup c$.

Let Y = the smallest $Y \prec N$ s.t. $\alpha \cup C \subset Y$. Then $Y \in L_\delta(N)$. Hence $\sigma^*, \tilde{N}' \in L_\delta(N)$, where $\sigma^* : \tilde{N}' \overset{\sim}{\longleftrightarrow} Y$ and $\tilde{N}' = \langle \tilde{N}, \tilde{\theta}, \tilde{N}', \tilde{\pi}, \tilde{s} \rangle$ is transitive. Hence $\tilde{N}' \in H_{\omega_1}$. Set: $\tilde{\sigma} = (\sigma^*)^{-1}\sigma$. Letting D = the set of N'-definable $X \subset \omega_1$ and $\tilde{D} = (\sigma^*)^{-1}{}''D$, $F = \sigma^* \upharpoonright \tilde{D}$, we again have that $D \in H_\theta$ is countable in N (arguing in $L_\delta(N)$). It follows as before that $\tilde{D}, F$ are countable in N. We define the notion of *smooth* function $f : \tilde{\omega}_1 \to \omega_1$ as before and again conclude that each smooth f extends uniquely to an smooth $f^* : \tilde{N}' \prec N'$. As before, we have $f^*(\tilde{C}) = C$, where $C = C^N_{\omega_1}(\text{rng}\,\pi)$, $\tilde{C} = C^{\tilde{N}}_{\tilde{\omega}_2}(\text{rng}\,\pi)$. Now set: $\tilde{\sigma} = (\sigma^*)^{-1} \circ \sigma$. Then $\tilde{\sigma} : \bar{N} \prec \tilde{N}$. Moreover, $\tilde{C} = C^{\tilde{N}}_{\tilde{\omega}_2}(\text{rng}\,\tilde{\sigma})$. (To see this note that $C^N_{\omega_2}(\text{rng}\,\sigma) = C$; hence $C^{\tilde{N}}_{\omega_2}(\text{rng}\,\tilde{\sigma}) = \sigma^{*-1}{}''C^N_{\omega_2}(\text{rng}\,\sigma) = \sigma^{*-1}{}''C = \sigma^{*-1}(C) = \tilde{C}$.) If f is smooth and $\sigma' = f^* \circ \tilde{\sigma}$, we can conclude as before that $C^N_{\omega_2}(\mathrm{rng}\,\sigma') = C$.

Now let $\bar{c} = \sigma^{-1}{}''c$. Define $A_{\bar{c}}$ as before (relativized to N).

<u>Def</u> Let $A \subset A_{\bar{c}}$, $\bar{c} = \langle \bar{\gamma}_i \mid i < \omega \rangle$. A is <u>dispersed</u> at $n < \omega$ iff for every $l < \omega$ there is $\bar{G} \in A_{\bar{c}}$ s.t., letting $c_{\bar{G}} = \langle \gamma_i \mid i < \omega \rangle$, we have:

$\gamma_n > \bar{\gamma}_l \wedge \bigwedge_{i \geq n} \bigvee_h \; \gamma_i \leq \bar{\gamma}_h < \bar{\gamma}_{h+1} \leq \gamma_{i+1}$.

By the proof of Lemma 4.1 (b) we know that:

<u>Lemma 5.6</u> Let $T \in \bar{\mathbb{N}}'$, $s = \mathrm{stem}(T)$, $n = |s|$. Then $\{G \in A_{\bar{c}} \mid T \in G\}$ is dispersed at n.

We now prove a technical lemma.

<u>Lemma 5.7</u> Let $A \subset A_{\bar{c}}$ be dispersed at n. Let $\sigma(\bar{s}) = s$. There are $\bar{G} \in A$, $T \in \mathbb{N}'$ s.t.

(a) $\mathrm{stm}(T) = \sigma(c_{\bar{G}} \upharpoonright n)$

(b) If $G \ni T$ is $\mathbb{N}'$-generic over N, there is $\sigma' \in N[G]$ s.t.

- $\sigma' : \bar{N} \prec N$
- $\sigma'(\bar{\theta}, \bar{\mathbb{N}}', \bar{s}) = \theta, \mathbb{N}', s$
- $C^N_{\omega_1}(\mathrm{rng}\,\sigma') = C^N_{\omega_1}(\mathrm{rng}\,\sigma)$
- $\sigma'{}''\bar{G} \subset G$

proof of 5.7.

Define $\mathcal{L}$ on $\langle H_\theta, < \rangle$ as before:

Predicate: $\in$; Constants: $\mathring{f}$, $\underline{x}$ $(x \in H_\theta)$

Axioms: ZFC$^-$, $H_{\omega_1} = \underline{H}_{\omega_1}$, For $x \in H_\theta$:

$\bigwedge v \, (v \in \underline{x} \longleftrightarrow \bigvee\limits_{z \in x} v = \underline{z})$, $\mathring{f} : \tilde{\omega}_2 \to \underline{\omega}_2$ cofinally;

"$\mathring{f}$ is smooth" - i.e. for $n < \omega$:

$\bigwedge v_1 \dots v_n < \tilde{\omega}_2 \bigwedge X \in \underline{\tilde{D}} \, (\langle \vec{v} \rangle \in X \longleftrightarrow \langle \mathring{f}(\vec{v}) \rangle \in \underline{F(X)})$.

Set $A^* = \{ d \mid \bigvee \bar{\sigma} \in A \; d = c_{\bar{\sigma}} \}$. For $d \in A^*$

set: $\tilde{d} = \tilde{\sigma} {}'' d$ and define $T_d \subset (\omega_2)^{<\omega}$

as before:

Def $t \in T_d \longleftrightarrow \mathcal{L} + \bigwedge\limits_{i < |t|} \underline{t_i} = \mathring{f}(\tilde{\delta}_i)$ is consistent,

where $\hat{d} = \langle \tilde{\delta}_i \mid i < \omega \rangle$.

<u>Claim</u> There is $d \in A^*$, $T \subset T_d$ s.t.

- $T \in \mathbb{N}'$
- $t = \sigma(d \upharpoonright n)$ where $t = \mathrm{stm}(T)$

pf.

For each $d \in A^*$ define a game $g(d)$ as follows. At stage i:

I plays an $\eta_i < \omega_2$ s.t. $\eta_i > \sup\limits_{h < i} \eta_h$

II plays, if possible, a ν_i s.t. $\langle \nu_0, \dots, \nu_i \rangle \in T_d$.

If $i < n$, II must choose $\nu_i = \sigma(\tilde{\delta}_i)$.

If $i \geq n$, II must choose $\nu_i > \eta_i$.

If at some i, II has no move, then I wins.

Otherwise II wins.

<u>Subclaim</u> II has a winning strategy for some $d \in A^*$.

prf.

Suppose not. For every $d \in A^*$, I then has a winning strategy S_d. Let C be the set of $\lambda < \omega_2$ s.t. for all $d \in A^*$, if $\nu_0, \dots, \nu_n < \lambda$ is any sequence of moves by II, then $S_d(\vec{\nu}) < \lambda$.

C is then club in ω_2. But then, setting $F(\xi) =$ the least $\lambda \in C$ s.t. $\xi < \lambda$, there is $m < \omega$ s.t. $\bigwedge i \geq m \; \gamma_i > \sup_{h<i} F(\gamma_h)$ ($\langle \gamma_i \mid i < \omega \rangle = c$ being our good sequence with $\bar{c} = \sigma^{-1}{}''c$.) Since A is dispersed at m, there is $d \in A^*$ s.t.

$\delta_m > \bar{\gamma}_m \wedge \bigwedge i \geq m \; \bigvee_j \; (\delta_i \leq \bar{\gamma}_j < \bar{\gamma}_{j+1} \leq \delta_{i+1})$.

But then $\langle \sigma(\delta_i) \mid i < \omega \rangle$ is a sequence of moves which defeats S_d. Contr!

QED (Subclaim)

Now let S be II's winning strategy for $d \in A^*$. Let $T =$ the set of finite sequences obtained by applying S to a finite sequence of moves by I. T has the desired properties.

QED (Claim)

Now let $G \ni T$ be $\mathbb{N}'$-generic over N. As before, letting $c_G = \langle \delta_i' \mid i < \omega \rangle$, $\tilde{d} = \langle \tilde{\delta}_i \mid i < \omega \rangle$, there is a unique smooth f s.t. $f(\tilde{\delta}_i) = \delta_i'$ for $i < \omega$. Moreover, $f \in N[G]$ (hence $f^* \in N[G]$). But then $\sigma' = f^* \circ \tilde{\sigma}$ has the desired properties.

QED (Lemma 5.7)

We are now ready to prove:

Thm 6 $\mathbb{N}'$ is ω_1-subproper.

The proof will again stretch over several sublemmas. In the following let N be viable, $\alpha < \omega_1$, and let $\pi = \langle \pi^i \mid i < \alpha \rangle$ be a tower for N with $N^i = N^{\pi_i}$ and $\pi^i(\theta^i, \mathbb{N}^{\prime i}) = \theta, \mathbb{N}'$ for $i < \alpha$. We must show that, for any finite $u \subset N^\alpha$ and any $T \in \mathbb{N}'^{\,0}$, π has a $\langle \theta, \mathbb{N}' \rangle, \mathbb{N}; G$ - revision σ which coincides with π on u and is s.t. $T \in G^0$.

We first show the existence of a somewhat simpler sort of revision:

Def By an <u>excellent</u> <u>revision</u> of π we mean a pair $\langle c, \sigma \rangle$ s.t.

- $c = \langle c^i \mid i \leq \alpha \rangle$
- $\sigma = \langle \sigma^i \mid i \leq \alpha \rangle$ is a revision of π
- $\langle c^\alpha, \sigma^\alpha \rangle$ is excellent for N
- $\langle c^i, \sigma^{i,i+1} \rangle$ is excellent for N^{i+1} $(i < \alpha)$
- $\sigma \in N[c^\alpha]$
- $\langle \sigma^{h,i+1} \mid h \leq i \rangle \in N^{i+1}[c^i]$ $(i < \alpha)$.

(Hence $c^\alpha \subset \omega_2$ is good and tame for N and $c^i \subset \omega_2^{N^{i+1}}$ is good and tame for N^{i+1}. It follows that $\langle c^i \mid i < \alpha \rangle \in N$ and $\langle c^h \mid h < i \rangle \in N^{i+1}$ for $i < \alpha$.)

Lemma 6.1 Let $c' \subset \omega_2$ be good and tame for N. Let $u \subset N^\alpha$ be finite. There is $\langle c, \sigma \rangle \in N[c']$ which is an excellent revision of π coinciding with π on u.
(Hence $N[c'] = N[c]$ by Lemma 5.1.)

proof. By induction on α.

Case 1 $\alpha = 0$. Trivial by Cor. 5.5

Case 2 $\alpha = \beta + 1$
Let $\langle c', \sigma' \rangle$ be excellent for N, N^α with $\sigma' \upharpoonright u = \pi^\alpha \upharpoonright u$. (This exists in $N[c']$ by Cor 5.5.)
Set $\tilde{u} = (\pi^\alpha)^{-1} '' u$, $\tilde{\pi} = \langle \pi^{i\alpha} \mid i \leq \beta \rangle$.
There exists $c' \in H_{\omega_1}$ which is good and tame for N^α (e.g. an N'^α - generic sequence). Hence, by the induction hypothesis there is $\langle \tilde{c}, \tilde{\sigma} \rangle \in H_{\omega_1}$ which is an excellent revision of $\tilde{\pi}$ coinciding with $\tilde{\pi}$ on $\tilde{u}$.

$$c^i = \begin{cases} c' & \text{if } i = \alpha \\ \tilde{c}^i & \text{if } i < \alpha \end{cases} \quad ; \quad \sigma^i = \begin{cases} \sigma' & \text{if } i = \alpha \\ \sigma' \circ \tilde{\sigma}^i & \text{if } i < \alpha. \end{cases}$$

$\langle c, \sigma \rangle$ has the desired properties.

QED (Case 2)

Case 3 α is a limit ordinal.

Let $\langle \alpha_i \mid i<\omega \rangle$ be monotone and cofinal in α s.t. $\alpha_0 = 0$ and $\alpha_i = \beta_i + 1$ for $i>0$. We successively pick $\langle c(i), \sigma(i)\rangle$ $(i<\omega)$ s.t. $\langle c(i), \sigma(i)\rangle$ is an excellent revision of the tower $\pi(i) = \langle \pi^{\nu, \alpha_{i+1}} \mid \alpha_i \leq \nu < \alpha_{i+1}\rangle$ for $N^{\alpha_{i+1}}$. Setting
$\tilde{\pi}(i) = \langle \pi^{\nu, \alpha_{i+1}} \mid \nu \leq \beta_{i+1}\rangle$ and defining $\langle \tilde{c}(i), \tilde{\sigma}(i)\rangle$ by:

$$\langle \tilde{c}(0), \tilde{\sigma}(0)\rangle = \langle c(0), \sigma(0)\rangle$$

$$\tilde{c}(i+1)^\nu = \begin{cases} c(i+1)^\nu & \text{if } \alpha_i \leq \nu < \alpha_{i+1} \\ \tilde{c}(i)^\nu & \text{if } i < \alpha_i \end{cases}$$

$$\tilde{\sigma}(i+1)^\nu = \begin{cases} \sigma(i+1)^\nu & \text{if } \alpha_i \leq \nu < \alpha_{i+1} \\ \sigma(i+1)^{\alpha_i} \circ \tilde{\sigma}(i)^\nu & \text{if } \nu < \alpha_i, \end{cases}$$

we see that $\langle \tilde{c}(i), \tilde{\sigma}(i)\rangle$ is an excellent revision of $\tilde{\pi}(i)$ for $i<\omega$.
While doing this construction we ensure that $\sigma(i)$ coincides with $\pi(i)$ on u_i, where $u_i \subset N^{\beta_{i+1}}$ is a finite set chosen so as to satisfy the conditions:

- $\mathcal{U}_h \subset \mathcal{U}_i$ for $h \le i$
- $(\pi^{\beta_{i+1}, \alpha})^{-1} {}'' \mathcal{U} \subset \mathcal{U}_i$
- Letting $\langle x_i \mid i < \omega \rangle$ be a fixed enumeration of N^α we have:
  $$(\pi^{\beta_{i+1}, \alpha})^{-1}(x_h) \in \mathcal{U}_i$$
  for $h < i$, whenever defined.
- Letting $\langle \langle z_i, \nu_i \rangle \mid i < \omega \rangle$ be a fixed enumeration of $\{\langle z, \nu \rangle \mid \nu < \alpha \wedge z \in N^\nu\}$, we have: If $h < i$, $\nu_h < \alpha_i$, then
  $$\pi^{\alpha_i, \beta_{i+1}}(\tilde{\sigma}(i-1)^{\nu_h}(z_h)) \in \mathcal{U}_i.$$

It is clear that we can set:

$$\tilde{C}^\nu = \tilde{C}(i)^\nu \text{ for } \nu < \alpha_{i+1}$$
$$\tilde{\sigma}^{\nu\tau} = \tilde{\sigma}(i)^{\nu\tau} \text{ for } \nu \le \tau < \alpha_{i+1}.$$

$\langle \tilde{\sigma}^{\nu\tau} \mid \nu \le \tau < \alpha \rangle$ is then a commutative, continuous system. By our construction we have ensured that:

- If $x \in N^\alpha$, there are $\nu < \alpha$, $\bar{x} \in N^\nu$ s.t. $x = \pi^{\nu\alpha}(\bar{x})$ and
  $$\pi^{\nu\tau}(\bar{x}) = \tilde{\sigma}^{\nu\tau}(\bar{x}) \text{ for } \nu \le \tau < \alpha,$$

- If $x \in N^\nu$, $\nu < \alpha$, there is $\tau \geq \nu$ s.t. $\tau < \alpha$ and for all ξ, if $\tau \leq \xi < \alpha$, then
  $\tilde{\sigma}^{\nu\xi}(x) = \pi^{\tau\xi}(\tilde{\sigma}^{\nu\tau}(x))$.

Arguing as before (cf. the proof of §2 Theorem 5), it follows that
$\langle N^\nu \mid \nu < \alpha \rangle$, $\langle \tilde{\sigma}^{\nu\tau} \mid \nu \leq \tau < \alpha \rangle$ has a unique direct limit of the form:
N^α, $\langle \tilde{\sigma}^\nu \mid \nu < \alpha \rangle$.

Now let $\langle c^*, \sigma^* \rangle \in N[c']$ be excellent.

Set:

$$c^\nu = \begin{cases} c^* & \text{if } \nu = \alpha \\ \tilde{c}^\nu & \text{if } \nu < \alpha \end{cases}$$

$$\sigma^\nu = \begin{cases} \sigma^* & \text{if } \nu = \alpha \\ \sigma^* \circ \tilde{\sigma}^\nu & \text{if } \nu < \alpha . \end{cases}$$

Then $\langle c, \sigma \rangle$ has the desired properties.

QED (Lemma 6.1)

We now define:

<u>Def</u> $\langle G, \sigma\rangle$ is a <u>superb</u> <u>revision</u> of π iff the following hold:

- σ is a revision of π
- G is $\mathbb{N}'$-generic over N
- $G^i = \sigma^{-1}{}''G$ is $\mathbb{N}'^i$-generic over N^i $(i \leq \alpha)$
- Set: $c^\alpha = c_G$; $c^i = c_{G^{i+1}}$ for $i < \alpha$. Then $\langle c, \sigma\rangle$ is an excellent revision of π.

<u>Remark</u> If $\langle G, \sigma\rangle$ is a superb revision of π, then σ is a $\langle \Theta, \mathbb{N}'\rangle, \mathbb{N}', G$ - -revision of π. Hence the assertion of Thm 6 will follow from the conjunction of Lemma 6.1 and:

<u>Lemma 6.2</u> Let $\langle c, \sigma\rangle$ be an excellent revision of π. Set $\bar{c}^\alpha = (\sigma^\alpha)^{-1}{}''c^\alpha$; $\bar{c}^i = (\sigma^{i,i+1})^{-1}{}''c^{i+1}$ $(i<\alpha)$. Define $A_{\bar{c}^\alpha}$ relative to N and $A_{\bar{c}^i}$ relative to N^{i+1} for $i < \alpha$. Let $u \subset N^\alpha$ be finite. Let $\bar{T} \in \mathbb{N}'^0$. Then there exist $T \in \mathbb{N}'$, $\bar{G} \in A_{\bar{c}^0}$ s.t.

- $stm(T) = \sigma^0(stm(\bar{T}))$
- If $G \ni T$ is $\mathbb{N}'$-generic over N, then there is $\sigma' \in N[G]$ s.t. $\langle G, \sigma'\rangle$ is a superb revision of π s.t. $G^0 = \bar{G}$, and σ' coincides with σ on u.

We prove Lemma 6.2 by induction on α.

Case 1 $\alpha = 0$.

Then $c = \langle c^0 \rangle$, $\sigma = \langle \sigma^0 \rangle$ where $\langle c^0, \sigma^0 \rangle$ is excellent for N. We know that $A = \{G \in A_{\bar{c}^0} \mid \bar{T} \in G\}$ is dispersed at $n = |\mathrm{stm}(\bar{T})|$. The conclusion follows by Lemma 5.7.

Case 2 $\alpha = \beta + 1$.

By the induction hypothesis there are $\bar{G} \in A_{\bar{c}^0}$, $\tilde{T} \in \mathbb{N}'^{\alpha}$ s.t. $\bar{T} \in \bar{G}$ and

- $\mathrm{stm}(\tilde{T}) = \sigma^{0\alpha}(\mathrm{stm}(\bar{T}))$
- If $\tilde{G} \ni \tilde{T}$ is $\mathbb{N}'^{\alpha}$ - generic over N^{α}, there is $\tilde{\sigma}' \in N^{\alpha}[\tilde{G}]$ s.t. $\langle \tilde{G}, \tilde{\sigma}' \rangle$ is a superb revision of $\tilde{\pi} = \langle \pi^{i\alpha} \mid i \leq \beta \rangle$ and $\tilde{\sigma}'$ coincides with $\tilde{\sigma} = \langle \sigma^{i\alpha} \mid i \leq \beta \rangle$ on $\tilde{u} = (\sigma^{\beta\alpha})^{-1} \text{''} u$.

But $\tilde{A} = \{G \in A_{\tilde{c}^{\alpha}} \mid \tilde{T} \in G\}$ is dispersed at $n = |\mathrm{stm}(\bar{T})|$. Since $\langle c^{\alpha}, \sigma^{\alpha} \rangle$ is excellent for N, there are $\hat{G} \in \hat{A}$, $T \in \mathbb{N}'$ s.t.

- $\mathrm{stm}(T) = \sigma^{\alpha}(\mathrm{stm}(\hat{T})) = \sigma^0(\mathrm{stm}(\bar{T}))$
- If $G \ni T$ is $\mathbb{N}'$ - generic over N, then there is $\sigma' \in N[G]$ s.t. $\sigma' : N^{\alpha} \prec N$; $\sigma'(\theta^{\alpha}, \mathbb{N}'^{\alpha}) = \theta, \mathbb{N}'$; $\sigma' \restriction u = \sigma^{\alpha} \restriction u$, $C^N_{\omega_1}(\mathrm{rng}\, \sigma') = C^N_{\omega_1}(\mathrm{rng}\, \sigma^{\alpha}) = C^N_{\omega_1}(\mathrm{rng}\, \pi)$, and $\sigma' \text{''} \tilde{G} \subset G$.

Now let $G \ni T$ be $\mathbb{N}'$-generic over N.
Let $\sigma' \in N[G]$, $\tilde{G} \in \tilde{A}$ be as above. Since $\tilde{G} \ni \tilde{T}$, there is $\tilde{\sigma} \in N^{\alpha}[\tilde{G}]$, $\bar{G} \in A_{\tilde{\sigma}^0}$ as above.

Set: $\sigma''^{\nu} = \begin{cases} \sigma' & \text{if } \nu = \alpha \\ \sigma' \tilde{\sigma}^{\nu} & \text{if } \nu < \alpha \end{cases}$

Then $\langle G, \sigma'' \rangle$ is a superb revision of π and σ'' coincides with σ on u. QED (Case 2)

<u>Case 3</u> α is a limit ordinal.

We know that there is $d \subset \omega_1$ which is good and tame for N and s.t.

$$d \in N[c^{\alpha}], \quad d \subset \operatorname{rng}(\sigma^0).$$

Hence $N[c^{\alpha}] = N[d]$ and we may suppose w.l.o.g. that $c^{\alpha} = d$. Set:

$$d = \langle \delta_i \mid i < \omega \rangle$$

$$d^{\nu} = (\sigma^{\nu})^{-1} {}'' d = \langle \delta_i^{\nu} \mid i < \omega \rangle \quad (\nu \leq \alpha),$$

The notion of a <u>superb</u> α-<u>revision</u> $\langle \tilde{G}, \tilde{\sigma} \rangle$ is defined like that of a superb revision with N^{α} in place of N and $\tilde{\pi} = \langle \pi^{i\alpha} \mid i \leq \alpha \rangle$ in place of π.

Equivalently:

<u>Def</u> $\langle \tilde{G}, \tilde{\sigma} \rangle$ is a <u>superb</u> α - <u>revision</u> of $\tilde{\pi} = \langle \pi^{i\alpha} \mid i \leq \alpha \rangle$ iff

- $\tilde{G}$ is $\mathbb{N}^{\prime\alpha}$ - generic over N
- $\tilde{\sigma} = \langle \tilde{\sigma}^i \mid i < \alpha \rangle$ where $\tilde{\sigma}^i : N^i \prec N$
- Set $\tilde{G}^i = (\tilde{\sigma}^i)^{-1} {}'' \tilde{G}$ $(i < \alpha)$ and $\tilde{\sigma}^{ij} = (\tilde{\sigma}^j)^{-1} \circ \tilde{\sigma}^i$ $(i \leq j < \alpha)$; Then $\langle \tilde{G}^{i+1}, \langle \tilde{\sigma}^{h, i+1} \mid h \leq i \rangle \rangle$ is a superb revision of $\langle \pi^{h, i+1} \mid h \leq i \rangle$ for $i < \alpha$.

<u>Lemma 6.2.1</u> Let $\overline{T} \in \mathbb{N}^{\prime 0}$. Let $u \subset N^\alpha$ be finite. There are $\overline{G} \in A_{\overline{c}^0}$ and $\langle \tilde{G}, \tilde{\sigma} \rangle$ s.t.

(i) $\overline{T} \in \overline{G}$

(ii) $\langle \tilde{G}, \tilde{\sigma} \rangle$ is a superb α - revision of $\tilde{\pi}$

(iii) Let $n = \mathrm{lth}(\overline{T})$. Let $c_{\tilde{G}} = \langle \tilde{\gamma}_i \ (i < \omega) \rangle$ and Then $c_{\tilde{G}} \restriction n = \sigma^{0,\alpha}(c_{\overline{G}} \restriction n)$ $\bigwedge i \geq n \bigvee j \left(\sup_{h < i} \tilde{\gamma}_h \leq \delta_j^\alpha < \delta_{j+1}^\alpha \leq \tilde{\gamma}_i \right)$.

(iv) $\tilde{\sigma}$ coincides with $\langle \sigma^{i\alpha} \mid i < \alpha \rangle$ on u (i.e. $(\sigma^{i\alpha})^{-1} \restriction u \subset (\tilde{\sigma}^{i\alpha})^{-1} \restriction u$ for $i < \alpha$),

(Note By (c) we have: $\tilde{G} \in A_{\overline{c}^\alpha}$.)

proof of Lemma 6.2.1

We successively construct $\langle T_i, G_i, \sigma_i \rangle$ $(i<\omega)$ s.t.

(a) $T_0 = \bar{\bar{T}} \in G_0$, $\sigma_0 = \emptyset$

(b) $T_{i+1} \in G_{i+1}$ and $\langle G_{i+1}, \sigma_{i+1} \rangle$ is a superb revision of $\pi_i =_{\mathrm{Df}} \langle \pi^{h, \alpha_{i+1}} \mid \alpha_i \leq h < \alpha_{i+1} \rangle$ s.t. $G_{i+1}^{\alpha_i} = G_i$ and σ_{i+1} coincides with $\hat{\sigma}_i = \langle \sigma^{h, \alpha_{i+1}} \mid \alpha_i \leq h < \alpha_{i+1} \rangle$ on $u_i \subset N^{\beta_{i+1}}$. ($u_i$ must still be determined.)

(c) $|stm(T_i)| = n + i$

(d) $c_{G_{i+1}} \restriction n+i = \sigma^{\alpha_i, \alpha_{i+1}}(stm(T_i))$

$T_0 = \bar{\bar{T}}$ is given. Now let T_i be given and let $\langle T_h, G_h, \sigma_h \rangle$ be given for $h<i$ satisfying (a)-(d). We pick G_i, T_i' s.t. $G_i \in A_{\bar{C}} \alpha_i$ and $T_i' \in \mathbb{N}^{\prime \alpha_{i+1}}$ s.t.

(e) $stm(T_i') = \sigma^{\alpha_i, \alpha_{i+1}}(stm(T_i))$

(f) Let $G \ni T_i'$ be $\mathbb{N}^{\prime \alpha_{i+1}}$ - generic over $N^{\alpha_{i+1}}$. There is $\sigma' \in N^{\alpha_{i+1}}[G]$ s.t.

- $\langle G, \sigma' \rangle$ is a superb revision of π_i
- $G^{\alpha_i} = G_i$
- σ' coincides with $\hat{\sigma}_i$ on u_i.

It is possible to pick G_i, T_i' by the induction hypothesis on d.

If $i = 0$, we set $\sigma_i = \emptyset$. If $i = h+1$, let $\sigma_i \in N^{d_i}[G_i]$ s.t. $\langle G_i, \sigma_i \rangle$ is a superb revision of π_h, $G_i^{d_h} = G_h$, and σ_i coincides with $\hat{\sigma}_h$ on u_h. (σ_i exists, since (b) holds at h.)

We then define T_{i+1} as a subtree of T_i' as follows: Let $s' = \mathrm{stm}(T_i')$. Let $s'^\frown\langle\nu\rangle \in T_i'$ s.t.

$$\delta_i^{d_{i+1}} < \delta_{i+1}^{d_{i+1}} \le \nu.$$

$$\bigvee_i \sup_{h<i} s'(h) \le$$

Set $T_{i+1} = T_i'(s'^\frown\langle\nu\rangle) =_{\mathrm{Df}}$ the set of $t \in T_i'$ s.t. $t \le s'^\frown\langle\nu\rangle$ or $s'^\frown\langle\nu\rangle \le t$ in T_i'.

This defines $\langle T_i, G_i, \sigma_i, T_i' \rangle$ $(i < \omega)$ modulo the choice of the finite sets $u_i \subset N^{\beta_{i+1}}$, which we must still determine.

If we then set:

$$\tilde{\sigma}^h_{i+1} = \begin{cases} \sigma^h_{i+1} & \text{if } \alpha_i \leq h < \alpha_{i+1} \\ \sigma^{\alpha_i}_{i+1} \circ \tilde{\sigma}^h_i & \text{for } h < \alpha_i \end{cases},$$

we see that $\langle G_{i+1}, \tilde{\sigma}_{i+1} \rangle$ is a superb revision of $\tilde{\pi}_i = \langle \pi^{h,\alpha_{i+1}} \mid h \leq \beta_{i+1} \rangle$.

It is easily seen that we can define: $\tilde{G}^h = \tilde{G}^h_i$ for $h < \alpha_i$,

$$\tilde{\sigma}^{h,i'} = (\tilde{\sigma}^{i'}_i) \circ \tilde{\sigma}^h_i \quad \text{for } h \leq i' < \alpha_i,$$

the choice of i being immaterial.

The finite sets $u_i \subset N^{\beta_{i+1}}$ are chosen so as to satisfy:

- $u_h \subset u_i$ for $h < i$
- $\tilde{u}_i \subset u_i$, where $\tilde{u}_i = (\sigma^{\beta_{i+1}, \alpha})^{-1} {}'' u$
- Let $\langle x_i \mid i < \omega \rangle$ enumerate N^α. Then $(\sigma^{\beta_{i+1}, \alpha})^{-1}(x_h) \in u_i$ if defined and $h < i$.
- Let $\langle \langle z_i, \nu_i \rangle \mid i < \omega \rangle$ enumerate $\{\langle z, \nu \rangle \mid \nu < \alpha \wedge z \in N^\nu\}$. If $h < i$ and $\nu_h < \beta_{i+1}$, then $\tilde{\sigma}^{\nu_h, \beta_{i+1}}(z_i) \in u_i$.

Just as before the directed system $\langle N^h \mid h < \alpha \rangle$, $\langle \tilde{\sigma}^{hi} \mid h \leq i < \alpha \rangle$ has a direct limit of the form:

$$N^\alpha, \langle \tilde{\sigma}^h \mid h < \alpha \rangle.$$

Set $\tilde{G} = \bigcup_{h<\alpha} \tilde{\sigma}^h{}''\tilde{G}^h$, $\bar{G} = \tilde{G}^0$

<u>Claim</u> $\bar{G}, \tilde{G}, \tilde{\sigma} = \langle \tilde{\sigma}^h \mid h < \alpha \rangle$ satisfy (i) — (iv) of Lemma 6.2.1

proof

$\tilde{G}$ is easily seen to be $\mathbb{N}'^\alpha$ - generic over N^α, given that $\tilde{G}^h$ is $\mathbb{N}'^h$ - generic over N^h. But then (ii) follows trivially, (i), (iv) are also straightforward. We prove (iii)

Let $c_{\tilde{G}} = \langle \tilde{\gamma}_i \mid i < \omega \rangle$, $c_{\tilde{G}^h} = \langle \tilde{\gamma}_i^h \mid i < \omega \rangle$ for $h < \alpha$. Then $\tilde{\sigma}^h(\tilde{\gamma}_i^h) = \tilde{\gamma}_i$. Now let:

(1) $\nu = \tilde{\gamma}^{\alpha_{i+1}}_{m+1}$ = the maximal point in $\operatorname{stm}(T_{i+1})$.

By our construction:

(2) $\tilde{\sigma}^{\alpha_{i+1}, h}(\nu) = \sigma^{\alpha_{i+1}, h}(\nu)$

for $\alpha_{i+1} \leq h < \alpha$.

Let $\nu_h = \sigma^{\alpha_{i+1} h}(\nu)$ for $\alpha_{i+1} \leq h < \alpha$.

Then for $h < \alpha$:

(3) $\tilde{\sigma}^h(\nu_h) = \sigma^h(\nu_h) = \nu'_\alpha =_{Df} \tilde{\gamma}_{m+i}$

Hence:

(4) $\tilde{\sigma}^h \upharpoonright (\nu_h + 1) = \sigma^h \upharpoonright (\nu_h + 1)$ for $h < \alpha$

(To see this, let f_h = the N^h-least map of $\omega_1^{N^h}$ onto $\nu_h + 1$ for $h \leq \alpha$. Then $\tilde{\sigma}^h(f_h) = \sigma^h(f_h) = f_\alpha$. Hence $\tilde{\sigma}^h \upharpoonright (\nu_h + 1) = \sigma^h \upharpoonright (\nu_h + 1) = f_\alpha \circ (f_h^{-1})$.)

We arranged, however, that
$\sup\limits_{l<i} \tilde{\gamma}^{\alpha_{i+1}} \leq \delta_l^{\alpha_{i+1}} < \delta_{l+1}^{\alpha_{i+1}} \leq \nu$.

By (4) we conclude:
$\sup\limits_{l<i} \tilde{\gamma}_{m+l} \leq \delta_l^\alpha < \delta_i^\alpha < \nu_\alpha = \tilde{\gamma}_{m+i}$.

QED (Lemma 6.2.1)

Now fix $T \in \mathbb{N}'^0$, $u \subset N^\alpha$ as in Lemma 6.2.1 and set:

A = the set of $\tilde{G}$ s.t. for some $\tilde{\sigma}$, the pair $\langle \tilde{G}, \tilde{\sigma} \rangle$ satisfies (i) – (iv) of Lemma 6.2.1.

Then $A \subset A_{\bar{c}^\alpha}$, since $\bar{c}^\alpha = \langle \delta_i \mid i < \omega \rangle$.

We prove:

<u>Lemma 6.2.2</u> A is dispersed at n (where $n = |\mathrm{stm}(\bar{T})|$).

Proof.

Let $s = \mathrm{stm}(\bar{T})$. Let $l < \omega$. Pick ν s.t. $s^\frown\langle\nu\rangle \in \bar{T}$, $\nu > \delta_l^0$, and there ~~is~~ $j < \omega$ s.t. $\sup_{h<n} s(h) \leq \delta_j^0 < \delta_{j+1}^0 < \nu$.

Set $\bar{T}' = \bar{T}_{(s^\frown\langle\nu\rangle)}$ = the tree of $t \in \bar{T}$ s.t. $t \leq s^\frown\langle\nu\rangle$ or $s^\frown\langle\nu\rangle \leq t$ in $\bar{T}$.

Apply Lemma 6.2.1 to $\bar{T}'$, getting $\langle \tilde{G}', \tilde{\sigma}' \rangle$. Then $\tilde{\sigma}^0(\nu) > \delta_l^\alpha$ and $\sup_{h<n} \tilde{\gamma}_h^\alpha \leq \delta_j^\alpha < \delta_{j+1}^\alpha < \tilde{\sigma}^0(\nu)$, where $\nu = \tilde{\gamma}_n$ (and $C_{\tilde{G}} = \langle \tilde{\gamma}_i \mid i < \omega \rangle$).

Since $\mathrm{stm}(\bar{T}') = n+1$, it follows readily that $\tilde{G} \in A$ with $\tilde{\gamma}_n > \delta_l^\alpha$. QED (6.2.2)

But then by Lemma 5.7 there is $\tilde{G} \in A$, $T \in \mathbb{N}'$ s.t.

(a) $stm(T) = \sigma^{\alpha}(c_{\tilde{G}} \upharpoonright m)$

(b) If $G \ni T$ is $\mathbb{N}'$-generic over N, there is $\sigma'' \in N[G]$ s.t. $\sigma'' : N^{\alpha} \prec N$, $\sigma'' \upharpoonright u = \sigma^{\alpha} \upharpoonright u$, $\sigma''(\theta^{\alpha}, \mathbb{N}'^{\alpha}) = \theta, \mathbb{N}'$, $C^{N}_{\omega_1}(rng\, \sigma'') = C^{N}_{\omega_1}(rng\, \sigma^{\alpha})$, and $\sigma'' {}'' \tilde{G} \subset G$.

Let $G \ni T$ be $\mathbb{N}'$-generic over N. Let $\tilde{G} \in A$ be as above. By the definition of A there are $\bar{G} \in A_{\bar{c}}{}^{\alpha}$ and $\tilde{\sigma}$ s.t. (i) — (iv) of Lemma 6.2.1 hold. Define $\sigma' \in N[G]$ by: $\sigma'^{\nu} = \begin{cases} \sigma'' & \text{if } \nu = \alpha \\ \sigma'' \circ \tilde{\sigma}^{\nu} & \text{if } \nu < \alpha \end{cases}$. Then σ' verifies Lemma 6.2.

QED (Thm 6)