§ 3 The Subcomplete Forcing Axiom (SCFA)

There are many forcings – e.g. Namba forcing – which add new countable sets of ordinals without adding new reals. One can hope to encompass some of these forcings by changing the clause $\pi''\bar{G} \subset G$ in the definition of "complete forcing" to:

(*) There is $\sigma \in V[G]$ s.t. $\sigma : \bar{N} \prec N$ and $\sigma''\bar{G} \subset G$.

In the case e.g. of Namba forcing it is clear that we must have $\sigma \neq \pi$. This means that the condition: "$\bar{N}$ is countable and transitive" in the definition of "complete forcing" must be strengthened. Otherwise we could take $\bar{N}$ as the transitivization of the set X of N-definable elements of N. But then the transitivizing function π would be the only possible $\pi : \bar{N} \prec N$. Hence $\pi = \sigma$. To avoid this, we impose on $\bar{N}$ the further require-ment of fullness:

Def $\bar{N} = L^{\bar{A}}_{\bar{\tau}}$ is *full* iff $\bar{N}$ is a ZFC$^-$ model and there is $\gamma > 0$ s.t. $L_\gamma(\bar{N})$ is a ZFC$^-$ model and $\bar{N}$ is regular in $L_\gamma(\bar{N})$ (i.e. if $f \in L_\gamma(\bar{N})$, $x \in \bar{N}$, and $f : x \to \bar{N}$, then $f \in \bar{N}$).
(Note: If $\bar{N}$ is a ZFC$^-$ model, then saying that $\bar{N}$ is regular in $L_\gamma(\bar{N})$ is equivalent to saying that $\bar{N}$ models second order ZFC$^-$ in $L_\gamma(\bar{N})$.)

Our intention is to broaden the notion of complete forcing to a notion for which we still have a good iteration theorem. To achieve this we shall have to impose some strong connection between the old map π and the new map σ. For one thing, given any finite $u \subset \bar{N}$, we want to be able to choose σ s.t. it coincides with π on u. We also impose the following requirement.

Def For $X \subset N$, $\delta \in N$ set:
$C_\delta(X)$ = the smallest $Y \prec N$ s.t. $X \cup \delta \subset Y$.

Def Let $\mathbb{B}$ be a complete BA.

$\delta(\mathbb{B})$ = the smallest cardinality of a dense subset of $\mathbb{B}$.

We require:

$C_\delta(\operatorname{rng}\sigma) = C_\delta(\operatorname{rng}\pi)$, where $\delta = \delta(\mathbb{B})$.

(Note If we were working with sets of conditions rather than Boolean algebras, we would normally take $\mathbb{P}$ to be of minimal cardinality and could write $\operatorname{card}(\mathbb{P})$ instead of $\delta(\mathbb{B})$.)

Putting all of this together:

Def $\mathbb{B}$ is <u>subcomplete</u> iff there is θ s.t. $\mathbb{B} \in H_\theta$ and whenever $N = L^A_\tau$ is a ZFC$^-$ model with $H_\theta \subset N$ and $\theta < \tau$, then:

Let $\pi : \bar{N} \prec N$, where $\bar{N}$ is countable and full. Let $\pi(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}) = \theta, \mathbb{B}, s$. Let $\bar{G}$ be $\bar{\mathbb{B}}$-generic over $\bar{N}$. Then there is $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic, then there is $\sigma \in V[G]$ s.t.

(a) $\sigma : \bar{N} \prec N$

(b) $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}) = \theta, \mathbb{B}, s$

(c) $C^N_\delta(\operatorname{rng}\sigma) = C^N_\delta(\operatorname{rng}\pi)$

(d) $\sigma''\bar{G} \subset G$.

We again say that θ <u>witnesses</u> the subcompleteness of $\mathbb{B}$ if θ is as in the definition. We say that θ <u>verifies</u> the subcompleteness of $\mathbb{B}$ if every cardinal $\theta' \geq \theta$ witnesses subcompleteness. It is clear that subcompleteness is locally based in the same sense as before and that, if θ witnesses subcompleteness, then $(2^{\theta})^{+}$ verifies it.

[<u>Note</u> This definition differs from the definition given in [IT] in the following respects:

(i) In the earlier definition we required that $N = L_{\tau}^{A}$ for a regular cardinal $\tau > \theta$. We weakened this requirement (thus making the definition harder to satisfy) in order to make the definition locally based.

(ii) The requirement:

$$C_{\delta}^{N}(\operatorname{rng}\sigma) = C_{\delta}^{N}(\operatorname{rng}\pi)$$

in the conclusion replaces a weaker one, (thus again making our new definition harder to meet). We made the change because in all cases in which we have hitherto verified the subcompleteness of a specific forcing, we essentially verified the weaker condition by first verifying the stronger one. The stronger condition also seems to work more smoothly in proofs of iteration theorems.]

[Note If we formulated this for sets of conditions rather than complete Boolean algebras, we would normally choose the set $\mathbb{P}$ to be of minimal cardinality. Thus we could write 'card($\mathbb{P}$)' in place of '$\delta(\mathbb{B})$'.]

The two step iteration theorem for subcomplete forcing reads:

<u>Thm 1</u> Let $\mathbb{A}$ be subcomplete and $\Vdash_{\mathbb{A}} \mathring{\mathbb{B}}$ is subcomplete. Then $\mathbb{A} * \mathring{\mathbb{B}}$ is subcomplete.

For longer iterations we must use the <u>revised countable supports</u> (RCS) rather than countable supports. The iteration theorem reads:

Thm 2 Let $\langle \mathbb{B}_i \mid i < \alpha \rangle$ be an RCS iteration s.t. $\mathbb{B}_0 = 2$ and

(a) $\mathbb{B}_i \neq \mathbb{B}_{i+1}$

(b) $\Vdash_i (\check{\mathbb{B}}_{i+1} / \dot{G}$ is subcomplete$)$

(c) $\Vdash_{i+1} (\delta(\check{\mathbb{B}}_i)$ has cardinality $\leq \omega_1)$

Then every $\mathbb{B}_i$ is subcomplete.

This is proven in [IT] §3. (However, the proof needs slight changes because of our change in the definition of "subcomplete".)

The subcomplete forcing axiom (SCFA) says that MA($\mathbb{B}$) holds for every subcomplete $\mathbb{B}$, SCFA$^+$ says that MA$^+$($\mathbb{B}$) for all subcomplete $\mathbb{B}$. By the iteration theorems and the fact that subcompleteness is locally based, we get:

Thm 3 SCFA$^+$ + CH is consistent relative to a supercompact.

proof (sketch)

We repeat the previous proof, using

an RCS iteration $\langle \mathbb{B}_i \mid i \le \kappa \rangle$ whose successor stages are defined exactly as before. Letting G be $\mathbb{B}_\kappa$-generic, it follows as before that $V[G]$ models SCFA$^+$ + CH.

QED (Thm 3)

We again refer to $V[G]$ as the "natural model". As before, we could have done a prior application of Silver forcing to make GCH hold in V, in which case GCH also holds in $V[G]$. $\diamondsuit$ also holds in $V[G]$. This follows as before, using:

<u>Lemma 4</u> Assume $\diamondsuit$. Let $\mathbb{B}$ be subcomplete. Then $\Vdash_{\mathbb{B}} \diamondsuit$.

prf.

Let $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. Define a $\diamondsuit'$-sequence $S' = \langle S'_\alpha \mid \alpha < \omega_1 \rangle$ by: Choose $f_\alpha : \omega \xrightarrow{\text{onto}} \omega + \alpha$. Let τ_α be least s.t. $L_{\tau_\alpha}[S_\alpha, f_\alpha]$ is a ZFC$^-$ model. Set:

$$S'_\alpha = \mathcal{P}(\alpha) \cap W_\alpha .$$

Let G be $\mathbb{B}$ - generic.

Claim S' remains a $\diamondsuit'$ - sequence in $V[G]$.

prf. Suppose not.

Then there are $C, A \in V[G]$ s.t. C is club in ω_1, $B \subset \omega_1$ and $B \cap \alpha \notin S'_\alpha$ for $\alpha \in C$.

Let $C = \dot{C}^G$, $B = \dot{B}^G$ and let $b \in G$ force the above statement. Let θ verify the subcompleteness of $\mathbb{B}$. Let $N = L^A_\tau$ be a ZFC^- model s.t. $H_\theta \subset N$ and $\theta < \tau$. (where θ verifies the subcompleteness of $\mathbb{B}$) Assume w.l.o.g. that $\dot{C}, \dot{B} \in N$. Let $X \prec N$ s.t. $\omega_1 \subset X$, $\bar{\bar{X}} = \omega_1$, and $\dot{C}, \dot{B}, \theta, \mathbb{B}, b \in X$. Let $f : \omega_1 \leftrightarrow X$ and set: $D = \{\alpha \mid f''\alpha \prec N.\}$

Then D is club in ω_1. Set:

$$E = \{\langle\nu,\tau\rangle \mid f(\nu) \in f(\tau)\}$$

$$\tilde{A} = \{\nu \mid f(\nu) \in A\};$$

$$\tilde{E} = \{\langle\nu,i\rangle \mid (\nu \in E \wedge i = 0) \vee (\nu \in \tilde{A} \wedge i = 1)\}.$$

Let $\alpha \in D$ s.t. $\dot{C}, \dot{B}, \theta, \mathbb{B}, b \in f''\alpha$ and $\tilde{E} \cap \alpha \in S'$. Let $\pi : \bar{N} \overset{\sim}{\leftrightarrow} f''\alpha$. Then $\bar{N} \in W_\alpha$. Let $\pi(\bar{\dot{C}}, \bar{\dot{B}}, \bar{\theta}, \bar{\mathbb{B}}, \bar{b}) = \dot{C}, \dot{B}, \theta, \mathbb{B}, b$.

We can easily choose α s.t. $\bar{N}$ is full and $\alpha = \omega_1^{\bar{N}}$.

Since $\bar{N}$ is countable in W_α there is $\bar{G} \in W_\alpha$ s.t. $\bar{G}$ is $\bar{\mathbb{B}}$-generic over $\bar{N}$ and $\bar{b} \in \bar{G}$. Set: $\bar{C} = \dot{\bar{C}}^{\bar{G}}$, $\bar{B} = \dot{\bar{B}}^{\bar{G}}$. Let G be $\mathbb{B}$-generic s.t. there is $\sigma \in V[G]$ with:

(a) $\sigma : \bar{N} \prec N$

(b) $\sigma(\dot{\bar{C}}, \dot{\bar{B}}, \bar{\Theta}, \bar{\mathbb{B}}, \bar{b}) = C, B, \Theta, \mathbb{B}, b$

(c) $\sigma '' \bar{G} \subset G$.

Then σ extends uniquely to:

$$\sigma' : \bar{N}[\bar{G}] \prec N[G] \text{ s.t. } \sigma'(\bar{G}) = G.$$

Let $C = \dot{C}^G$, $B = \dot{B}^G$. Then $\sigma'(\bar{C}, \bar{B}) = C, B$. But $\bar{C} = C \cap \alpha$ is unbounded in α; hence $\alpha \in C$. Hence $\bar{B} = B \cap \alpha \notin S'_\alpha$, since $b = \sigma(\bar{b}) \in G$. But $\bar{B} = \dot{\bar{B}}^{\bar{G}} \in W_\alpha$; hence $\bar{B} \in S'_\alpha$. Contr! QED (Lemma 4)

Hence;

<u>Corollary 5</u> $SCFA^+ + GCH + \diamondsuit$ is consistent relative to a supercompact.

We also note that by a similar proof:

Lemma 7 Let T be a Souslin tree. Then T remains Souslin under subcomplete forcing.

pf.

Let $\mathbb{B}$ be subcomplete. Let G be $\mathbb{B}$-generic.

Claim T remains Souslin in $V[G]$.

Suppose not. Then there is $D \in V[G]$ which is a maximal antichain of size ω_1 in T. Let $D = \dot{D}^G$ and let $b \in G$ force $\dot{D}$ to be a maximal antichain of size ω_1. Let θ verify the subcompleteness of $\mathbb{B}$ and let $N = L_\tau^A$ be a ZFC$^-$ model s.t. $H_\theta \subset N$ and $\theta < \tau$. Let $\dot{D} \in N$. Let $\pi : \bar{N} \prec N$ s.t. $\bar{N}$ is countable and full and $\pi(\bar{\theta}, \bar{\mathbb{B}}, \bar{\dot{D}}, \bar{b}, \bar{T}) = \theta, \mathbb{B}, \dot{D}, b, T$.

Then $\bar{T} = T|\alpha$ is the restriction of T to points of rank $< \alpha$, where $\alpha = \omega_1^{\bar{N}}$. Let W be a countable, transitive ZFC$^-$ model, where $\bar{N}, T|(\alpha+1) \in W$. Let $\bar{G}$ be $\bar{\mathbb{B}}$ generic over W (hence over $\bar{N}$). Let x be a point of rank α in T. Set $b_x = \{z \mid z < x \text{ in } T\}$. Then $\bar{G}$ is $\bar{\mathbb{B}}$-generic over $N[b_x]$, where b_x is $\bar{T}$-generic over N. By the product lemma b_x is $\bar{T}$-generic over $N[\bar{G}]$.

Set $\bar{D} = \dot{D}^{\bar{G}}$. Since $\bar{D}$ is a maximal antichain in $\bar{T}$, we conclude: $b_x \cap \bar{D} \neq \emptyset$. Thus every point of rank α in T lies above an element of $\bar{D}$. Hence $\bar{D}$ is a maximal antichain in T.

Now let G be $\mathbb{B}$-generic and $\sigma \in V[G]$ s.t. $\sigma : \bar{N} \prec N$, $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{T}, \dot{\bar{D}}, \bar{b}) = \theta, \mathbb{B}, T, \dot{D}, b$, and $\sigma '' \bar{G} \subset G$. σ then extends to $\sigma' : \bar{N}[\bar{G}] \prec N[G]$ s.t. $\sigma'(\bar{G}) = G$. Hence $\sigma'(\bar{D}) = D$, where $D = \dot{D}^G$ is a maximal antichain of size ω_1. But $D = \bar{D}$ since $\bar{D} \subset D$ is maximal. Hence D is countable. Contr!

QED (Lemma 7)

It follows as before that the natural model is particularly rich in Souslin trees.

We now consider some consequences of SCFA and SCFA$^+$.

Since complete forcings are subcomplete, the above mentioned consequences of CFA$^+$ follow from SCFA$^+$. We now show that some of the most striking consequences of MM also follow from SCFA.

Lemma 8 SCFA implies Friedman's property:
Let $\tau > \omega_1$ be regular. Let $A \subset \tau$ be a stationary set of ω-cofinal points. There is a normal function $f : \omega_1 \to A$.

proof (sketch).
Let $\mathbb{P}_A$ be the natural conditions for adding a normal $f : \omega_1 \to A$ s.t. $\sup f''\omega_1 = \tau$. In [SPSC] we define $\mathbb{P}_A$ and prove that it is subcomplete. By §1 Lemma 1, there exist $\pi : \bar{H} \prec H_{\alpha^+}$, G s.t.

- $\bar{H}$ is transitive, $\omega_1 \subset \bar{H}$
- $A \in \operatorname{rng}(\pi)$
- G is $\mathbb{P}_{\bar{A}}$-generic over $\bar{H}$, where $\pi(\mathbb{P}_{\bar{A}}) = \mathbb{P}_A$.

Then $\dot{f} \in \operatorname{rng}(\pi)$, where $\dot{f}$ is the canonical name for the function f to be added. Set: $f = \pi \circ \pi^{-1}(\dot{f})^G$. Then f has the desired property.

QED (Lemma 8)

<u>Corollary 8.1</u> Assume SCFA. Then

(a) $\square_\beta$ is false for cardinals $\beta > \omega$.

(b) Global $\square$ is false below κ, where κ is inaccessible.

prf.

Suppose not. Let $\kappa = \beta^+$ in case (a), or be inaccessible in case (b). Then there is a club $D \subset \kappa$ and a sequence $\langle C_\lambda \mid \lambda \in D \wedge cf(\lambda) = \omega \rangle$ s.t.

- $C_\lambda \subset \lambda$ is club in λ
- $otp(C_\lambda) < \lambda$
- If μ is a limit point of C_λ, then $C_\mu = \mu \cap C_\lambda$

Then $A = \{\lambda \in D \mid otp(C_\lambda) = \eta\}$ is stationary for some η. Let $f: \omega_1 \to A$ be normal. Set $C = rng(f)$. Let $\mu = \sup C$. Then C, C_μ are club in μ, but C, C_μ have at most one point in common. Contr! QED

By essentially the same proof as Lemma 8:

<u>Lemma 9</u> Assume SCFA. Let $\tau > \omega_1$ be regular. Let $A_i \subset \tau$ be a stationary set of ω-cofinal points for $i < \omega_1$. Let $\langle D_i \mid i < \omega_1 \rangle$ be a partition of ω_1 into disjoint stationary sets. Then there is a normal function $f: \omega_1 \to \bigcup_i A_i$ s.t. $f(j) \in A_i$ for $j \in D_i$.

Hence:

<u>Corollary 9.1</u> Assume SCFA. Let τ be as above. Then $\tau^{\omega_1} = \tau$.

proof.

Let $\langle A_\xi \mid \xi < \tau \rangle$ partition $\{\lambda < \tau \mid cf(\lambda) = \omega\}$ into τ many disjoint ~~stationary~~ sets. For each $a \in [\tau]^{\omega_1}$ let $\langle \xi_i \mid i < \omega_1 \rangle$ enumerate a and let $f : \omega_1 \longrightarrow \bigcup_{i<\omega_1} A_{\xi_i}$ s.t. $f(j) \in A_{\xi_i}$ if $j \in D_i$, where f is normal and $\langle D_i \mid i < \omega_1 \rangle$ partitions ω_1 into stationary sets. Set: $\lambda = \sup f''\omega_1$.

$$a = B_\lambda =_{Df} \{\xi \mid A_\xi \cap \lambda \text{ is stationary in } \lambda\}$$

Hence $[\tau]^{\omega_1} \subset \{B_\lambda \mid \lambda < \tau\}$. QED (9.1)

Assuming SCFA we then get

<u>Corollary 9.2</u> If $cf(\beta) \leq \omega_1 < \beta$, $2^{\underline{\beta}} \leq \beta^+$, Then $2^\beta = \beta^+$.

pf. $2^\beta = (2^{\underline{\beta}})^{cf(\beta)} \leq (\beta^+)^{\omega_1} = \beta^+$

Hence by Silver's theorem:

<u>Corollary 9.3</u> If β is a strong limit cardinal and $\omega < cf(\beta) < \beta$, then $2^\beta = \beta^+$.

All of the above consequences of SCFA were known consequences of MM. We now make a new application of SCFA + CH.

Problem Let $\lambda > \omega_1$ be a cardinal. Let $X \prec H_\lambda$ s.t. $\overline{\overline{X}} = \omega_1 \subset X$. Set:

$$R_X = \{\tau \mid \omega_1 < \tau < \lambda \wedge \tau \text{ is regular} \wedge \tau \in X\}$$

For limit $\alpha \in X$ set: $cf_X(\alpha) = cf(otp(\alpha \cap X))$

(Hence $cf_X(\tau) \in \{\omega, \omega_1\}$.)

What forms can the function $cf_X \upharpoonright R_X$ take?

If we assume $V = L$, the possibilities are very limited, since it can be shown by a fine structural argument that:

Fact Assume $V = L$. Let α be a limit ordinal with $\overline{\overline{\alpha}} < \alpha$. There is $\rho < \alpha$ s.t. $cf(\alpha) = cf(\tau)$ for all $\tau \in (\rho, \alpha)$ which are regular in L_α.

Transitivising X then gives:

Lemma 10 Assume $V = L$. Let X be as above and let $\alpha \in X$ be a cardinal. There is $\rho < \alpha$ s.t. $cf_X(\tau) = cf_X(\alpha)$ for all regular $\tau \in (\rho, \alpha)$.

Hence $cf_X(\tau)$ can change only finitely often as τ ranges over R_X.

Using measurable cardinals, Magidor and Foreman proved a result in the opposite direction, which we state at the end of this chapter. However, the assumption SCFA + GCH below λ yields an optimal result:

<u>Lemma 11</u> Assume SCFA. Let λ be as above and let GCH hold below λ.
Let σ be any map of $\{\tau \mid \omega_1 < \tau < \lambda \wedge \tau \text{ is regular}\}$ into $\{\omega, \omega_1\}$.
Let $Z \subset H_\lambda$ s.t. $\overline{\overline{Z}} \leq \omega_1$. There is $X \prec H_\lambda$ s.t. $Z \subset X$, $\overline{\overline{X}} = \omega_1 \subset X$, and $cf_X(\tau) = \sigma(\tau)$ for all $\tau \in R_X$.

proof (sketch)

In [EN] we showed that if κ is inaccessible and GCH holds below κ, then, letting σ be any map of $\{\tau \mid \omega_1 < \tau < \kappa \wedge \tau \text{ is regular}\}$ into $\{\omega, \omega_1\}$, there is a subcomplete forcing $\mathbb{B}$ s.t. whenever G is $\mathbb{B}$-generic, then in $V[G]$ we have: $cf(\tau) = \sigma(\tau)$ whenever $\omega_1 < \tau < \kappa$ and τ is regular in V. (Hence $\kappa = \omega_2$ in $V[G]$.) A cursory examination of the proof reveals that it goes through for arbitrary cardinals $\lambda > \omega_1$

in place of κ, although λ might then be collapsed to ω_1. Now let $\theta > \lambda$ s.t. $\mathbb{B}, \sigma \in H_\theta$ and θ is regular. By §1 Lemma 1 there are $\pi : \bar{H} \prec H_\theta$, $\bar{G}$ s.t.

- $\bar{H}$ is transitive, $\mathrm{card}(\bar{H}) = \omega_1 \subset \bar{H}$, $Z \cup \{\mathbb{B}, \sigma\} \subset \mathrm{rng}(\pi)$
- $\bar{G}$ is $\bar{\mathbb{B}}$-generic over $\bar{N}$, where $\pi(\bar{\sigma}, \bar{\mathbb{B}}) = \sigma, \mathbb{B}$.

Let $X = \mathrm{rng}(\pi)$. If $\tau \in R_X$, $\pi(\bar{\tau}) = \tau$, we then have: $\sigma(\tau) = \bar{\sigma}(\bar{\tau}) = \mathrm{cf}_X(\tau)$.

QED (Lemma 11)

If GCH does not hold below λ we can make it hold by a prior forcing which is ω-closed (hence complete). Set:

Def Γ = the set of regular $\tau > \omega$ s.t. whenever $\alpha^+ < \tau$, there is $\gamma \in (\alpha, \tau]$ s.t $2^{\check{\gamma}} = \gamma$.

(This is just the set of regular cardinals which will survive if – in the most economical way possible – we collapse to make GCH true.)

Then $\omega_1 \in \Gamma$ and $\omega_2 \in \Gamma$ iff CH holds.

Lemma 12 Assume SCFA. Let $\sigma : \{\tau < \lambda \mid \tau \in \Gamma\} \to \{\omega, \omega_1\}$ s.t. $\sigma(\omega_1) = \omega_1$. For each $Z \subset \lambda$ of size $\leq \omega_1$ there is $X \prec H_\lambda$ s.t $Z \subset X$, $\omega_2 = \bar{\bar{X}} \subset X$, and

- $cf_X(\tau) = \sigma(\tau)$ for $\tau \in R_X \cap \Gamma$
- If $\tau \in R_X \setminus \Gamma$, then $cf_X(\tau) = cf_X(\tau')$, where τ' = the maximal $\tau' \in \Gamma$ s.t $\tau' \leq \tau$.

proof (sketch)

First force with the ~~BA~~ BA $\mathbb{C}$ which makes GCH true below λ and leaves Γ as the set of regular $\tau < \check{\lambda}$ s.t. $\tau > \omega$. There is then $\mathbb{B}$ s.t whenever G is $\mathbb{C}$-generic, then $\mathring{\mathbb{B}}^G$ is subcomplete and forces $cf(\tau) = \sigma(\tau)$ for $\tau \in \Gamma$. Set $\mathbb{B} = \mathbb{C} * \mathring{\mathbb{B}}$ and repeat the above argument. Letting θ be as above we get $\pi : \bar{H} \prec H_\theta$, $\bar{G}$ s.t. $\bar{G}$ is $\bar{\mathbb{B}} = \pi^{-1}(\mathbb{B})$ generic. Let $\bar{\sigma} = \pi^{-1}(\sigma)$, $\bar{\Gamma} = \pi^{-1}(\Gamma)$. Let $\bar{G}_0 = \bar{G} \cap \bar{\mathbb{C}}$, where $\pi(\bar{\mathbb{C}}) = \mathbb{C}$, $\pi(\bar{\mathring{\mathbb{B}}}) = \mathring{\mathbb{B}}$. Then

$\bar{H}[\bar{G}] = \bar{H}[\bar{G}_0][\bar{G}_1]$ where $\bar{G}_1$ is $\bar{\mathbb{B}} = \bar{\mathring{\mathbb{B}}}^{\bar{G}_0}$-generic over $\bar{H}[\bar{G}_0]$. But then,

$\bar{\Gamma} = \{\tau \mid \omega_1 < \tau < \bar{\lambda} \wedge \tau \text{ is reg. in } \bar{H}[\bar{G}_0]\}$,

where $\pi(\bar{\lambda}) = \lambda$. Hence

$$cf_X(\tau) = cf_{\bar{H}[\bar{G}_0][\bar{G}_1]}(\bar{\tau}) = \bar{\sigma}(\bar{\tau}) = \sigma(\tau)$$

for $\tau \in \Gamma$, $\pi(\bar{\tau}) = \tau$. Moreover, if $\tau \in X \setminus \Gamma$ is regular and $\pi(\bar{\tau}) = \tau$, then $\bar{G}_0$ collapses $\bar{\tau}$ to $\bar{\tau}' =$ the max $\bar{\tau}' \in \bar{\Gamma}$ s.t. $\bar{\tau}' \leq \bar{\tau}$. But then $cf(\bar{\tau}) = cf(\bar{\tau}')$ in $\bar{H}[\bar{G}_0]$. Hence

$cf_X(\tau) = cf(\bar{\tau}) = cf(\bar{\tau}') = \bar{\sigma}(\bar{\tau}') = \sigma(\tau')$ in $\bar{H}[\bar{G}_0][\bar{G}_1]$.

QED (Lemma 12)

MM of course also implies the conclusion of Lemma 12, but it also implies $\omega_2 \notin \Gamma$, which means that $cf_X(\omega_2) = \omega_1$ in the application. Hence SCFA + CH gives results which do not necessarily carry over to MM. Fr. ins. if κ is a strong limit cardinal which is regular or has cofinality ω_1, then Lemma 12 gives $X \prec H_{\kappa^+}$ s.t.

- $\overline{\overline{X}} = \omega_1 \subset X$
- $cf(X \cap \kappa) = \omega_1$
- $cf(X \cap \tau) = \omega$ for regular $\tau \in X \cap (\omega_1, \kappa)$.

<u>Question</u> Does this follow from (or is even compatible with) MM?

<u>Note</u> Menachem Magidor has informed me that he and Matt Foreman have proven the following theorem, which goes in the same direction as the above results but uses a much weaker hypothesis:

Let $\langle \kappa_i \mid i < \alpha \rangle$ be any monotone sequence of measurable cardinals. Let the generic extension $V[G]$ be the result of making successive Levy collapses in such a way that:

- κ_0 becomes ω_2
- κ_{i+1} becomes κ_i^{++}
- If λ is a limit and $\kappa_\lambda > \tau = \sup_{i<\lambda} \kappa_i$,

  then κ_λ becomes τ^{++}. *)

Then in $V[G]$ the following statement holds:

Let $\theta > \sup_i \kappa_i$. Let $\sigma : \alpha \to \{\omega, \omega_1\}$.

There is $X \prec H_\theta$ s.t.

- $\sigma, \langle \kappa_i \mid i < \alpha \rangle \in X$
- $\bar{\bar{X}} = \omega_1 \subset X$
- $\mathrm{cf}(X \cap \kappa_i) = \sigma(i)$ whenever $\kappa_i \in X$.

*) If λ is regular, we could have $\tau = \lambda^+$.