§5 Semi-subproper Forcing

When dealing with semi-subproper BA's, we must replace the notion "full" by "almost full" as defined in [SPSC] §3.1. Recall that a (possibly ill founded) ZFC^- model $\mathfrak{A}$ is called *solid* iff its well founded core $wfc(\mathfrak{A})$ is transitive and $\in_{\mathfrak{A}} \cap wfc(\mathfrak{A})^2 = \in \cap wfc(\mathfrak{A})^2$. A transitive ZFC^- model W is called *almost full* iff there is a solid model $\mathfrak{A}$ of ZFC^- s.t. $W \in wfc(\mathfrak{A})$, $\mathfrak{A} \models V = L(W)$, and W is regular in $\mathfrak{A}$ (i.e. if $f: x \to W$, $x \in W$, and $f \in \mathfrak{A}$, then $f \in W$). (W is then called *full* if there is a transitive $\mathfrak{A}$ with these properties.)

It is clear from [SPSC] §3 that the specific arguments used to verify that specific algebras are subcomplete or subproper would work equally well if, in the definition of those concepts, we replaced "full" by "almost full". There seemed no reason to do so, however, since the modified definitions seem harder to fulfill, and the iteration theorems were provable for the concepts as given. In the case of semi-subproper forcing, however, we unfortunately do not know how to prove iterability without making the change. (This means that Lemma 7 of [SPSC] §2 is unproven as stated - and possibly unprovable.)

To facilitate the treatment of semi-subproper forcing we also define:

Def Let W be a transitive ZFC^- model. Let $\mathbb{B} \in W$ be a complete BA in the sense of W. $G \subset \mathbb{B}$ is <u>weakly</u> $\mathbb{B}$-<u>generic</u> <u>over</u> W iff the following hold:

- G is an ultrafilter on $\mathbb{B}$
- Whenever $\Delta \in W$ is a set predense in $\mathbb{B}$ (i.e. $\bigcup\Delta = 1$ in $\mathbb{B}$) s.t. $W \models \overline{\overline{\Delta}} = \omega_1$, then $\Delta \cap G \neq \emptyset$.

We then define:

Def $\mathbb{B}$ is semi-subproper iff it satisfies the definition of "subproper" with two changes:

- "full" is replaced by "almost full"
- Clause (d) is replaced by:

(d) $\bar{G} = \sigma_0^{-1}{}''G$ is weakly $\bar{\mathbb{B}}$ generic over $\bar{W}$.

(Note (d) can be equivalently be replaced by: Whenever $t \in \bar{W}^{\bar{\mathbb{B}}}$ and $\sigma_0(t)^G < \omega_1$, then $\sigma_0(t)^G < \omega_1^{\bar{W}}$.

We leave this to the reader.)

We now consider the implications of (a) - (d)

Recall that $W = L_\tau^A = \langle L_\tau[A], \in, A \rangle$. Since G is generic we can form the ZFC^- model

$$W[G] = \langle W[G], A, W \rangle$$

whose language has the predicates $\in, \dot{A}, \dot{W}$.

By a $W^{\mathbb{B}_\lambda}$ - sentence let us mean a sentence $\varphi(t_1, \ldots, t_n)$, where φ is a fmla of this language and $t_1, \ldots, t_n \in W^{\mathbb{B}_\lambda}$.

If we assign each $W^{\mathbb{B}_\lambda}$ - sentence ψ a truth value $[\![\psi]\!] = [\![\psi]\!]^W \in \mathbb{B}_\lambda$, we have:

(1) $W[G] \models \varphi[\vec{t}^G] \longleftrightarrow [\![\varphi(\vec{t})]\!] \in G$

for $t_1, \ldots, t_n \in W^{\mathbb{B}_\lambda}$. Now assign $[\![\psi]\!]^{\bar{W}}$ for $\bar{W}^{\bar{\mathbb{B}}_{\bar{\lambda}}}$ sentences the same way.

Clearly:

(2) $\sigma_0([\![\varphi(\vec{t})]\!]^{\bar{W}}) = [\![\varphi(\sigma_0(\vec{t}))]\!]^{W}$.

Define a structure:

$\bar{W}'' = \langle \bar{w}'', I, E, A', w' \rangle$ by:

$\bar{w}'' = \bar{w}^{\mathbb{B}_{\bar{\lambda}}}$, $t I t' \leftrightarrow [\![t = t']\!] \in \bar{G}$,

$t E t' \leftrightarrow [\![t \in t']\!] \in \bar{G}$, $A' t \leftrightarrow [\![\mathring{A} t]\!] \in \bar{G}$,

$W' t \leftrightarrow [\![\dot{W} t]\!] \in \bar{G}$.

Noting that for any $\bar{W}^{\mathbb{B}_\lambda}$-sentence $\bigvee v\, \varphi(v, \vec{t})$, there is $s \in \bar{W}^{\mathbb{B}_\lambda}$ s.t. $[\![\bigvee v\, \varphi(v, \vec{t})]\!] = [\![\varphi(s, \vec{t})]\!]$, we see that:

(3) $[\![\bigvee v\, \varphi(v, t)]\!] \in \bar{G} \leftrightarrow \bigvee s \in \bar{W}^{\mathbb{B}_{\bar{\lambda}}}\, [\![\varphi(s, t)]\!] \in \bar{G}$.

Using this we get:

(4) $\bar{W}'' \Vdash \varphi \leftrightarrow [\![\varphi]\!] \in \bar{G}$ for $\bar{W}^{\mathbb{B}_{\bar{\lambda}}}$-sentences φ.

Clearly E is well founded, since

$$t E t' \rightarrow \sigma_0(t)^G \in \sigma_0(t')^G.$$

By (2), $\bar{W}''/I$ models the axiom of extensionality. Hence there is:

$$\bar{g} : \bar{W}''/I \overset{\sim}{\longleftrightarrow} \bar{W}^{**} = \langle \bar{W}^{**}, A^*, W^* \rangle,$$

where $\bar{W}^{**}$ is transitive. Set: $t^{\bar{G}} = \bar{g}(t/I)$. Then:

(5) $t^{\bar{G}} = \{ s^{\bar{G}} \mid [\![s \in t]\!]^{\bar{W}} \in \bar{G} \}$,

(6) $\bar{W}^{**} \models \varphi[\vec{t}^{\bar{G}}] \longleftrightarrow [\![\varphi(\vec{t})]\!] \in \bar{G}$

for $t_1, \ldots, t_n \in \bar{W}^{\mathbb{B}_{\bar{\lambda}}}$.

By (1), (2), (6) we then have:

(7) $\bar{W}^{**} \models \varphi[\vec{t}^{\bar{G}}] \leftrightarrow W[G] \models \varphi[\sigma_0(\vec{t})^G]$.

Hence there is $\sigma_0^*: \bar{W}^{**} \prec W[G]$ defined by: $\sigma_0^*(t^{\bar{G}}) = \sigma_0(t)^G$.

Since G is generic over W, we have $\check{x}^G = x$ for $x \in W$. $\bar{G}$, however, may not be fully generic, so we may not have $\check{x}^{\bar{G}} = x$ for $x \in \bar{W}$. However we do have:

(8) $\check{\xi}^{\bar{G}} = \xi$ for $\xi \leq \omega_1^{\bar{W}}$

by weak genericity, since $[\![t < \check{\xi}]\!] =$
$= \bigcup_{\nu < \xi} [\![t = \check{\nu}]\!]$.

If $\dot{\vec{G}}$ is the canonical $\mathbb{B}_\lambda$-generic name, then $\sigma_0^*(\dot{\vec{G}}^{\bar{G}}) = G$; hence, setting, $\bar{G}^* = \dot{\vec{G}}^{\bar{G}}$, we have:

(9) $\bar{W}^{**} = \bar{W}^*[\bar{G}^*]$ and $\bar{G}^* = \sigma_0^{*-1} G$ is $\sigma_0^{*-1}(\mathbb{B}_\lambda)$-generic over W^*,

Hence $\bar{W}^*$, σ_0^* to some extent play the role of $\bar{W}$, σ_0 in the case of subproper forcing. We refer to $\bar{W}^*$, σ_0^* as the <u>canonical extension</u> of $\bar{W}$, σ_0.

Set: $k(x) = \check{x}^{\bar{G}}$ for $x \in \bar{W}$. Then

(10) $k : \bar{W} \prec \bar{W}^*$

prf.

$\bar{W} \models \varphi[x_1, \ldots, x_n] \longleftrightarrow [\![\varphi_{\dot{W}} (\check{x}_1, \ldots, \check{x}_n)]\!] = 1$

$\longleftrightarrow [\![\quad \text{"} \quad]\!] \in \bar{G}$

$\longleftrightarrow \bar{W}^{**} \models \varphi_{\dot{W}} [\check{x}_1^{\bar{G}}, \ldots, \check{x}_n^{\bar{G}}]$

$\longleftrightarrow \bar{W}^* \models \varphi[k(x_1), \ldots, k(x_n)]$,

QED(10)

But:

(11) $\sigma_0^* k = \sigma_0$,

since $\sigma_0^*(\check{x}^{\bar{G}}) = \sigma_0(\check{x})^G = \check{\sigma_0(x)}^G = \sigma_0(x)$.

Set: $k(\bar{\lambda}_0) = On \cap \bar{W}^*$. Then:

(12) $k(\bar{\lambda}_i) = \sup k''\bar{\lambda}_i \quad (i = 0, 1, \ldots, n)$.

prf.

Let $t \in \bar{W}^{\mathbb{B}}$ s.t. $t^{\bar{G}} < k(\bar{\lambda}_i)$.

<u>Claim</u> $t^{\bar{G}} < k(\xi)$ for a $\xi < \bar{\lambda}_i$.

We have $[\![t < \check{\bar{\lambda}}_i]\!] \in \bar{G}$, where

$[\![t < \check{\bar{\lambda}}_i]\!] = \bigcup_{\xi < \bar{\lambda}_i} [\![t = \check{\xi}]\!]$. Let

$X = \{ \xi < \bar{\lambda}_i \mid [\![t = \check{\xi}]\!] \neq 0 \}$. Then

X is bounded in $\bar{\lambda}_i$ since $\mathrm{card}(\mathbb{B}) < \bar{\lambda}_i$

in $\bar{W}$. Let $X \subset \xi < \bar{\lambda}_i$. Then

$[\![t < \check{\xi}]\!] \in \bar{G}$. Hence $t^{\bar{G}} < k(\xi)$.

(Here we took $[\![t < \check{\bar{\lambda}}_0]\!] =_{df} 1$.) QED(12)

But then $k : \bar{W} \prec W^*$ cofinally. Hence $\bar{W}^*$ is almost full by [SPSC] §3.1 Fact 9.

But then

(13) $\bar{W}^{**} = \bar{W}^*[\bar{G}^*]$ is almost full.

prf Let $\mathfrak{A}$ verify the almost fullness of $\bar{W}^*$. Then $\bar{G}^*$ is $k(\mathbb{B}_{\bar{\lambda}})$-generic over $\mathfrak{A}$, since $\bar{W}^*$ is regular in $\mathfrak{A}$. Hence $\mathfrak{A}[\bar{G}^*]$ verifies the almost completeness of $\bar{W}^{**}$.

QED (13)

Using this we prove a technical lemma which will save time later

Lemma 6 Let $\mathbb{A} \subseteq \mathbb{B}$ s.t.

$\Vdash_{\mathbb{A}} (\check{\mathbb{B}}/\dot{G}$ is semi-subproper$)$.

Let $\theta > \overline{\overline{\mathbb{B}}}$ s.t.

$\Vdash_{\mathbb{A}} (\check{\theta}$ verifies the semi-subproperness of $\check{\mathbb{B}}/\dot{G})$

Let $\lambda_1, \dots, \lambda_n < \theta$ s.t. $\lambda_i > \overline{\overline{\mathbb{B}}}$ is regular for $i = 1, \dots, n$. Let $W = L^{A}_{\tau}$ where $\tau > \theta$ is regular and $H_\theta \subset W$. Let $\overline{W} = L^{\bar{A}}_{\bar{\tau}}$ be countable and almost full. Let

$\bar{\theta}, \bar{\mathbb{A}}; \bar{\mathbb{B}}, \bar{\lambda}_1, \dots, \bar{\lambda}_n, e \in \overline{W}; t \in V$,

where $e \in \bar{\mathbb{B}}$. Let $a \in \mathbb{A}$, $\dot{\sigma}_0 \in V^{\mathbb{A}}$ s.t. $a \neq 0$ and whenever $G_0 \ni a$ is $\mathbb{A}$-generic and $\sigma_0 = \dot{\sigma}_0^{G_0}$, then

(a) $\sigma_0 : \overline{W} \prec W$

(b) $\sigma_0(\bar{\theta}, \bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{\lambda}_i) = \theta, \mathbb{A}, \mathbb{B}, \lambda_i \quad (i = 1, \dots, n)$

(c) $t^{\sigma_0} \in \overline{W}$

(d) $\bar{G}_0 = \sigma_0^{-1}{}''G$ is weakly $\bar{\mathbb{A}}$-generic over $\overline{W}$ and $h_{\bar{\mathbb{A}}}(e) \in \bar{G}_0$.

Then there are $b \in \mathbb{B}$, $\dot{\sigma} \in V^{\mathbb{B}}$ s.t. $a = h_{\mathbb{A}}(b)$ and whenever $G \ni b$ is $\mathbb{B}$-generic, $\sigma = \dot{\sigma}^G$, $G_0 = G \cap \mathbb{A}$, $\sigma_0 = \dot{\sigma}_0^G = \dot{\sigma}^{G_0}$, then:

(e) $\sigma : \bar{W} \prec W$

(f) $\sigma(\bar{\theta}, \bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{\lambda}_i) = \theta, \mathbb{A}, \mathbb{B}, \lambda_i \quad (i = 1, \ldots, n)$

(g) $\sigma(t^{G_0}) = \sigma_0(t^{G_0})$

(h) $\sup \sigma''\bar{\lambda}_i = \sup \sigma_0''\bar{\lambda}_i \quad (i = 0, \ldots, n)$

where $\bar{\lambda}_0 =_{df} On \cap \bar{W}$

(j) $\bar{G} = \sigma^{-1}{}''G$ is weakly $\mathbb{B}$-generic over $\bar{W}$ and $e \in \bar{G}$.

proof.

Let $G_0 \ni a$ be $\mathbb{A}$-generic, $\sigma_0 = \dot{\sigma}_0^{G_0}$. Our assumptions justify the analysis in (1)–(13) above. Let $\bar{W}^*, \sigma_0^*$ be the canonical extension of $\bar{W}, \sigma_0$. Then

$\sigma_0^* : \bar{W}^*[\bar{G}_0^*] \prec W[G_0]$, $\sigma_0^*(\bar{G}_0^*) = G_0$,

where $\bar{W}^*[\bar{G}_0^*]$ is full in $V[G_0]$. If we set $k(x) = \check{x}^{\bar{G}}$ for $x \in \bar{W}$, we get $k : \bar{W} \prec \bar{W}^*$ and $\sigma_0^* \circ k = \sigma_0$. Moreover $k(\bar{\lambda}_i) = \sup k''\bar{\lambda}_i$ for $i = 0, \ldots, n$ (where $k(\bar{\lambda}_0) =_{df} On \cap \bar{W}^*$). Since $\sigma_0(h_{\bar{\mathbb{A}}}(e)) = h_{\mathbb{A}}(\sigma_0(e)) \in G_0$, we have: $\sigma_0(e)/G_0 \neq 0$. Since $\mathbb{B}' = \mathbb{B}/G_0$ is semiproper in $V[G_0]$, we conclude that there is $b' \in \mathbb{B}' \setminus \{0\}$

s.t. $b' \subset \sigma_0(e)/G_0$ and whenever $G' \ni b'$ is $\mathbb{B}'$-generic over $V[G_0]$, there is $\sigma' \in V[G_0][G']$ s.t.

(i) $\sigma' : \overline{W}^*[\overline{G}_0^*] \prec W[G_0]$, $\sigma'(\overline{G}_0^*) = G_0$

(ii) $\sigma'(k(x)) = \sigma_0^*(k(x)) = \sigma_0(x)$
for $x = \overline{\theta}, \overline{\mathbb{A}}, \overline{\mathbb{B}}, e, \overline{\lambda}_i$ $(i = 1, \dots, n)$

(iii) $\sup \sigma' `` k(\overline{\lambda}_i) = \sup \sigma_0^* `` k(\overline{\lambda}_i) = \sup \sigma_0 `` \overline{\lambda}_i$
$(i = 0, \dots, n)$

(iv) $\overline{G}' = \sigma'^{-1} `` G'$ is weakly $k(\overline{\mathbb{B}})/\overline{G}_0^*$-generic over $\overline{W}^*[\overline{G}_0^*]$.

Set: $\sigma = \sigma' \circ k$.
Then σ satisfies (e), (f), (g), (h) by (i)–(iii).
Set $G = G_0 * G' = \{ b \in \mathbb{B} \mid b/G_0 \in G' \}$. Then G is $\mathbb{B}$-generic and we prove:

(14) σ satisfies (j) for $G = G_0 * G'$ – i.e.
$\overline{G} = \sigma^{-1} `` G$ is weakly $\overline{\mathbb{B}}$-generic over $\overline{W}$ and $e \in \overline{G}$.

proof.
$\overline{G}$ is obviously an ultrafilter, since G is an ultrafilter, $\sigma : \overline{W} \prec W$, and $\sigma(\overline{\mathbb{B}}) = \mathbb{B}$. Moreover $\sigma(e) = \sigma_0(e) \in G$ since $\sigma_0(e)/G_0 \in G'$. Hence $e \in \overline{G}$. Now let Δ be predense in $\mathbb{B}$ s.t. $\overline{\overline{\Delta}} \leq \omega_1$ in $\overline{W}$.

<u>Claim</u> $G \cap \Delta \neq \emptyset$

$k(\Delta) = k``\Delta$ since $k \upharpoonright (\omega_1 + 1) = \mathrm{id}$. But $k(\Delta)$ is predense in $k(\overline{\mathbb{B}})$; hence

$\Delta' = \{ b/\bar{G}_0^* \mid b \in k(\Delta) \}$ is predense in $k(\mathbb{B})/\bar{G}_0^*$. But $\Delta' = \{ k(b)/\bar{G}_0^* \mid b \in \Delta \}$. Hence $k(b)/\bar{G}_0^* \in \bar{G}'$ for a $b \in \Delta$. Hence $\sigma_0^*(k(b)/\bar{G}_0^*) = \sigma_0(b)/G_0' \in$ $\in G'$. Hence $\sigma_0(b) \in G$. Hence $b \in \bar{G}$.

QED (14)

We have seen that for every $\mathbb{A}$-generic $G_0 \ni a$ there is a $b' \in \mathbb{B}/G_0$ with certain properties. Hence it is forced that there is such a b'. Hence there is $\mathring{b}' \in V^{\mathbb{A}}$ s.t. the above holds of $b' = \mathring{b}'^{G_0}$ whenever $G_0 \ni a$ is $\mathbb{A}$-generic. We may also assume w.l.o.g. that $[\![\mathring{b}' \neq 0]\!] = a$. Hence $\Vdash_{\mathbb{A}} \mathring{b}' \in \check{\mathbb{B}}/\mathring{G}$. But then there is a unique $b \in \mathbb{B}$ s.t. $\Vdash_{\mathbb{A}} \check{b}/\mathring{G} = \mathring{b}'$. Hence $h_{\mathbb{A}}(b) = [\![\check{b}/\mathring{G} \neq 0]\!] = a$.

Now let $G \ni b$ be $\mathbb{B}$-generic. Set: $G_0 = G \cap \mathbb{A}$, $\sigma_0 = \dot{\sigma}_0^G = \dot{\sigma}_0^{G_0}$, $\mathbb{B}' = \mathbb{B}/G_0$. Then $b' = b/G_0 \in \mathbb{B}'$ and $G = G_0 * G'$, where $G' = G/G_0 = \{ b/G_0 \mid b \in G \}$. Hence there is $\sigma \in V[G_0][G'] = V[G]$ satisfying (e)–(g) ((g) follows by (14)).

Since this holds for every $\mathbb{B}$-generic $G \ni b$, there is a $\dot\sigma \in V^{\mathbb{B}}$ s.t. (e)-(j) hold for $\sigma = \dot\sigma^G$ whenever $G \ni b$ is $\mathbb{B}$-generic.

QED (Lemma 6)

Using this we prove the two step theorem for semi-subproper forcing:

Thm 7 Let $\mathbb{A} \subseteq \mathbb{B}$ s.t. $\mathbb{A}$ is semi-subproper and

$$\Vdash_{\mathbb{A}} (\check{\mathbb{B}}/\dot{G} \text{ is semi-subproper}).$$

Then $\mathbb{B}$ is semi-subproper.

prf.

Let θ be a cardinal big enough to verify the semi-subproperness of $\mathbb{A}$ and s.t. $\Vdash_{\mathbb{A}} (\check\theta$ verifies the semi-subproperness of $\check{\mathbb{B}}/\dot{G})$.

Let $\tau > \theta$ be regular, $W = L_\tau^A$, where $H_\theta \subset W$. Let $\sigma : \bar W \prec W$, where $\bar W$ is transitive, countable and almost full. Let

$\sigma(\bar\theta, \bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar s, \bar\lambda_i) = \theta, \mathbb{A}, \mathbb{B}, s, \lambda_i$ $(i = 1, \dots, n)$

where $\bar{\bar{\mathbb{B}}} < \lambda_i$ and $\lambda_i < \theta$ is regular for $i = 1, \dots, n$.

Let $\sigma(\bar e) = e \in \mathbb{B} \setminus \{0\}$.

<u>Claim</u> There is $b \in \mathbb{B} \setminus \{0\}$ s.t. $b \subset e$ and whenever $G \ni b$ is $\mathbb{B}$-generic, then there is $\sigma' \in V[G]$ s.t.

(a) $\sigma' : \bar{W} \prec W$

(b) $\sigma'(\bar{\Theta}, \bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{s}, \bar{e}, \bar{\lambda}_i)$

(c) $\sup \sigma'' \bar{\lambda}_i = \tilde{\lambda}_i \quad (i = 0, \ldots, n)$ where

$\bar{\lambda}_0 = On \cap \bar{W}$; $\tilde{\lambda}_i = \sup \sigma'' \bar{\lambda}_i$.

(d) $\bar{G} = \sigma'^{-1}{}'' G$ is weakly $\bar{\mathbb{B}}$-generic over $\bar{W}$,

prf.

Since Θ verifies the semi-subproperness of $\mathbb{A}$, there are $a, \dot{\sigma}_0$ s.t. $a \in \mathbb{A} \setminus \{0\}$, $a \subset h_{\mathbb{A}}(c)$, and whenever $G_0 \ni a$ is $\mathbb{A}$-generic and $\sigma_0 = \dot{\sigma}_0^{G_0}$, then

- (a)–(c) hold with σ_0 in place of σ'
- $\bar{G}_0 = \sigma_0^{-1}{}'' G_0$ is weakly $\mathbb{A}$-generic over $\bar{W}$.

By Lemma 6 there is then $b \subset c$ with the desired properties. (Take $e = \bar{e}$, $t = \check{\bar{s}}$).

QED (Thm 7)

We are now ready to prove:

<u>Thm 8</u> Thm 1 holds with "semi-subproper" in place of "subcomplete".

proof (sketch)

The proof is very much like that of Thm 5. We are given an RCS iteration $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ satisfying (a)–(c) of Thm 1 with "semi-subproper" in place of "subcomplete". By induction on i we prove:

<u>Claim</u> Let $h \le i$. Let G be $\mathbb{B}_h$-generic. Then $\mathbb{B}_i / G$ is semi-subproper in $V[G]$.

The cases $h = i$, $i = 0$, and $i = j+1$ are as before. Let $i = \lambda$ with λ a limit ordinal. We consider the same two cases:

<u>Case 1</u> $\mathrm{cf}(\lambda) \le \overline{\overline{\mathbb{B}_i}}$ for an $i < \lambda$.

Just as before it suffices to prove:

<u>Claim</u> Assume $\mathrm{cf}(\lambda) \le \omega_1$ in V. Then $\mathbb{B}_\lambda$ is semi-subproper.

We again fix $f : \omega_1 \to \lambda$ s.t. $\sup f''\omega_1 = \lambda$.
Let $\theta > \lambda$ as before be a cardinal s.t. $\overline{\overline{\mathbb{B}}} < \theta$ and H_θ ($\check{\theta}$ witnesses the semi-subproperness of $\check{\mathbb{B}}_j/\dot{G}$) for $i \leq j < \lambda$. Let $W = L^A_\tau$, where $\tau > \theta$ is regular and $H_\theta \subset W$. Let $\sigma : \overline{W} \prec W$ s.t. $\overline{W}$ is countable, transitive, and almost full. Suppose moreover that: $\sigma(\bar{f}, \bar{\theta}, \bar{\lambda}, \overline{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = f, \theta, \mathbb{B}, s, \lambda_i$ $(i = 1, \ldots, n)$ where s codes an $a \in \mathbb{B}_\lambda \setminus \{0\}$ and $\lambda_i < \theta$ is regular with $\overline{\overline{\mathbb{B}}}_\lambda < \lambda_i$ for $i = 1, \ldots, n$.

It suffices to show:

<u>Claim</u> There is $c \in \mathbb{B}_\lambda \setminus \{0\}$ s.t. $c \subset a$ and whenever $G \ni c$ is $\mathbb{B}_\lambda$-generic, then there is $\sigma' \in V[G]$ s.t.

(a) $\sigma' : \overline{W} \prec W$

(b) $\sigma'(\bar{f}, \bar{\theta}, \bar{\lambda}, \overline{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = f, \theta, \lambda, \mathbb{B}, s, \lambda_i$ $(i = 1, \ldots, n)$

(c) $\sup \sigma''\bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, n)$
where $\bar{\lambda}_0 = \mathrm{On} \cap \overline{W}$, $\tilde{\lambda}_i = \sup \sigma''\bar{\lambda}_i$ $(i = 0, \ldots, n)$

(d) $\bar{G} = \sigma'^{-1}{}''G$ is weakly $\overline{\mathbb{B}}_{\bar{\lambda}}$-generic over $\overline{W}$

Making use of f we again define $\langle \bar{\xi}_i \mid i<\omega\rangle$ monotone and cofinal in $\bar{\lambda}$. We again set $\xi_i = \sigma(\bar{\xi}_i)$ and conclude:

(1) $\langle \bar{\xi}_i \mid i<\omega\rangle \in \bar{W}$ if $cf(\lambda)=\omega$

(2) $\sigma'(\bar{\xi}_i) = \xi_i \quad (i<\omega)$ whenever $\sigma' : \bar{W} \prec W$, $\sigma'(\bar{f}) = f$.

As before, our strategy is to define sequences $c_i \in \mathbb{B}_{\xi_i}$, $\dot{\sigma}_i \in V^{\mathbb{B}_{\xi_i}}$ s.t. $\langle c_i \mid i<\omega\rangle$ is a thread in $\langle \mathbb{B}_{\xi_i} \mid i<\omega\rangle$ and c_i forces that $\sigma_i : \bar{W} \prec W$ and $\bar{G}_i = \sigma_i^{-1}{}'' G_i$ is weakly $\bar{\mathbb{B}}_{\bar{\xi}_i}$-generic over $\bar{W}$ whenever $G_i \ni c_i$ is $\mathbb{B}_{\xi_i}$-generic and $\sigma_i = \dot{\sigma}_i^{G_i}$.

We then set $c = \bigcap_i c_i$ and will have built enough pointwise correspondence between the σ_i that we can define:

$\sigma'(x) =_{\text{Df}} \sigma_i(x)$ for suff. large i,

where $\sigma_i = \dot{\sigma}_i^G$ and $G \ni c$ is $\mathbb{B}_\lambda$-generic.

The pointwise correspondence will also guarantee that (b), (c) of the claim hold and

(d′) $\bar{G}_i = \sigma'^{-1}{}'' G_i$ is weakly $\bar{\mathbb{B}}_{\bar{\xi}_i}$-generic over $\bar{W} \quad (i<\omega)$

As before we define a dense subset X of $\mathbb{B}_{\bar{\lambda}}$ in two cases:

<u>Case A</u> $cf(\lambda)=\omega$. Then $cf(\bar{\lambda})=\omega$ in $\bar{W}$ and we have $\langle \bar{\xi}_i \mid i<\omega\rangle \in \bar{W}$. We let X = the set of $b=\bigcap_{i<\omega} b_i$ s.t. $\langle b_i \mid i<\omega\rangle \in \bar{W}$ is a thread in $\langle \mathbb{B}_{\bar{\xi}_i} \mid i<\omega\rangle$. (Hence $b=\bigcap_{i<\omega} h_{\bar{\xi}_i}(b)$ in $\bar{W}$).

<u>Case B</u> $cf(\lambda)>\omega$. Then $cf(\bar{\lambda})>\omega$ in $\bar{W}$ and we set: $X=\bigcup_{i<\bar{\lambda}} \mathbb{B}_i \setminus \{0\}$. (Hence $b=h_{\bar{\xi}_i}(b)$ for some i if $b\in X$)

We then face the problem of getting:

(d) $\bar{G}=\sigma^{-1}{}''G$ is weakly $\mathbb{B}_{\bar{\lambda}}$-generic over $\bar{W}$

from (d') above. We again solve this by guaranteeing that a "master sequence" lies in $\bar{G}$:

<u>Def</u> By a <u>master sequence</u> we mean a sequence $\langle b_i \mid i<\omega\rangle$ s.t.

(a) $b_i \in X$, $b_i \subset b_h$ and $h_{\bar{\xi}_h}(b_i)=h_{\bar{\xi}_h}(b_h)$ for $h\le i$

(b) Whenever $G\subset \mathbb{B}_{\bar{\lambda}}$ is an ultrafilter s.t. $\{b_i \mid i<\omega\}\subset G$ and $G\cap \mathbb{B}_{\bar{\xi}_i}$ is weakly $\mathbb{B}_{\bar{\xi}_i}$-generic over $\bar{W}$ for $i<\omega$, then G is weakly $\mathbb{B}_{\bar{\lambda}}$-generic over $\bar{W}$.

We prove:

(3) There is a master sequence $\langle b_i \mid i<\omega \rangle$ s.t. $b_0 \subset \bar{a}$.

Pf.

Let $b_0 \subset \bar{a}$ s.t. $b_0 \in X$. Let $\langle \Delta_i \mid i<\omega \rangle$ enumerate the $\Delta_i \in \bar{W}$ s.t. Δ_i is predense in $\bar{\mathbb{B}}_{\bar{\lambda}}$ (i.e. $\bigcup \Delta_i = 1$) and $\bar{\bar{\Delta}}_i \leq \omega_1$ in $\bar{W}$. Let b_i be given. By (4) of the proof of Thm 5 Case 1, there is $b \subset b_i$ s.t. $h_{\bar{\xi}_i}(b) = h_{\bar{\xi}_i}(b_i)$ and $\Delta^i = \{a \in \bar{\mathbb{B}}_{\bar{\xi}_i} \mid \bigvee d \in \Delta_i \; a \cap b \subset d\}$ is strongly dense below $h_{\bar{\xi}_i}(b)$ in $\bar{\mathbb{B}}_{\bar{\xi}_i}$. Set $b_{i+1} = b$. This defines $\langle b_i \mid i<\omega \rangle$. Now let G be an ultrafilter on $\bar{\mathbb{B}}_{\bar{\lambda}}$ s.t. $G \cap \bar{\mathbb{B}}_{\bar{\xi}_i}$ is weakly generic over $\bar{W}$ for $i<\omega$ and $\{b_i \mid i<\omega\} \subset G$. Let Δ be predense in $\bar{\mathbb{B}}_{\bar{\lambda}}$ s.t. $\bar{\bar{\Delta}} = \omega_1$ in $\bar{W}$. Then $\Delta = \Delta_i$. For $d \in \Delta$ set $a_d = \bigcup\{a \in \Delta^i \mid a \cap b_{i+1} \subset d\}$. Then $\{a_d \mid d \in \Delta\}$ is predense above $h_{\bar{\xi}_i}(b_i)$ and has cardinality $\leq \omega_1$ in $\bar{W}$. Hence there is d with $a_d \in G$. Hence $a_d \cap b_{i+1} \subset d$. Hence $d \in \Delta \cap G$. QED(3)

From now on let $\langle \bar{b}_i \mid i<\omega \rangle$ be a fixed master sequence s.t. $\bar{b}_0 \subset \bar{a}$, where $\sigma(\bar{a})=a$, and let $\langle x_i \mid i<\omega \rangle$ be a fixed enumeration of $\bar{W}$ with infinite repetitions.

As before we construct $c_k \in \mathbb{B}_{\bar{\xi}_k}$, $\sigma_k^{\cdot} \in V^{\mathbb{B}_{\bar{\xi}_k}}$ $(k<\omega)$ s.t. $\langle c_i \mid i<\omega \rangle$ is a thread in $\langle \mathbb{B}_{\bar{\xi}_i} \mid i<\omega \rangle$ and $(*)$ holds as in Case 1 of the proof of Thm 5 – except that $(*)(d)$ is now changed to:

(d) $\bar{G}_k = \sigma^{-1}{}'' G_k$ is weakly $\mathbb{B}_{\bar{\xi}_k}$ – generic over $\bar{W}$.

The proof that $(*)$ implies the <u>Claim</u> is exactly as in Thm 5 with minor verbal changes (e.g. "weakly generic" instead of "generic").

It remains only to construct $c_i, \dot{\sigma}_i$ and verify (*). We set: $c_0 = 1$, $\dot{\sigma}_0 = \check{\sigma}$. Given $c_j, \dot{\sigma}_j$ with $k = j+1$, we use Lemma 6 to get $c_k, \dot{\sigma}_k$ (with $a = c_j$, $\mathbb{A} = \mathbb{B}_{\xi_j}$, $b = c_k$, $\mathbb{B} = \mathbb{B}_{\xi_k}$, $\dot{\sigma}_0 = \dot{\sigma}_j$, $\dot{\sigma} = \dot{\sigma}_k$, $e = h_{\xi_k}(b_k)$ and t appropriately defined). QED (Case 1)

<u>Case 2</u> Case 1 fails.

Then λ is regular and $\overline{\overline{\mathbb{B}}}_i < \lambda$ for $i < \lambda$. We closely follow the proof of Thm 5. We again let $\bar{W}, W, \theta, \sigma$ be as before with $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}, \bar{\lambda}_i) = \theta, \mathbb{B}, s, \lambda, \lambda_i$ $(i = 1, \ldots, n)$ where $a \in \mathbb{B}_\lambda \setminus \{0\}$ and $\lambda_1, \ldots, \lambda_n$ are as before. We set: $\lambda_{n+1} = \lambda$, $\bar{\lambda}_{n+1} = \bar{\lambda}$, $\bar{\lambda}_0 = On \cap \bar{W}$ and $\tilde{\lambda}_i = \sup \sigma '' \bar{\lambda}_i$ $(i = 0, \ldots, n+1)$ as before. (We also write: $\tilde{\lambda} = \tilde{\lambda}_{n+1}$.) We again fix an enumeration $\langle x_i \mid i < \omega \rangle$ of $\bar{W}$ with infinite repetitions and a master sequence $\langle \bar{b}_i \mid i < \omega \rangle$ with $\bar{b}_0 \subset \bar{a}$. (Here Case B holds, so our dense set X is $\bigcup_{i<\lambda} \mathbb{B}_\lambda$.)

<u>Claim</u> There is $c \in \mathbb{B}_\lambda$ s.t. $c \subset a$, $c \neq 0$, and whenever $G \ni c$ is $\mathbb{B}_\lambda$ - generic there is $\sigma' \in V[G]$ s.t

(a) $\sigma' : \bar{W} \prec W$

(b) $\sigma'(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, s, \lambda_i$ $(i = 1, \ldots, n+1)$

(c) $\sup \sigma' '' \bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, n+1)$

(d) $\bar{G} = \sigma'^{-1} '' G$ is weakly $\bar{\mathbb{B}}_{\bar{\lambda}}$ - generic over $\bar{W}$.

As before we choose $\langle \bar{\xi}_i^{j} \mid i<\omega\rangle$ monotone and cofinal in $\bar{\lambda}_j$ for $j = 0, m, m+1$ and set $\xi_i^{j} = \sigma(\bar{\xi}_i^{j})$. We also write: $\xi_i = \xi_i^{n+1}$, $\bar{\xi}_i = \bar{\xi}_i^{n+1}$.

We inductively construct $c_k \in \mathbb{B}_{\bar{\xi}_k}$, $\dot{\sigma}_k \in V^{\mathbb{B}_{\bar{\xi}_k}}$ s.t. I of Case 2 in the proof of Thm 5 holds and II of Case 2 in the proof of Thm 5 holds except that II(d) is replaced by:

II(d) Let $\sigma_k(\bar{\xi}_m) \leq \xi_k < \sigma_k(\bar{\xi}_{m+1})$. Then $\bar{G} = \sigma_k^{-1}{}'' G_{\sigma_k(\bar{\xi}_m)}$ is weakly $\mathbb{B}_{\bar{\xi}_m}$-generic over $\bar{W}$.

The proof that this implies the Claim is exactly as before. ~~At~~ remains to construct $c_k, \dot{\sigma}_k$ and verify I, II. As before we shall in fact construct $b_k, \dot{\sigma}_k$ with $b_k \in \mathbb{B}_{\bar{\xi}_k}$ in an intermediate step before constructing $c_k \subset b_k$. We inductively verify I - IV, where III, IV are exactly as before. Suppose now that I - IV hold below k, and that $b_k, \dot{\sigma}_k$ are given satisfying III(a)-(c) and IV. We must construct c_k and verify I, II and III(d). We proceed exactly as before. We define $a^{i\nu\mu}$ ($\nu \leq \xi_k < \mu < \tilde{\lambda}$, $\sup_{i<\lambda} \xi_i < \nu$) and A_k exactly as before. IV then gives $\dot{\sigma}_a$ $(a \in A_k)$ s.t.

(4) $\dot{\sigma}_a^{G_\nu} = \dot{\sigma}_k^G$ for $a = a^{i\nu\mu} \in A_k$ whenever $G \ni a$ is $\mathbb{B}_{\xi_k}$ - generic and $G_\nu = G \cap \mathbb{B}_\nu$.

As before, we conclude:

(5) If $G \ni a$ is $\mathbb{B}_\nu$ - generic, $a = a^{i\nu\mu} \in A_k$, then II (a)-(g) hold with $\sigma_a = \dot{\sigma}_a^G$ in place of σ_k, $\sigma_j = \dot{\sigma}_j^G = \dot{\sigma}_j^{G_{\xi_j}}$ for $j < k$, where $G_\eta = G \cap \mathbb{B}_\eta$ for $\eta \leq \nu$.

We then apply Lemma 6 to get:

(6) Let $a \in A_k$, $a = a^{i\nu\mu}$. There are $\tilde{a} \in \mathbb{B}_\mu$, $\dot{\sigma}_a' \in V^{\mathbb{B}_\mu}$ s.t. $h_\nu(\tilde{a}) = a$ and whenever $G \ni \tilde{a}$ is $\mathbb{B}_\mu$ - generic, $\sigma_a = \dot{\sigma}_a^G$, $\sigma_a' = \dot{\sigma}_a'^G$ and $\sigma_j = \dot{\sigma}_j^G = \dot{\sigma}_j^{G_{\xi_j}}$ for $j < k$ $(G_\eta = G \cap \mathbb{B}_\eta)$, then:

(a) $\sigma_a' : \bar{W} \prec W$

(b) $\sigma_a'(\bar{\theta}, \bar{\mathbb{B}}, \bar{r}, \bar{\lambda}_i) = \theta, \mathbb{B}, r, \lambda_i$ $(i = 1, \dots, n+1)$

(c) $\sup \sigma_a' `` \bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \dots, n+1)$

(d) $\bar{G} = \sigma_a'^{-1} `` G$ is weakly $\mathbb{B}_{\bar{\xi}_{i+1}}$ - generic over $\bar{W}$

(e) Let r be least s.t. $\mu \leq \xi_r$. Then $\sigma_a'(x_i, b_i, d_i) = \sigma_a(x_i, b_i, d_i)$ for $i < r$, where d_j is as before for $j < k$ and

$$d_j = \begin{cases} \text{the } \bar{W}\text{-least } d \in x_j \text{ s.t. } \sigma_a(d) \in G_\nu \\ \quad \text{if such exists;} \\ 0 \text{ if not} \end{cases}$$

for $k \leq j < r$.

(f) Let r be as above. Let $i = 0, m, m+1$.
Let $\sigma_a(\bar{\xi}^i_m) \leq \xi_r < \sigma_a(\bar{\xi}^i_{m+1})$, Then
$\sigma'_a(\bar{\xi}^i_l) = \sigma_a(\bar{\xi}^i_l)$ for $l \leq m+1$

(g) $\sigma'_a(h_{\bar{\xi}_{i+1}}(\bar{b}_{i+1})) \in G$.

This follows by Lemma 6, taking
$\mathbb{A} = \mathbb{B}_\nu$, $\bar{\mathbb{A}} = \bar{\mathbb{B}}_{\bar{\xi}_i}$, $\mathbb{B} = \mathbb{B}_\lambda$, $\bar{\mathbb{B}} = \bar{\mathbb{B}}_{\bar{\xi}_{i+1}}$,
$a = a^{i\nu\mu}$, $b = \tilde{a}$, $\sigma_0 = \sigma_a$, $\sigma = \sigma'_a$, $e = h_{\bar{\xi}_{i+1}}(\bar{b}_{i+1})$
and defining t appropriately.
The rest of the proof is virtually identical to that of Thm 5.

QED (Thm 8)

It is not difficult to reformulate and reprove Lemma 2 - Lemma 4 for "semi-subproper" in place of "subcomplete".