§3 The full iterability proof

In §1 we proved Lemma 5.3 only for iterations of length ω in which no truncation occured. We now prove the general case. Our proof is a straight-forward modification of Steel's proof in §9 of the printed version of [S].

For the readers convenience we begin by recapitulating the definition of the "resurrection sequence". Let M be a weak mouse. Let $E_\nu^M \neq \emptyset$. Set:

$$\bar{\beta}(M, \nu) = \text{the maximal } \beta \geq \nu \text{ s.t. } \beta \in M \text{ and } \omega\rho^\omega_{M\|\beta} < \omega\rho^\omega_{M\|\xi} \text{ for } \nu \leq \xi < \beta.$$

We define $\bar{\beta}_i = \bar{\beta}_i(M, \nu)$ for $i \leq p = p(M, \nu)$ as follows: $\bar{\beta}_0 = ht(M)$. If $\bar{\beta}_i$ is defined and $\bar{\beta}_i > \nu$, we set:

$\bar{\beta}_{i+1} = \bar{\beta}(M\|\beta_i, \nu)$. Otherwise $\bar{\beta}_{i+1}$ is undefined and $p = i$. Clearly, we have: $\beta_0 > \beta_1 > \cdots > \beta_p = \nu$.

Now let $\vec{N}$ be an array $M = N_\xi$. We set: $\beta = \beta[\xi, \nu] = \bar{\beta}(M, \nu)$. It follows that there is $\eta < \xi$ s.t. $N_\xi \| \beta = M_\eta$. We denote this η by $\eta[\xi, \nu]$. Thus $M_\eta = \text{core}(N_\eta)$. We let $\sigma = \sigma[\xi, \nu]$ be the core map. Thus $\sigma : N_\xi \| \beta \to N_\eta$.

The "resurrection sequence" or "trace" of $\langle \xi, \nu \rangle$ is defined by:

$S(\xi, \nu) = \langle \langle \eta_i, \beta_i, \sigma_i, \nu_i \rangle | i \leq \tilde{p} \rangle$, where:

$\eta_0 = \xi$, $\beta_0 = \text{ht}(N_\xi)$, $\sigma_0 = \text{id} \upharpoonright M_\xi$, $\nu_0 = \nu$.

If $\beta_i > \nu_i$, then we set:

$\beta_{i+1} = \bar{\beta}(N_{\eta_i}, \nu_i) = \beta[\eta_i, \nu_i]$

$\eta_{i+1} = \eta[\eta_i, \nu_i]$; $\sigma_{i+1} = \sigma[\eta_i, \nu_i]$,

$\nu_{i+1} = \sigma_i(\nu_i)$. [Here $\sigma(\nu_i) = \text{ht}(N_{\eta_{i+1}})$ if $\nu_i = \beta_{i+1} = \text{ht}(M_{\eta_{i+1}})$.]

Otherwise $\beta_{i+1}, \eta_{i+1}, \sigma_{i+1}, \nu_{i+1}$ are undefined.

Set: $\sigma^{(i)} = \sigma_i \circ \sigma_{i-1} \circ \dots \circ \sigma_0$. Then $\sigma^{(i)} : N_\xi \| \overline{\beta_i} \xrightarrow{\Sigma^*} N_{\eta_i}$; $\nu_i = \sigma^{(i)}(\nu)$.

We of course also set:

$\eta_i[\xi,\nu] = \eta_i$, $\beta_i[\xi,\nu] = \beta_i$ etc.

Also $\sigma^{(i)}[\xi,\nu] = \sigma^{(i)}$. We set:

$\eta^* = \eta^*[\xi,\nu] = \eta_p$, $\beta^* = \beta^*[\xi,\nu] = \beta_p$,

$\sigma^* = \sigma^*[\xi,\nu] = \sigma^{(p)}$ etc.

Then:

<u>Lemma 0.1</u> $S(\xi,\nu) = \widehat{S(\xi,\nu)\restriction i \; S}(\eta_i,\nu_i)$ $(i \le p)$

In particular, $\eta_{i+1}[\xi,\nu] = \eta[\eta_i,\nu_i]$ etc. Thus, if $\sigma' = \sigma^*[\eta_i,\nu_i]$, we have: $\sigma^* = \sigma'\sigma^{(i)}$!

<u>Lemma 0.2</u> Let $\gamma < \nu$ be a successor cardinal in N_ξ. Then $\sigma^{(i)}\restriction\gamma+1 = id$ and γ is a successor cardinal in N_{η_i} for $i \le p$.

proof.

Ind. on i. $i=0$ is trivial. Now let it hold at $i<p$.

<u>Claim</u> $\sigma_{i+1}\restriction(\gamma+1) = id$ and γ is a successor cardinal in $N_{\eta_{i+1}}$.

$\sigma_{i+1} : N_{\eta_i} \| \beta_{i+1} \longrightarrow N_{\eta_{i+1}}$ is the core map, where $\gamma < \nu \leq \beta_{i+1} < ht(N_{\eta_i})$. Thus $\omega\rho^{\omega}_{N_{\eta_i}\|\beta_{i+1}} \geq \gamma$, where $\sigma_{i+1} \upharpoonright \omega\rho^{\omega}_{N_{\eta_i}\|\beta_{i+1}} = id$. Thus $\sigma_{i+1} \upharpoonright \gamma = id$ and it suffices to show: $\sigma_{i+1}(\gamma) = \gamma$. Suppose not. Then $\omega\rho^{\omega}_{N_{\eta_i}\|\beta_{i+1}} = \gamma$. Let $\gamma = \tau^{+N_{\eta_i}}$.

Then $\gamma = \tau^{+N_{\eta_i}\|\beta_{i+1}}$ and $\sigma_{i+1}(\gamma) = \tau^{+N_{\eta_{i+1}}} > \gamma > \tau$. Hence γ is not a cardinal in $N_{\eta_{i+1}}$. But $\gamma = \omega\rho^{\omega}_{N_{\eta_{i+1}}}$ is a cardinal in $N_{\eta_{i+1}}$.

(Note: If $\bar{\beta} = \bar{\beta}(M, \nu)$, then $\omega\rho^{\omega}_{M\|\bar{\beta}} < \nu$, since $\omega\rho^{\omega}_{M\|\nu} < \nu$ and $\omega\rho^{\omega}_{M\|\beta} \leq \omega\rho^{\omega}_{M\|\nu}$. Thus $\omega\rho^{\omega}_{N_{\eta_i}\|\beta_{i+1}} < \nu_i$, since $\beta_{i+1} = \bar{\beta}(N_{\eta_i}, \nu_i)$.) QED (2)

As a corollary, we get:

Cor 0.3 Let $\gamma < \nu$ be a cardinal in N_ξ. Then $\sigma^{(i)} \upharpoonright \gamma = id$ and γ is a cardinal in N_{η_i}.

We now return to our specific array $\vec{N} = \langle N_i \mid i \leq \xi \rangle$. We suppose that $\sigma : P \xrightarrow[\Sigma^*]{} N_\gamma \min(\vec{\rho})$, where P is countable. Let $\mathcal{I} = \langle\langle P_i \rangle, \ldots, T \rangle$ be a direct putative normal iteration of P of countable length ("putative" meaning that the last element, if there is one, might be ill founded). We claim that one of the following holds.

(A) $\mathcal{I}$ has a last element h and there is $\sigma' : P_h \xrightarrow[\Sigma^*]{} N_{\gamma'} \min(\vec{\rho}')$ s.t.

(i) If π_{0h} is not total, then $\gamma' < \gamma$.

(ii) If π_{0h} is total, then $\gamma' = \gamma$, $\vec{\rho}' = \vec{\rho}$ and $\sigma' \pi_{0h} = \sigma$.

(B) $\mathcal{I}$ has a maximal branch b of limit length s.t. there is $\sigma' : P_b \xrightarrow[\Sigma^*]{} N_{\gamma'} \min(\vec{\rho}')$ s.t.

(i) If π_b is not total, then $\gamma' < \gamma$

(ii) If π_b is total, then $\gamma' = \gamma$, $\vec{\rho}' = \vec{\rho}$ and $\sigma' \pi_b = \sigma$.

Suppose not. Let γ be the least counter-example. Then γ is definable from $\vec{N}$, P, δ in V_θ. Let $\mathcal{Y} = \langle\langle P_i\rangle, \langle \nu_i\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be a counterexample. Following Steel we define:

Def Let $\Gamma = lh(\mathcal{Y})$, $n^*: \Gamma \xrightarrow{1-1} \omega$.
Set: $n(i) = \min\{n^*(j) \mid i \leq_T j\}$ for $i < \Gamma$.

<u>Lemma 1.1</u>
(a) $(n(i) = n(j) \wedge i < j) \longrightarrow i \leq_T j$
(b) Let b be a branch in $\mathcal{Y}$ of limit length. Then b is maximal iff $\sup n''b = \omega$.

pf. trivial.

Def i <u>survives at</u> j (in symbols: $i \operatorname{surv} j$) iff $i \leq j$, $n(i) = n(j)$, and there is no $h \in (i, j)$ s.t. $n(h) < n(i)$.

(<u>Note</u> ~~It~~ follows that if $h \in (i, j)$ and $h \notin (i, j)_T$, then $n(i) < n(h)$, since otherwise $n(i) = n(h) = n^*(k)$, where $k \geq_T h \geq_T i$. But then $k \not\leq_T j$. Hence $n(j) \neq n(i)$. Contr!)

Lemma 1.2

(a) i surv j surv $k \longrightarrow i$ surv k

(b) $(i$ surv $j \wedge i \leq_T h \leq_T j) \longrightarrow i$ surv h surv j

(c) Let b be a branch of limit length. b is maximal iff for all $i \in b$ there is $j \in b$ s.t. $i < j$ and i does not survive at j.

prf.
(a), (b) are trivial, as is $(\longrightarrow)$ of (c). We prove $(\longleftarrow)$ of (c). Suppose not. Then b is not maximal; hence $b = \{i \mid i <_T \lambda\}$, where $\lambda = \sup b$. But there are arbitrarily large $j < \lambda$ s.t. $j \not\leq_T \lambda$ and $n(j) < n(\lambda)$. Thus there are arbitrarily large $j < \lambda$ s.t. $n(j) = n$ for a fixed $n < n(\lambda)$. Let $n = n^*(h)$. Then $j \leq_T h$ for arbitrarily large $j < \lambda$. But $\{j \mid j \leq_T h\}$ is closed. Hence $\lambda \leq_T h$ and $n(\lambda) \leq n(h) \leq n < n(\lambda)$, Contr!

QED (Lemma 1.2)

Our assumption that (B) fails says that a certain tree $U = U(\delta, \gamma, \vec{\rho})$ is ill founded. We define:

Def Let $\delta : P \xrightarrow[\Sigma^*]{} N_\gamma$ $\min(\vec{\rho})$.

$U = U(\delta, \gamma, \vec{\rho})$ is the set of sequences $\langle \langle l_i, \delta_i, \gamma_i, \vec{\rho}^{\,i} \rangle \mid i \leq n \rangle$ s.t.

(a) $\delta_0 = \delta$, $\gamma_0 = \gamma$

(b) $l_i \leq_T l_{i+1}$, but does not survive at l_{i+1}

(c) $\delta_i : P_{l_i} \xrightarrow[\Sigma^*]{} N_{\gamma_i}$ $\min(\vec{\rho}^{\,i})$

(d) If $\pi_{i,i+1}$ is not total, then $\gamma_{i+1} < \gamma_i$

(e) If $\pi_{i,i+1}$ is total, then $\gamma_i = \gamma_{i+1}$, $\vec{\rho}^{\,i} = \vec{\rho}^{\,i+1}$, and $\delta_{i+1} \pi_{i,i+1} = \delta_i$.

We order U by: $s <_U s' \longleftrightarrow s \supsetneq s'$.

Thus $U = U(\delta, \gamma, \vec{\rho})$ is well founded by the fact that (B) fails.

We now recall the definition of a coarse premouse $R = \langle R, \in, \Theta_R \rangle$: R is a transitive model satisfying:

(a) nullset, pairing, union, powerset, separation, infinity and choice (in the form $\bigwedge x \bigvee \alpha\ x \sim \alpha$).

(b) Σ_2 collection: $\bigwedge x \bigvee y \varphi \to \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v \varphi$ for Σ_2 fmlae φ.

(c) V_θ collection, where $\theta = \theta^R \in R$:

$$\bigwedge x \in V_\theta \bigvee y \varphi \to \bigvee v \bigwedge x \in V_\theta \bigvee y \in v \varphi$$

for arbitrary fmlae φ.

We also require:

(d) ${}^\omega R \subset R$.

Note that by (d) we have: $P \in R$ and, in fact, $\mathcal{T} \in R$.

Def Let $R = \langle R, \in, \theta \rangle$ be a coarse premouse. We call R correct iff $Q \in V_\theta^R$ and:

(i) There are $\langle \mathcal{T}^i \mid i \leq \xi \rangle$, $\langle Q_i \mid i \leq \xi \rangle$, $\langle N_i \mid i \leq \xi \rangle$ which are "defined" from Q in R as above. In particular, $\xi < \theta$.

(ii) There is $\gamma \leq \xi$ defined as above — i.e. γ is minimal s.t. for some $\delta : P \to N_\gamma \min(\vec{p})$, (A) and (B) fail in R.

(Note If $M \in R$ and $\delta : P \to M \bmod (\vec{p})$, then $\delta, \vec{p} \in R$ by (d))

The word "defined" is placed in quotes in (i), since $\langle \mathcal{T}^i \mid i \leq \xi \rangle$ may not be uniquely defined from Q in R.

This will happen if for some limit λ, $\mathcal{J}^\lambda|\lambda$ has two distinct cofinal well founded branches. In this case the branch $b_\lambda = \{i \mid i \leq_{T_\lambda} \lambda\}$ chosen will be economical and Q_λ will be a simple iterate of Q. $\mathcal{J}^\lambda|\lambda$, $\langle \mathcal{J}^i \mid i<\lambda \rangle$, $\langle Q_i \mid i<\lambda \rangle$ are then uniquely defined and $\lambda =$ the unique λ s.t. $\mathcal{J}^\lambda|\lambda$ has distinct cofinal well founded branches is uniquely defined from Q in R. But then $\langle N_i \mid i \leq \lambda \rangle$ is also uniquely defined. If $\xi > \lambda$, then $Q_i = Q_\lambda$ for $\lambda \leq i \leq \xi$ and $\mathcal{J}^i$ is obtained by repetition of Q_λ for $\lambda \leq i \leq \xi$. Moreover $\langle N_i \mid \lambda \leq i \leq \xi \rangle$ has a canonical definition, since extenders are never applied to the sequence after λ. This in fact canonically defines a sequence $\vec{N}^R = \langle N_i^R \mid i<\theta \rangle$, if we carry out the canonical extender free procedure to θ. $\vec{N}^R$ is thus uniquely defined from Q. But then so is $\gamma = \gamma^R$. Hence for each correct coarse premouse R we have a uniquely defined $\vec{N}^R$ and a unique $\gamma = \gamma^R$ s.t. N_γ^R gives a counterexample to (A), (B).

<u>Def</u> Let $\vec{N}$ be an array. Let $i < lh(\mathcal{T})$. $\vec{\delta} = \langle \delta_h \mid h \leq_T i \rangle$ is a <u>good sequence</u> <u>for</u> $\delta, \gamma, \vec{\rho}$ wrt $\vec{N}$ iff there are $\langle \gamma_h \mid h \leq_T i \rangle$, $\langle \vec{\rho}^{\,h} \mid h \leq_T i \rangle$ s.t.

(a) $\delta_h : P_h \longrightarrow_{\Sigma^*} N_{\gamma_h} \quad \min(\vec{\rho}^{\,h})$

(b) $\delta_0 = \delta$, $\vec{\rho}^{\,0} = \vec{\rho}$, $\gamma_0 = \gamma$

(c) If π_{hj} is total, then $\vec{\rho}^{\,h} = \vec{\rho}^{\,j}$ and
$$\delta_h = \delta_j \pi_{hj}$$

(d) Let $h = T(j+1)$, $j+1 \leq_T i$ s.t. $\eta_j < ht(P_h)$. Let $\bar{\beta} = \bar{\beta}_m(P_h, \nu_h) = \eta_j$, where $m \leq p(P_h, \nu_h)$. Let $\beta = \beta_m[\gamma_h, \nu]$, where $\nu = \delta_h(\nu_h)$. Set: $\sigma = \sigma^{(m)}[\gamma_h, \nu]$, $\gamma = \eta_m[\gamma_h, \nu]$.

Then $\gamma_{j+1} = \gamma$, $\delta_{j+1} \pi_{h,j+1} = \sigma \delta_h \restriction P_j^*$ ($P_j^* = P_h \| \eta_j$) and $\vec{\rho}^{\,j+1} =$
$$= \min(\delta_{j+1}, N_{\gamma_{j+1}}, \langle \rho^n_{N_{\gamma_{j+1}}} \mid n < \omega \rangle).$$

(<u>Note</u> $\langle \gamma_h \mid h \leq_T i \rangle$, $\langle \vec{\rho}^{\,h} \mid h \leq_T i \rangle$ are uniquely determined by $\vec{\rho}, \gamma$ and $\vec{\delta}$.)

<u>Def</u> Let $\vec{\delta} = \langle \delta_h \mid h \leq_T i \rangle$ be good for $\delta, \vec{\rho}, \gamma$.
Let $j \leq_T i$. Set:
$u = u_j = \{ l \mid l \leq_T j$ and no $h < l$ survives at $l \}$
Let $\langle l_i \mid i < n \rangle$ be the monotone enumeration of u. Set:
$p_j = p_j(\vec{\delta}) = \langle \langle l_i, \delta_{l_i}, \gamma_{l_i}, \vec{\rho}^{\,l_i} \rangle \mid i \leq n \rangle$.

We obviously get:

<u>Lemma 1.3</u> Let $\vec{\delta}, i, j, \delta, \vec{\rho}, \gamma$ be as above. Then $p_j \in U(\delta, \gamma, \vec{\rho})$.

(<u>Note</u> $p_0 = \langle \langle 0, \delta, \gamma, \vec{\rho} \rangle \rangle$ is maximal in $U(\delta, \gamma, \vec{\rho})$.)

<u>Def</u> $\mathbb{E} = \langle \mathbb{E}_i \mid i < lh(\mathcal{T}) \rangle$ is a <u>prerealization</u> of $\mathcal{T}$ iff $\mathbb{E}_i = \langle R_i, \vec{\delta}^i, \vec{\rho}^{\,i,i}$ for $i < lh(\mathcal{T})$, where:

(i) R_i is a correct coarse premouse

(ii) $\vec{\delta}^i = \langle \delta^i_h \mid h \leq_T i \rangle$ is a good sequence for $\delta^i_0, \gamma^{R_i}, \vec{\rho}^{\,i,0}$ wrt $\vec{N}^{R_i}$ inducing $\langle \gamma^i_h \mid h \leq_T i \rangle, \langle \vec{\rho}^{\,i,h} \mid h \leq_T i \rangle$

(iii) $U_i = U(\delta^i_0, \gamma^i_0, \vec{\rho}^{\,i,0})$ is well founded.

If $\mathbb{E}$ is a prerealization we set:
$\delta_i = \delta_i^{\mathbb{E}} = \delta_i^i$; similarly: $\gamma_i = \gamma_i^i$, $\vec{\rho}^{\,i} = \vec{\rho}^{\,i,i}$.
We also set: $S_i = N_{\gamma_i}$, $\sigma_i^* = \sigma^*[\gamma_i, \delta_i(\nu_i)]$
$\gamma_i^* = \eta[\gamma_i, \delta_i(\nu_i)]$, $S_i^* = N_{\gamma_i^*}$.
Thus $\sigma_i^* : S_i \| \delta_i(\nu_i) \xrightarrow[\Sigma^*]{} S_i^*$. We set: $\delta_i^* = \sigma_i^* \delta_i$, $\lambda_i^* = \delta_i^*(\lambda_i)$, $\nu_i^* = \mathrm{ht}(S_i^*) = \delta_i^*(\nu_i)$.

<u>Def</u> $\mathbb{E}$ is a <u>protorealization</u> of $\mathcal{T}$ iff iff $\mathbb{E}$ is a prerealization and:

(iv) Let $h < i$. Then $\lambda_h^* < \lambda_i^*$,
$$V_{\lambda_h^*+1}^{R_h} = V_{\lambda_h^*+1}^{R_i}, \quad \delta_i \restriction \lambda_h = \delta_h^* \restriction \lambda_h,$$
and $V_{\lambda_h^*+2}^{R_h} \subset R_i$.

We now develop some properties of protorealizations.

<u>Def</u> Let $\mathbb{E}$ be a protorealization. Let $h = T(i+1)$. Set: $\bar{\beta} = \bar{\beta}_i = \eta_i^{\mathcal{T}} = \bar{\beta}_m(P_h, \nu_h)$ for some m. Set: $\beta_i = \bar{\beta}_m(S_h, \delta_h(\nu_h))$
$\tilde{\gamma}_i = \eta_m[\gamma_h, \delta_h(\nu_h)]$, $\tilde{\sigma}_i = \sigma^{(m)}[\gamma_h, \delta_h(\nu_h)]$.

Set: $\tilde{S}_1 = N^{P_1}_{\tilde{\gamma}_1}$. Then $\tilde{\sigma}_1 : S_h \| \beta_1 \to_{\Sigma^*} \tilde{S}_1$.

Set: $\tilde{\delta}_1 = \tilde{\sigma}_1 \sigma_h$. Then $\tilde{\delta}_1 : P_1^* \to \tilde{S}_1$.

(Note that if $\bar{\beta}_1 = ht(P_h)$, then $m = 0$, $\beta_1 = ht(S_h)$, $\tilde{\gamma}_1 = \gamma_h$, $\tilde{S}_1 = S_h$, $\tilde{\sigma}_1 = id$.)

Set: $\tilde{p}^1 = \begin{cases} \vec{p}^{\,h} & \text{if } \bar{\beta}_1 = ht(P_h) \\ \min(\tilde{\delta}_1, \tilde{S}_1, \langle p^n_{\tilde{S}_1} \mid n < \omega \rangle) & \text{if not.} \end{cases}$

Clearly we have:

<u>Lemma 1.4</u> $\tilde{\delta}_1 : P_1^* \to_{\Sigma^*} \tilde{S}_1 \ \min(\tilde{p}^1)$.

Set: $\nu_1^* = \delta_1(\nu_1)$, $\kappa_1^* = \delta_1(\kappa_1)$, $\tau_1^* = \delta_1(\tau_1)$, where $\tau_1 = \kappa^+_{J^{E P_1}_{\nu_1}}$. We also set:

$\tilde{\kappa}_1 = \tilde{\delta}_1(\kappa_1)$, $\tilde{\tau}_1 = \tilde{\delta}_1(\tau_1)$. Let

$\sigma' = \sigma_1' = \sigma^*[\tilde{\gamma}_1, \tilde{\delta}_1(\nu_h)]$. Then

$\sigma_h^* = \sigma_1' \tilde{\sigma}_1$ and $\delta_h^* = \sigma_1' \tilde{\delta}_1$.

<u>Lemma 1.5</u> $\sigma_1' \upharpoonright (\tilde{\tau}_1 + 1) = id$

(Hence $\tilde{\kappa}_1 = \kappa_1^*$ and $\tilde{\tau}_1 = \tau_1^*$).

proof.

τ_1 is a successor cardinal in $P_1^* = P_h \| \bar{\beta}_1$. Hence $\tilde{\tau}_1$ is a successor in $\tilde{S}_1$. But $\tilde{\tau}_1 < \tilde{\delta}_1(\nu_h)$. The conclusion follows by Lemma 0.2.

QED (Lemma 1.5)

<u>Lemma 1.6</u> $\tilde{\delta} \upharpoonright (\tau_i + 1) = \delta_h^* \upharpoonright (\tau_i + 1) = \delta_i^* \upharpoonright (\tau_i + 1)$.

proof.

The first equality follows by Lemma 1.5. The second is trivial if $h = i$. Now let $h < i$. Then $\delta_i(\tau_i)$ is a successor cardinal in $J^{ES_i}_{\delta_i(\lambda_h)}$ where $\delta_i(\lambda_h)$ is a cardinal in S_i. Hence $\delta_i(\tau_i) < \delta_i(\nu_i)$ is a successor cardinal in S_i. Hence $\sigma_i \upharpoonright (\delta(\tau_i) + 1) = id$. Hence $\delta_i^* \upharpoonright (\tau_i + 1) = \delta_i \upharpoonright (\tau_i + 1) = \delta_h^* \upharpoonright (\tau_i + 1)$.

QED (Lemma 1.6)

<u>Lemma 1.7</u> $J^{E\tilde{S}_i}_{\tilde{\tau}_i} = J^{ES_h^*}_{\tilde{\tau}_i} = J^{ES_i^*}_{\tilde{\tau}_i}$

prf. Let $u = J^{EP_i}_{\tau_i}$. Then

$\tilde{\delta}_i(u) = \delta_h^*(u) = \delta_i^*(u)$ by Lemma 1.6.

QED (Lemma 1.7)

<u>Note</u> Let $\tau_i^{(n)}$ = the n-th cardinal successor of τ_i in $J^{EP_i}_{\lambda_i}$. Set $\tilde{\tau}_i^{(n)} = \tilde{\delta}_i(\tau_i^{(n)})$, $\tau_i^{*(n)} = \delta_i^*(\tau_i^{(n)})$.

Then Lemmas 1.5 – 1.7 hold with $\tau_i^{(n)}$ in place of τ_i.

<u>Def</u> $\mathbb{E}$ is a <u>realization</u> of $\mathcal{J}$ iff

(a) $\mathbb{E}$ is a protorealization of $\mathcal{J}$

(b) Let $h <_T i$ s.t. π_{hi} is total. Let $\mu < \delta_h(\beta)$ where $\beta < \kappa_h$. Then:

$$V_{\theta_h}^{R_h} \models \varphi(\mu, \delta_h(x), \vec{\rho}^{\,h}, S_h) \longleftrightarrow$$

$$\longleftrightarrow V_{\theta_i}^{R_i} \models \varphi(\mu, \delta_i \pi_{hi}(x), \vec{\rho}^{\,i}, S_i),$$

where $x \in P_h$ and φ is 1-st order.

(c) Let $h = T(i+1)$. There is $G = G_i : P_{i+1} \to \tilde{S}_i$ s.t.

(i) $G\pi_{h,i+1} = \tilde{\delta}_i$

(ii) Let $\mu < G(\beta)$, $\beta < \kappa_i$. Then

$$V_{\theta_h}^{R_h} \models \varphi(\mu, G(x), \tilde{\rho}^{\,i}, \tilde{S}_i) \longleftrightarrow$$

$$\longleftrightarrow V_{\theta_{i+1}} \models \varphi(\mu, \delta_{i+1}(x), \vec{\rho}^{\,i+1}, S_{i+1})$$

where $x \in P_{i+1}$ and φ is 1-st order.

(iii) Let f be a partial map of $\tau < \tilde{\kappa}_i$ to $\mathcal{P}(\tilde{\kappa}_i) \cap \tilde{S}_i$ which is $\Sigma^*(\tilde{S}_i)$ in parameters from $\mathrm{rng}(G) \cup \{\tilde{\rho}^{\,i}_m \mid m < \omega \wedge \tilde{\rho}^{\,i}_m \in \tilde{S}_i\}$. Let $X = f(\xi)$ be defined. Let $\alpha_1, \ldots, \alpha_n < \lambda_i$. Then

$$\langle G(\vec{\alpha})\rangle \in X \longleftrightarrow X \in F_{\delta^*_{i+1}(\langle \vec{\alpha}\rangle)}$$

where $F = E_{\nu^*}^{S_i^*}$.

We easily get:

<u>Lemma 1.8</u> Let $\mathbb{E}$ be a realization of $\mathcal{I}$ and let $h = T(j+1)$. Then

(a) $G_j : P_{j+1} \xrightarrow[\Sigma^*]{} \tilde{S}_j \min(\tilde{\rho}^j)$

(b) Let $x = \pi_{h,j+1}(f)(\alpha)$, where $f \in \Gamma^*(P_j^*, \kappa_j$, $\alpha < \lambda_j$. Then $G_j(x) = \tilde{\delta}_j(f)(G_j(\alpha))$

<u>Lemma 1.9</u> Let $\mu < G_j(\beta)$, $\beta < \kappa_j$, where $h = T(j$

Then:

(a) $\tilde{S}_j \models \varphi(\mu, G(\vec{x})) \longleftrightarrow S_{j+1} \models \varphi(\mu, \delta_{j+1}(\vec{x}))$

for $\vec{x} \in P_{j+1}$, φ a 1-st order formula.

(b) $\tilde{S}_j \models \varphi(\mu, G(\vec{x})) \bmod (\tilde{\rho}^j) \longleftrightarrow$

$\longleftrightarrow S_{j+1} \models \varphi(\mu, G(\vec{x})) \bmod (\tilde{\rho}^{j+1})$

for $\vec{x} \in P_{j+1}$, φ a Σ^* formula.

<u>Lemma 1.10</u> Let μ, h, j be as above and let $x_\ell = \pi_{h,j+1}(f_\ell)(\alpha_\ell)$ $(\ell = 1, ..., n)$, where $f_\ell \in \Gamma^*(P_j^*, \kappa_j)$, $\alpha_\ell < \lambda_j$. Let φ be a $\Sigma_0^{(m)}$ fmla, where $\omega\rho^m_{P_j} > \kappa_j$. Then

$\tilde{S}_j \models \varphi(\mu, G(\vec{x})) \bmod (\tilde{\rho}^j) \longleftrightarrow$

$\longleftrightarrow \{\vec{\xi} \mid \tilde{S}_j \models \varphi(\mu, \tilde{\delta}_j(\vec{f})(\vec{\xi})) \bmod (\tilde{\rho}^j)\} \in F_{\delta_j^*(\vec{\alpha})}$

where $F = F^{S_j^*}$

proof of Lemma 1.10

Set $X_\mu = \{\vec{\xi} \mid \tilde{S}_i \models \varphi(\mu, \tilde{\delta}_i(\vec{f})(\xi)) \bmod \tilde{\rho}^i\}$

Then $\mu \mapsto X_\xi$ is a $\Sigma^*(\tilde{S}_i, \tilde{\rho}^i)$ function defined on $G(\beta)$. Hence

$\tilde{S}_i \models \varphi(\mu, G(\vec{x})) \bmod (\tilde{\rho}) \longleftrightarrow$

$\longleftrightarrow G(\vec{\alpha}) \in X \longleftrightarrow X \in F_{\delta_i^*(\vec{\alpha})}$.

QED (Lemma 1.10)

<u>Lemma 2</u> Let $\mathbb{E}$ be a realization of $\mathcal{I}$. Let $h = T(i+1)$. Set $F = E^{P_i}_{\nu_i}$, $\tilde{F} = E^{S_i^*}_{\nu_i^*}$.
Then

(a) $\langle \tilde{\sigma}_i, \sigma_i^* \upharpoonright \lambda_i \rangle : \langle P_i, F \rangle \xrightarrow{*} \langle \tilde{S}_i, \tilde{F} \rangle$

(b) $\langle \tilde{\sigma}_i, \sigma_i^* \upharpoonright \lambda_i \rangle : \langle P_i, F \rangle \xrightarrow{**} \langle \tilde{S}_i | \tilde{\rho}_0', \tilde{F} \rangle$.

The proof requires some sublemmas. The case $\nu_i < ht(P_i)$ will turn out to be easy, since then $F_\alpha \in P_i^*$ for $\alpha < \lambda_i$. We must essentially deal only with the case $\nu_i = ht(P_i)$ (hence $\sigma_i^* = \sigma_i$, $\sigma_i^* = id$, $S_i^* = S_i$).

For technical convenience, we first make the following definition:

<u>Def</u> Consider a structure:
$\mathcal{I} = \langle \langle P_l | l \leq i \rangle, \langle \nu_l | l \leq i \rangle, \langle \eta_l | l \leq i \rangle, \langle \pi_{hl} | h \leq_T l \leq i \rangle, T \rangle$, where T is an iteration tree of length $i+2$ (hence $T(i+1)$ is defined).
We call $\mathcal{I}$ a <u>preemptive</u> <u>iteration of length</u> $i+1$ iff the following hold:

(a) $\mathcal{J}' = \mathcal{J}|(i+1)$ is a direct normal iteration of length $i+1$, where:
$\mathcal{J}' = \langle\langle P_\ell \mid \ell \le i\rangle, \langle \nu_\ell \mid \ell < i\rangle, \langle \eta_\ell \mid \ell < i\rangle, \langle \pi_{h\ell} \mid h \le_T \ell \le i\rangle, T\cap(i+1)^2$

(b) $\nu_i, \eta_i, T(i+1)$ ... so chosen that $\mathcal{J}'$ can be continued — i.e.

(i) $\nu_i > \nu_\ell$ for $\ell < i$; $E^{P_i}_{\nu_i} \ne \emptyset$

(ii) $T(i+1) =$ the least h s.t. $\kappa_i < \lambda_h$

(iii) $\eta_i =$ the maximal η s.t. τ_i is a cardinal in $P_{T(i+1)} \| \eta$.

. . . . .

$\mathbb{E}$ is called a <u>realization</u> of the preemptive structure $\mathcal{J}$ iff it is a realization of $\mathcal{J}' = \mathcal{J}|(i+1)$.

Note that if $\mathcal{J}$ is a direct normal iteration and $i+2 < lh(\mathcal{J})$, then there is a unique preemptive iteration $\overline{\mathcal{J}}$ of length $i+1$ s.t. $\overline{\mathcal{J}}|(i+1) = \mathcal{J}|(i+1)$ and $\nu_i^{\overline{\mathcal{J}}} = \nu_i^{\mathcal{J}}$.

<u>Lemma 2.1</u> Let $\mathcal{I}$ be a preemptive countable iteration of length $i+1$. Let $\mathbb{E}$ be a realization of $\mathcal{I}$ s.t. (a), (b) of Lemma 2 hold for $j<i$. Let $\nu_i = ht(P_i)$. Then:

(+) Let $A \subset \tau_i$ in a parameter p and let $\tilde{A} \subset \tilde{\tau}_i$ be $\Sigma_1(S_i)$ in $\tilde{p} = \delta_i(p)$ by the same definition. Then A is $\Sigma_1(P_i^*)$ in a parameter q and $\tilde{A}$ is $\Sigma_1(\tilde{S}_i)$ in $\tilde{q} = \tilde{\delta}(q)$ by the same definition.

(<u>Remark</u> $\delta_i = \delta_i^*$, $S_i = S_i^*$, since $\nu_i = ht(P_i)$)

proof. Suppose not.
Let i be the least counterexample. Let $h = T(i+1)$. Then $h < i$. Moreover $i = j+1$.
(Otherwise Lim(i) and there is $j <_T i$ s.t. π_{ji} is total, crit$(\pi_{ji}) > \lambda_h$, and $\pi_{ji}(p') = p$ for some p'. Then A is $\Sigma_1(P_j)$ in p'. Moreover, by (b) in the def. of "realization", $\tilde{A}$ is $\Sigma_1(\tilde{S}_j)$ in $\tilde{p}' = \delta_j(p')$. Let $\overline{\mathcal{I}}$ be the <u>preemtive</u> iteration of <u>length</u> $j+1$ s.t. $\overline{\mathcal{I}}|(j+1) = \mathcal{I}|(j+1)$ and $\nu_j^{\overline{\mathcal{I}}} = ht(P_j) = \pi_{ji}^{-1}(\nu_i)$. Then $\overline{\mathcal{I}}$ is a shorter counterexample. Contr!) Now let $\xi = T(i)$.

(1) $\kappa_i < \kappa_j$

prf. Let $\kappa' = \pi_{\xi i}^{-1}(\kappa_i) = \text{crit}(F')$ where F' is the top extender of P_j^*. Then

$\kappa' < \kappa_j$, since otherwise $\kappa_i = \pi_{\xi_i}(\kappa') \geq$
$\geq \pi_{\xi_i}(\kappa_j) = \lambda_i$. Hence $h = i$. Contr!
Hence $\kappa_i = \pi_{\xi_i}(\kappa') = \kappa' < \kappa_j$. QED (1)

(2) $h \leq \xi$, since $\kappa_i < \kappa_j < \lambda_\xi$.

(3) $\omega\rho^1_{P_i} \leq \tau$.

prf.

Suppose not. Let $A \subset \tau_i$ be $\Delta_1(P_i)$ in p and $\tilde{A} \subset \tilde{\tau}_i = \sigma_i(\tau_i)$ be $\Delta_1(S_i)$ in $\tilde{p} = \sigma_i(p)$ by the same definition. Then

$$A\zeta \longleftrightarrow P_i \models \bigvee z\, \varphi_0(z, \zeta, p)$$

$$\neg A\zeta \longleftrightarrow P_i \models \bigvee z\, \varphi_1(z, \zeta, p),$$

where φ_0, φ_1 are Σ_0. Similarly for $\tilde{A}$ with $S_i, \tilde{p}$ for P_i, p. Since $\tau_i < \omega\rho^1_{P_i}$, we conclude:

$$A \in \mathbb{P}(\tau_i) \cap P_i \subset J^{E^{P_i}}_{\lambda_h} = J^{E^{P_h}}_{\lambda_h} \subset P_i^*.$$

$x = A$ is $\Pi^1_0(P_i)$ in p, τ_i by the definition:

(*) $x \subset \tau_i \wedge \bigwedge \xi \in \tau_i\, (\xi \in x \longleftrightarrow \bigvee z\, \varphi_0(z, \zeta, p))$.

But in P_i we have:

(**) $\bigwedge \zeta < \tau\, \bigvee z\, (\varphi_0(z, \zeta, p) \vee \varphi_1(z, \zeta, p))$.

(**) is a Π^1_0 statement. Since

$\delta_i : P_i \xrightarrow{\Sigma^*} S_i \min(\vec{p}^{\,i})$, (**) must hold in $S_i \bmod (\vec{p}^{\,i})$ with $\tilde{\tau}_i, \tilde{p}$ in place of τ_i, p. It follows easily that $x = \tilde{A}$ in $\Pi_0^1(S_i, \vec{p}^{\,i})$ in $\tilde{\tau}_i, \tilde{p}$ by the definition (*) (with $\tilde{\tau}_i, \tilde{p}$ in place of τ_i, p). Hence $\delta_i(A) = \tilde{A}$. But $\delta_i \upharpoonright (\tau_i^+ + 1) = \tilde{\delta}_i \upharpoonright (\tau_i^+ + 1)$ (where $\tilde{\tau}_i^+ = \tau_i^{+J_i}$, by the Note following Lemma 1.7.). Hence $\tilde{\delta}_i(A) = \tilde{A}$. Hence A is $\Delta_1(P_i^*)$ in A and $\tilde{A}$ is $\Delta_1(\tilde{S}_i)$ in $\tilde{A} = \tilde{\delta}_i(A)$ by the same definition. Hence $\mathcal{J}$ is not a counter-example. Contr! QED (3)

(4) $\omega\rho^1_{P_i^*} \le \tau$, since $\pi_{3i}(\tau) = \tau$, $\pi_{3i} : P_i^* \xrightarrow{\Sigma^*} P_i$.

(5) Let $A \subset \tau_i$ be $\Sigma_1(P_i)$ in p and $\tilde{A} \subset \tilde{\tau}_i$ be $\Sigma_1(S_i)$ in $\tilde{p} = \delta_i(p)$ by the same definition. Then A is $\Sigma_1(P_i^*)$ in some q and A is $\Sigma_1(\tilde{S}_i)$ in $\tilde{q} = \tilde{\delta}_i(q)$ by the same definition.

proof.

Let $A\xi \longleftrightarrow \bigvee z\, B(z, \xi, p)$ where B is $\Sigma_0(P_i)$, $\tilde{A}\xi \longleftrightarrow \bigvee z\, \tilde{B}(z, \xi, \tilde{p})$ where B is $\Sigma_0(S_i)$ by the same definition.

$\pi_{\xi, i} : P_{1}^{*} \xrightarrow[F]{} P_{i}$ is a Σ_0 ultrapower by (4), where $F = E_{\nu_1}^{P_i}$. Thus $p = \pi_{\xi i}(f)(\alpha)$, where $\alpha < \lambda_1$ and $f : \kappa_1 \to P_1^{*}$, $f \in P_1^{*}$. Thus:

$$A\xi \leftrightarrow \bigvee u \in P_1^{*} \bigvee z \in \pi_{\xi i}(u)\ B(z, \xi, \pi_{\xi i}(f)(\alpha))$$
$$\leftrightarrow \quad " \quad \{\mu < \kappa_1 \mid \bigvee z \in u\, B'(z, \xi, f(\mu))\} \in F_\alpha$$
$$\leftrightarrow \quad " \quad \bigvee X\, (X = \{\mu < \kappa_1 \mid \bigvee z \in u\, B'(z, \xi, f(\mu))\} \wedge X \in F_\alpha$$

where B' is $\Sigma_0(P_1^{*})$ by the same definition. Hence A is $\Sigma_1(P_1^{*})$ in $\langle \kappa_1, f, r \rangle$, where r is so chosen that F_α is $\Sigma_1(P_1^{*})$ in r and $\tilde{F}_{\sigma_1^{*}(\alpha)}$ is $\Sigma_1(\tilde{S}_1)$ in $\tilde{r} = \tilde{\sigma}_1(r)$, where $\tilde{F} = E_{\nu_1^{*}}^{S_1^{*}}$. (This is possible, since (a) of Lemma 2 holds at 1.) Our claim is that $\tilde{A}$ has the same definition in $\langle \tilde{\kappa}_1, \tilde{\sigma}_1(f), \tilde{r} \rangle$ — i.e.

<u>Claim</u> $A\xi \leftrightarrow \bigvee u \in \tilde{S}_1 \bigvee X\, (X \in \tilde{F}_{\sigma_1^{*}(\alpha)} \wedge$
$$\wedge\, X = \{\mu < \tilde{\kappa}_1 \mid \bigvee z \in u\, \tilde{B}'(z, \xi, \tilde{\sigma}_1(f)(\mu)\},$$

where $\tilde{B}'$ has the same $\Sigma_0(\tilde{S}_1)$ definition.

Let $G = G_1 : P_i \xrightarrow[\Sigma^{*}]{} \tilde{S}_1 \min(\tilde{\rho}^1)$ be as in the def. of "realization". Note that $\tilde{\tau}_i = \sigma_i(\tau_i) = \sigma_\xi^{*}(\tau_i) = \tilde{\sigma}_i(\tau_i)$, since

$\sigma^*_\xi \upharpoonright (\tau_i + 1) = \tilde{\sigma}_i \upharpoonright (\tau_i + 1)$. But then $\tilde{\tau}_i = \tilde{\sigma}_i(\pi_{\xi i}(\tau_i)) = G(\tau_i)$. By Lemma 1.9 we get:

$$A\varsigma \longleftrightarrow \bigvee z\ \tilde{B}'(z, \varsigma, G(p))',$$

where $G(p) = \tilde{\sigma}_i(f)(G(\alpha))$. Note that $\tilde{\kappa}_i = \tilde{\sigma}_i(\kappa_i) = \mathrm{crit}(\hat{F})$, where $\hat{F}$ is the top extender of $\tilde{S}_i$. Hence there is a $\Sigma_1(\tilde{S}_i)$ map g of $\tilde{\tau}_i$ cofinally to $\tilde{\nu} = ht(\tilde{S}_i)$ defined by:

$$g(\xi) = \text{the least } \varsigma \text{ s.t. } \bigwedge x \in \mathbb{P}(\tilde{\kappa}_i) \cap J^E_\xi \ \bigvee y \in J^E_\varsigma \ \ y = \hat{F}(x).$$

Set: $X_{\xi,\varsigma} = \{\mu < \tilde{\kappa}_j \mid \bigvee z \in J^{E^{\tilde{S}_j}}_{g(\xi)}\ \tilde{B}'(z, \varsigma, \tilde{\sigma}_i(f)(\mu))$

für $\xi, \varsigma < \tilde{\tau}_i$. Then $\xi, \varsigma \longmapsto X_{\xi,\varsigma}$ is a $\Sigma_1(\tilde{S}_i)$ map in $\tilde{\sigma}_i(f)$, $\tilde{\kappa}_j = \tilde{\sigma}_i(\kappa_j) =$ and $\tilde{\tau}_i = \tilde{\sigma}_i(\tau_i)$, where $\tilde{\sigma}_i(x) = G(\pi_{\xi i}(x))$.

By (c)(iii) in the def. of "realization" it follows that:

$$G(\alpha) \in X_{\xi,\varsigma} \longleftrightarrow X_{\xi,\varsigma} \in \tilde{F}_{\sigma_i(\dot{\alpha})},$$

where $\tilde{F} = E^{S_i}_{\nu_i}$. We can now prove the claim. To see $(\longleftarrow)$, assume the right side to hold.

Then $X_{\xi,\varsigma} \in \tilde{F}_{\delta_i(\alpha)}$ for some $\xi < \tilde{\tau}_i$. Hence $G(\alpha) \in X_{\xi,\varsigma}$. Hence $V \models \tilde{B}'(z, \varsigma, \tilde{\delta}_1(f)(G(\alpha)))$. Hence $A\varsigma$. We now prove $(\rightarrow)$. Let $A\varsigma$. Then there is z s.t. $\tilde{B}'(z, \varsigma, \tilde{\delta}_1(f)(G(\alpha)))$. Let $z \in J^{E^{\tilde{S}_i}}_{g(\xi)}$, where $\xi < \tilde{\tau}_i$. Then $G(\alpha) \in X_{\xi,\varsigma}$. Hence $X_{\xi,\varsigma} \in \tilde{F}_{\delta_i(\alpha)}$, which gives the desired conclusion.

QED (5)

(6) $\xi > h$

proof

Otherwise $\xi = h$ by (2). Hence $\eta_i \geq \eta_1$, since $\tau_i < \tau_1$.

<u>Case 1</u> $\eta_i = \eta_1$. Then $\tilde{S}_i = \tilde{S}_1$ and η is not a counterexample by (5). Contr!

<u>Case 2</u> $\eta_i > \eta_1$. Let $A, \tilde{A}, q, \tilde{q}$ be as in (5). Then $A \in P_i^*$. Let $\eta_i = \bar{\beta}_i = \bar{\beta}_m$, $\eta_1 = \bar{\beta}_1 = \bar{\beta}_n$. Then $n > m$. Set: $\beta' = \tilde{\delta}_i(\bar{\beta}_m)$; $\sigma' = \sigma^{(n-m)}[\tilde{\delta}_i, \tilde{\delta}_i(\nu_h)]$. Then $\sigma' : \tilde{S}_i \| \beta' \xrightarrow[\Sigma^*]{} \tilde{S}_1$ and $\sigma' \tilde{\delta}_i = \tilde{\delta}_1$.

But $\sigma' \upharpoonright (\tilde{\tau}_i + 1) = \mathrm{id}$, since $\tilde{\tau}_i$ is a successor cardinal in $\tilde{S}_i$, $\tilde{\tau}_i < \tilde{\delta}_i(\nu_h)$.

Since $\tilde{A}$ is $\Sigma_1(\tilde{S}_i)$ in $\tilde{q}$, by (5), it follows that $\tilde{A}$ is $\Sigma_1(\tilde{S}_i \| \beta')$ in $q' = \sigma^{-1}(\tilde{q}) = \tilde{\delta}_i(q)$ by the same definition. Since $\tilde{\delta}_i(P_i^*) = \tilde{S}_i \| \beta'$, $\tilde{\delta}_i(q) = q'$, we conclude that $\tilde{\delta}_i(A) = \tilde{A}$. Hence A is $\Sigma_1(P_i^*)$ in A and $\tilde{A}$ is $\Sigma_1(\tilde{S}_i)$ in $\tilde{A} = \tilde{\delta}_i(A)$ by the same definition. Contr!

QED (6)

(7) $\eta_i = ht(P_\xi)$ (i.e. $P_\xi = P_i^*$)

pf. Suppose not.

Then $\tau_i + P_\xi > \omega\eta_i = On \cap P_i^*$ by (4).

But $\tau_i < \lambda_h$, where λ_h is a limit cardinal in P_λ. Hence $\tau_i + P_\xi = \tau + J_\lambda^{P_\xi} = \tau + P_i^* < \omega\eta_i$. Contr! QED (7).

Now define a preemptive iteration $\bar{\mathcal{I}}$ of length $\xi + 1$ by: $\bar{\mathcal{I}} | (\xi+1) = \mathcal{I} | (\xi + 1)$, $\nu_\xi^{\bar{\mathcal{I}}} = ht(P_\xi)$. Then $T^{\bar{\mathcal{I}}}(\xi+1) = T^{\mathcal{I}}(i+1) = h$, $\eta_\xi^{\bar{\mathcal{I}}} = \eta_i^{\mathcal{I}}$. By (5) $\bar{\mathcal{I}}$ is then a counterexample of shorter length. Contr! QED (Lemma 2.1)

<u>Def</u> i is <u>bold</u> in $\mathcal{I}$ iff $\nu_i = ht(P_i)$ and whenever $A \subset \tau_i$ is $\Delta_1(P_i)$ in p and $\tilde{A} \subset \tilde{\tau}_i$ is $\Delta_1(P_i^*)$ in $\tilde{p} = \delta_i(p)$, then $A \in P_i^*$ and $\tilde{A} = \tilde{\delta}_i(A)$.

<u>Lemma 2.2</u> Let $\mathcal{I}$ be a preemptive countable iteration of length $i+1$. Let $\mathbb{E}$ be a realization of $\mathcal{I}$ s.t. (a), (b) of Lemma 2 hold for $j < i$. Let $\nu_i = ht(P_i)$ and assume that i is not bold. Then;

(++) Let $A \subset \tau_i$ be $\Sigma_1(P_i)$ in p and A be $\Sigma_1(S_i | \rho_0^i)$ in $\tilde{p} = \delta_i(p)$ by the same definition. Then A is $\Sigma_1(\tilde{P}_i)$ in some q and A is $\Sigma_1(\tilde{S}_i | \tilde{\rho}_0^i)$ in $\tilde{q} = \tilde{\delta}_i(q)$ by the same definition.

proof.

We again take i as a minimal counter-example and get: $h = T(i+1) < i$, $i = j+1$. Let $\xi = T(i)$. (1), (2) are proven exactly as before. (3) is proven essentially as before: Let $A \subset \tau_i$ be $\Sigma_1(P_i)$ in p and $\tilde{A} \subset \tilde{\tau}_i$ be $\Sigma_1(S_i | \rho_0^i)$ in $\tilde{p} = \delta_i(p)$ by the same def. By a similar (but easier) argument we get: $A \in \tilde{P}_i$, $\tilde{A} = \tilde{\delta}_i(A)$.

(4), (5) are then proven exactly as before. However, we also need:

(5.1) Let $A \subset \tau_i$ be $\Sigma_1(P_i)$ in p and $\tilde{A} \subset \tilde{\tau}_i$ be $\Sigma_1(\tilde{S}|\rho_0^i)$ in $\tilde{p} = \sigma_i(p)$ by the same def. Then A is $\Sigma_1(P_i^*)$ in some q and $\tilde{A}$ is $\Sigma_1(\tilde{S}_i|\tilde{\rho}_0^i)$ in $\tilde{q} = \tilde{\sigma}_i(q)$ by the same def.

prf.

We modify the proof of (5). Assume:
$A\xi \longleftrightarrow \bigvee z\, B(z,\xi,p)$, where B is $\Sigma_0(P_i)$;
$\tilde{A}\xi \longleftrightarrow \bigvee z\, \tilde{B}(z,\xi,p)$, where $\tilde{B}$ is $\Sigma_0(S_i|\rho_0^i)$
by the same def. Exactly as before we get:

$A\xi \longleftrightarrow \bigvee u \in P_i^* \bigvee X(X = \{\mu < \kappa_i \mid \bigvee z \in u\, B'(z,\xi,f(\mu))\} \wedge X \in F_\alpha$.

where B' is $\Sigma_1(P_i^*)$ by the same def. Since (b) of Lemma 2 holds at i, we know there are q, $\bar{H}$, $\bar{F}$, $\hat{H}$, $\hat{F}$ s.t.: $\bar{H} = P_i^* \cap {}^{\kappa_i}\mathcal{P}(\kappa_i)$;
$\bar{F} = F_\alpha$, where $F = E^{P_i}_{\nu_i}$; $\hat{F} \subset \tilde{F}_{\tilde{\sigma}_i^*(\alpha)}$, where $\tilde{F}$ is the top extender of S_i^*; $\bar{H}$, $\bar{F}$ are $\Sigma_1(P_i^*)$ in q and H, F are $\Sigma_1(\tilde{S}_i|\tilde{\rho}_i^0)$ in $\tilde{q} = \tilde{\sigma}_i(q)$ by the same definition.

Moreover, $H \subset \{X \in {}^{\tilde{\kappa}_i}\mathcal{P}(\tilde{\kappa}_i) \mid \bigwedge i < \tilde{\kappa}_i\, (X \cap \tilde{\kappa}_i \setminus X \in \hat{F}$

A is then $\Sigma_1(P_i^*)$ in $\langle \varkappa_i, f, q \rangle$. We wish to show that $\tilde{A}$ is $\Sigma_1(\tilde{S}_i | \tilde{\rho}_0^{i})$ in $\langle \tilde{\varkappa}_i, \tilde{\sigma}_i(f), \tilde{q} \rangle$ by the same definition. It suffices to show:

<u>Claim</u> $\tilde{A}\xi \longleftrightarrow \bigvee u \in \tilde{S}_i | \tilde{\rho}_0^{i} \bigvee X \; (X \in \hat{F} \wedge$

$\wedge \; X = \{\mu < \tilde{\varkappa}_i \mid \bigvee z \in u \; \tilde{B}'(z, \xi, f(\mu))\})$

where $\tilde{B}'$ is $\Sigma_0(\tilde{S}_i | \tilde{\rho}_0^{i})$ by the same def.

We recall that since $\tilde{\sigma}_i : P_i^* \xrightarrow[\Sigma^*]{} \tilde{S}_i \; \mathrm{min}(\tilde{\rho}^i)$, we actually have: $\tilde{\sigma}_i : P_i^* \xrightarrow[Q^*]{} \tilde{S}_i \; \mathrm{mod}(\tilde{\rho}^i)$. In particular, $\tilde{\sigma}_i : P_i^* \xrightarrow[Q]{} (\tilde{S}_i | \tilde{\rho}_0^{i})$.

For $u \in P_i^*$, $\xi < \tau_i$ set:

$$X(u, \xi) = \{\mu < \varkappa_i \mid \bigvee z \in u \; B'(z, \xi, f(\mu))\}$$

$$Y(u) = \langle X(u, \xi) \mid \xi < \tau_i \rangle.$$

Then the function $u \longmapsto \tilde{X}(u)$ is $\Sigma_0(P_i^*)$ and is defined everywhere in P_i^*. Clearly $u \subset v \to X(u, \xi) \subset X(v, \xi)$. Let $\tilde{X}(u, \xi)$, $\tilde{Y}(u)$ have the same definition in $\tilde{S}_i | \tilde{\rho}_0^{i}$, with $\tilde{\tau}_i, \tilde{\varkappa}_i$ in place of $\tau_i, \varkappa_i$. The same conclusions hold for $\tilde{S}_i | \tilde{\rho}_0^{i}$.

Define a predicate $D(\eta,\mu)$ on P_i^* by:

$D(\eta,\mu) \longleftrightarrow (\eta<\tau_i \wedge Y(J_\mu)\in H \wedge$

$\wedge\ \mu$ is least s.t. $\bigwedge x\in \mathbb{P}(\kappa_i)\cap J_\eta^E \bigvee y\in J_\mu^E\ y=F'(x$

where $E=E^{P_i^*}$ & F' is the top extender of P_i^*. Then D is $\Sigma_1(P_i^*)$ in τ_i,κ_i,q.

Let $\tilde{D}$ have the same def. in $\tilde{\tau}_i,\tilde{\kappa}_i,\tilde{q}$ over $\tilde{S}_i|\tilde{\rho}_0'$. There are arbitrarily large $\mu<\mathrm{ht}(P_i^*)$ s.t. $\bigvee\eta<\mu\ D(\eta,\mu)$. Hence there arbitrarily large $\mu<\tilde{\rho}_0'$ s.t. $\bigvee\eta<\mu\ \tilde{D}(\eta,\mu)$. For $\eta<\tilde{\tau}_i$ set:

$g(\eta)\simeq$ the least μ s.t. $\tilde{D}(\eta,\mu)$.

Then g is a partial $\Sigma_1(\tilde{S}_i|\tilde{\rho}_0')$ map in $\tilde{\tau}_i,\tilde{\kappa}_i,\tilde{q}$ and $\mathrm{rng}(g)$ is cofinal in $\tilde{\rho}_0'$. For $\eta\in\mathrm{dom}(g)$ set:

$$\tilde{Y}_\eta=\tilde{Y}(J_{g(\eta)}^{E^{\tilde{S}_i}})\ ;\ \tilde{X}_{\eta,\zeta}=\tilde{Y}_\eta(\zeta)$$

for $\zeta<\tilde{\tau}_i$. Then

$$\tilde{X}_{\eta\zeta}=\{\mu<\tilde{\kappa}_i\,|\,\bigvee z\in J_{g(\eta)}^{E^{\tilde{S}_i}}\ \tilde{B}'(z,\zeta,\tilde{\delta}_i(f)(\mu))\},$$

Since $\tilde{Y}_\eta\in H$, we know:

$\tilde{X}_{\eta\xi} \in \hat{F}$ or $\tilde{\kappa}_j \setminus \tilde{X}_{\eta\xi} \in \hat{F}$, whenever $\tilde{X}_\eta$ is defined. We also know $\hat{F} \subset \tilde{F}_{\delta_i^*(\alpha)}$, where $\tilde{F}$ is the top extender of S_i^*.

Letting $G = G_i : P_i \xrightarrow[\Sigma^*]{} \tilde{S}_i \min(\tilde{\rho}')$ be as before we get just as before:

$$G(\alpha) \in \tilde{X}_{\eta\xi} \longleftrightarrow \tilde{X}_{\eta\xi} \in \tilde{F}_{\delta_i^*(\alpha)}.$$

But then:

$$G(\alpha) \in \tilde{X}_{\eta\xi} \longleftrightarrow \tilde{X}_{\eta\xi} \in \hat{F},$$ since otherwise $\tilde{X}_{\eta\xi} \in F_{\delta_i^*(\alpha)}$ and $\tilde{\kappa}_i \setminus \tilde{X}_{\eta\xi} \in \hat{F} \subset F_{\delta_i^*(\alpha)}$. Contr!

The rest of the proof is exactly as in (5). QED (5.1)

(6) is as before: In Case 1 (5.1) shows that $\mathcal{T}$ is not a counterexample. In Case 2 we repeat the previous argument from (5) to show that $\mathcal{T}$ is bold. Contr!

(7) is exactly as before, as is the conclusion (using (5.1)).

QED (Lemma 2.2)

We are now ready to prove Lemma 2, proceeding by induction on j. We again set: $h = T(j+1)$, $F = E^{P_j}_{\nu_j}$, $\tilde{F} = E^{S_j^*}_{\nu_j^*}$

<u>Case 1</u> $F \in P_j$

Then $F_\alpha \in J^{E^{P_j}}_{\lambda_h} = J^{E^{P_j^*}}_{\lambda_h} \subset P_j^*$. Hence

$\tilde{F}_{\sigma_j^*(\alpha)} = \sigma_j^*(F_\alpha) = \tilde{\sigma}_j(F_\alpha) \in \tilde{S}_j$, since $\sigma_j^* \restriction (J^E_{\tau_j^+})^{P_j} = \tilde{\sigma}_j \restriction (J^E_{\tau_j^+})^{P_j}$ (cf. the remark following Lemma 1.7).

Thus (a) holds, since F_α is $\Sigma_1(P_j^*)$ in F_α and $\tilde{F}_{\sigma_j^*(\alpha)}$ is $\Sigma_1(\tilde{S}_j)$ in $\tilde{\sigma}(F_\alpha)$ by the same def. But $\tilde{\sigma}(F_\alpha) \in \tilde{S}_j | \tilde{\rho}_0'$, so the same argument shows:

$$\langle \tilde{\sigma}_j, \sigma_j^* \restriction \lambda_j \rangle : \langle P_j, F \rangle \xrightarrow{*} \langle \tilde{S}_j | \tilde{\rho}_0', \tilde{F} \rangle,$$

which implies (b).

<u>Case 2</u> $F \notin P_j$. Then F is the top extender. F_α is $\Delta_1(P_j)$ in α and $\tilde{F}_{\alpha^*}$ ($\alpha^* = \sigma_j^*(\alpha)$) is $\Delta_1(S_j)$ in $\alpha^* = \sigma_j(\alpha)$.

Hence by Lemma 2.1 F_α is $\Sigma_1(P_j^*)$ in some q and $\tilde{F}_{\alpha^*}$ is $\Sigma_1(\tilde{S}_j)$

in $\tilde{q} = \tilde{\delta}_i(q)$ by the same definition. This proves (a). We now prove (b)

<u>Case 2.1</u> i is bold.

Then $\tilde{F}_{\alpha *} = \tilde{\delta}_i(F_\alpha)$ and we proceed exactly as in Case 1.

<u>Case 2.2</u> Case 2.1 fails.

Set $F_\alpha = \overline{G}$ and let G be $\Sigma_1(\tilde{S}_i | \rho_0^{i})$ in by the usual def. of $\tilde{F}_{\alpha *}$. Then $G \subset \tilde{F}_{\alpha *}$. Define $\overline{H} \subset {}^{\varkappa_i}\mathbb{P}(\varkappa_i)$ by:

$X \in \overline{H} \leftrightarrow \bigvee \gamma \in P_i \bigwedge \xi < \varkappa_i \bigvee Y \in J_\gamma^{E^{P_i}} Y = F(X_\xi)$

Then $H = {}^{\varkappa_i}\mathbb{P}(\varkappa_i)$ is $\Sigma_1(Q_i)$. Let H be $\Sigma_1(S_i | \rho_0^{i})$ by the same definition. Then

$$X \in H \longrightarrow \bigwedge \xi < \tilde{\varkappa}_i \, (X_\xi \text{ or } \tilde{\varkappa}_i \setminus X_\xi \in G)$$

By Lemma 2.2, $\overline{G}, \overline{H}$ are $\Sigma_1(D_i^*)$ in some q and G, H are $\Sigma_1(\tilde{S}_i | \tilde{\rho}_0^{i})$ in $\tilde{q} = \tilde{\delta}_i(q)$ by the same definition. Hence $\overline{G}, G, \overline{H}, H$ verify (a).

QED (Lemma 2)

<u>Def</u> Let $i \leq j < lh(\mathcal{I})$.

$c(i,j) = \{h \mid j < h < lh(\mathcal{I}) \wedge h$ is a successor $\wedge$
$\wedge\, T(h) \leq i \wedge T(h)$ survives at $h\}$

<u>Lemma 3.1</u> $c(i,j)$ is finite

prf. Suppose not.

Let $n_0 \leq n(i)$ s.t. $n_0 = n(T(h)) = n(h)$ for infinitely many $h \in c(i,j)$. Let h, h' be two such with $h < h'$. Then $h \not\leq_T h'$, since otherwise $h \leq_T T(h')$. Hence $\{l \mid h \leq_T l\} \cap \{l \mid h' \leq_T l'\} = \emptyset$. Hence $n(h) \neq n(h$

Contr! QED (Lemma 3.1

<u>Def</u> Let $\mathfrak{R} = \langle R, \in, \theta \rangle$ be a coarse mouse. ξ is a <u>cutoff point</u> of $\mathfrak{R}$ iff $\theta < \xi < rn(R)$ and $\langle V_\xi^R, \in, \theta \rangle$ is a coarse mouse.

<u>Def</u> Let $\mathbb{E}$ be a realization of $\mathcal{I}$. For $i < lh(\mathcal{I})$ set: $|x|_{U_i} = \text{lub}\{|y|_{U_i} \mid y < x \text{ in } U_i\}$ for $x \in U_i$. Set: $|U_i| = \text{lub}\{|x|_{U_i} \mid x \in U_i\}$.

Set: $p_i = p_i^{\mathbb{E}} =_{\text{Df}} p_i(\vec{\delta}^i) =$ The point in U_i determined by the good sequence $\vec{\delta}^i$.

Def Let $\mathbb{E}$ be a realization of $\mathcal{T}|\gamma$, where $\gamma < lh(\mathcal{T})$. $\mathbb{E}$ <u>has room</u> iff for all $i < \gamma$, $R_i^{\mathbb{E}}$ has at least $\omega \cdot |p_i| + |C(i,\gamma)|$ many cutoff points (in order type).

(Note that $|p_0| = |M_0|$).

Def Let $i < \gamma \leq lh(\mathcal{T})$. i is a <u>break point</u> <u>at</u> γ iff whenever $i < h \leq \gamma$ s.t. $T(h) \leq i$, then $T(h)$ does not survive at h.
(In other words $C(i,i)^{\mathcal{T}|(\gamma+1)} = \emptyset$.)

Steel's main lemma reads:

<u>Lemma 3</u> Let $\mu < lh(\mathcal{T})$. There is a realization $\mathbb{F}$ of $\mathcal{T}|(\mu+1)$ with room. Moreover, if $i < \mu$ and $\mathbb{E}$ is a realization of $\mathcal{T}|(i+1)$ with room, then $\mathbb{F}$ can be so chosen that:

(a) If i is a break point at μ, then
$\mathbb{F}|(i+1) = \mathbb{E}$ and $R_\mu^{\mathbb{F}} \in R_i^{\mathbb{E}}$.

(b) Let $k \leq i$ be largest s.t. k survives at μ. Then $\mathbb{F}|k = \mathbb{E}|k$ and

(i) $R_\mu^{\mathbb{F}} = R_k^{\mathbb{E}}$, $\gamma_\mu^{\mathbb{F}} \leq \gamma_k^{\mathbb{E}}$

(ii) If $\pi_{k\mu}$ is total, then $\gamma_\mu^{\mathbb{F}} = \gamma_k^{\mathbb{E}}$ and
$$\delta_\mu^{\mathbb{F}} \circ \pi_{k\mu} = \delta_k^{\mathbb{E}} \text{ and } \vec{\rho}^{\,\mu,\mathbb{F}} = \vec{\rho}^{\,k,\mathbb{E}}$$

(iii) If $\pi_{k\mu}$ is not total, then $\gamma_\mu^{\mathbb{F}} < \gamma_k^{\mathbb{E}}$,

(iv) $\delta_l^\mu = \delta_l^k$, $\gamma_l^\mu = \gamma_l^k$, $\vec{\rho}^{\,\mu,l} = \rho^{k,l}$ for $l \lneqq k$.

From this we derive a contradiction. Recall that $\delta : P_0 \xrightarrow[\Sigma^*]{} N_\gamma \ \min(\vec{\rho})$ s.t. (A), (B) fail. Moreover, γ is the minimal ordinal which allows such a failure. Suppose that $\mathcal{I}$ is of successor length $\mu+1$. Assume the function n^* to be so chosen that $n^*(\mu) = 0$. Then 0 survives at h. $\mathcal{I}|1$ has a realization $\mathbb{E} = \langle R, \delta, \vec{\rho} \rangle$ defined by $R = \langle V_{\theta+\xi}, \in, \theta \rangle$, where $\xi =$ the $\omega \cdot |U|$-th $\bar{\xi}$ s.t. $\langle V_{\theta+\bar{\xi}}, \in, \theta \rangle$ is a coarse premouse. (θ is our fixed inaccessible. Note that $C(0,0)^{\mathcal{I}|1} = \emptyset$. U is $U_0^{\mathbb{E}} = U(\delta, \gamma, \vec{\rho})$. We of course have: $\vec{N} = \vec{N}^R, \gamma = \gamma^R$.)

Then $\mathbb{E}$ has room. Let $\mathbb{F}$ be as in (b) of Lemma 3 with $i = 0$. Set: $\delta' = \delta_\mu^{\mathbb{F}}$; $\vec{\rho}' = \vec{\rho}^{\mu\mathbb{F}}$, $\gamma' = \gamma_\mu^{\mathbb{F}}$. Then $\delta' : P_\mu \xrightarrow[\Sigma^*]{} N_{\gamma'} \ \min(\vec{\rho}')$ satisfies (A). Contr! Now let $\mathcal{I}$ have limit length. Define n_l, j_l $(l < \omega)$ by:

$n_0 = 0$; $n_{l+1} = \min\{n^*(j) \mid j > j_l\}$;

$j_l =$ that j s.t. $n^*(j) = n_l$.

It follows easily that j_l is a break point in $lh(\mathcal{I})$ (hence in j_{l+1}). By (a) there is a sequence $\mathbb{E}_l$ s.t. $\mathbb{E}_l$ is a realization of $\mathcal{I}|j_l+1$ and $R_{i_{l+1}} \in R_{i_l}$ for $l<\omega$. Contr!

We shall prove Lemma 3 by induction on μ after proving two preliminary lemma

__Lemma 3.2__ Let $\mathbb{E}$ be a realization of $\mathcal{I}|(i+1)$ with room. Let $h=T(i+1)$ and suppose that h does not survive at $i+1$. Then there is a realization $\mathbb{F}$ of $\mathcal{I}|(i+2)$ with room s.t. $\mathbb{F}|(i+1)=\mathbb{E}$ and $R^{\mathbb{F}}_{i+1} \in R^{\mathbb{E}}_i$.

proof.
Let $F=E^{P_i}_{\nu_i}$, $F^*=E^{S_i^*}_{\nu_i^*}$. Let $\langle N,\hat{F}\rangle$ be a background certificate for $\hat{F}$ in R_i s.t. $\mathcal{U}_{\kappa_i^*} \in N$ in R_i. Recall that $\mathcal{U}^{R_i}_{\kappa_i^*}=\mathcal{U}^{R_h}_{\kappa_i^*}$. (Since $E^{Q_i}_{\nu_i} \neq \phi$, the iteration $\mathcal{J}^{\nu_i^*}$ is uniquely defined from Q in $V^{R_i}_{\Theta_i}$. But $\bar{\bar{Q}}_j<\kappa_i^*$ for $j<\kappa_i^*$. Hence $\mathcal{J}^{\nu_i^*}|\kappa_i^*=\mathcal{J}^{\kappa_i^*}|\kappa_i^*$ is uniquely defined from Q in $V^{R_i}_{\kappa_i^*}=V^{R_h}_{\kappa_i^*}$. Hence $\mathcal{J}^{\kappa_i^*}$ is the same in R_i and R_h. Hence so is $\mathcal{U}_{\kappa_i^*}$.) We recall that
$\delta_i^* \upharpoonright (\tau_i+1)=\tilde{\delta}_i \upharpoonright (\tau_i+1)=\delta_h^* \upharpoonright (\tau_i+1)$.

Let $\pi: N \xrightarrow[\hat{F}]{} N'$. Then $V_{\lambda_i^*+2} \subset N'$ in R_i. Hence $\delta_i^*, \lambda_i^* \in N'$.

Select $\beta < lh(\hat{F})$; $d, l : \tilde{\kappa}_i \longrightarrow V_{\tilde{\kappa}_i}$ in N s.t. $\delta_i^* = \pi(d)(\beta)$, $\lambda_i^* = \pi(l)(\beta)$.

Set: $W_0 = \{\beta\} \cup rng(\delta_i^*)$.

W_1 = the union of the $rng(f)$ s.t. f maps some $\xi < \tilde{\kappa}_i$ partially to $\mathbb{P}(\tilde{\kappa}_i) \cap \tilde{S}_i$ and f is $\Sigma^*(\tilde{S}_i)$ in parameters from $rng(\tilde{\delta}_i) \cup \{\tilde{\rho}_m^i \mid m < \omega \wedge \tilde{\rho}_m^i < ht(\tilde{S}_i)\}$.

W_2 = the set of $X \in \mathbb{P}(\tilde{\kappa}_i) \cap N$ which are N-definable from parameters in $\sup \tilde{\delta}_i{}''\kappa_i \cup \{d, l, Q, J\}$

(Note If f is as above, then $f \in N$. There are only countably many such f and ${}^\omega N \subset N$. Hence $W_1 \in N$, $\overline{\overline{W}}_1 < \tilde{\kappa}_i$ in N. Since $\sup \tilde{\delta}_i{}''\kappa_i < \tilde{\kappa}_i$, we also have $W_2 \in N$, $\overline{\overline{W}}_2 < \tilde{\kappa}_i$ in N.)

By Lemma 2:

(1) $\langle \tilde{\delta}_i, \delta_i^* \upharpoonright \lambda_i \rangle : \langle P_n, F \rangle \xrightarrow{**} \langle \tilde{S}_i \| \tilde{\rho}_0^i, F^* \rangle$.

Let $r : W_0 \to \tilde{\kappa}_i$ fix $W = W_0 \cup W_1 \cup W_2$. Then $\langle r(\vec{\alpha}) \rangle \in X \longleftrightarrow \langle \vec{\alpha} \rangle \in \hat{F}(X)$ for $X \in W_1 \cup W_2$, $\alpha_1, \ldots, \alpha_n \in W_0$. By (1) we can define $G : P_{i+1} \xrightarrow{\Sigma^*} \tilde{S}_i \min(\tilde{\rho}^i)$ by: $G(\pi_{h,i+1}(f)(\alpha)) = \tilde{\delta}_i(f)(r\delta_i^*(\alpha))$, where $f \in \Gamma^*(P_i^*, \kappa_i)$, $\alpha < \lambda_i$.

Note that $\tilde{S} = N_{\tilde{\gamma}_i}^{R_n}$, where either

$\pi_{h,i+1}$ is total and $\tilde{\gamma}_i = \gamma_h$, $\tilde{\rho}^i = \vec{\rho}^h$,

or else $\pi_{h,i+1}$ is not total and $\tilde{\gamma}_i < \gamma_h$.

In either case, since h does not survive at $i+1$, we have:

(2) $p_h^\frown \gamma'$ extends p_h in U_h

(where $p_h = p_h(\delta^h)$).

Set: $p' = p_h^\frown \gamma'$. Then $|p'| < |p_h|$ in U_h.

<u>Main Claim</u> There are $\hat{F}_\beta$ many $\xi < \tilde{\kappa}_i$ s.t. there exist (in $V_{\tilde{\kappa}_i}^{R_i}$): $R, \vec{N}', \vec{\rho}', \langle \delta_h' \mid h \leq_T i+1 \rangle$, S', $\langle \vec{\rho}'^h \mid h \leq_T i+1 \rangle$, $\langle \gamma_h' \mid h \leq_T i+1 \rangle$, δ'', $\vec{\rho}''$, s.t.

(a) R is a coarse pm

(b) $V_{\ell(\xi)+2}^R = V_{\ell(\xi)+2}^{R_i}$

(c) $Q \in V_{\ell(\xi)}^R$ and $\vec{N}' = \vec{N}^R$ is defined from Q in R as $\vec{N}$ was defined from Q.

(d) $\delta'' = \delta_{i+1}'$, $\vec{\rho}'' = \vec{\rho}'^{i+1}$, $\gamma'' = \gamma_{i+1}'$, where $\vec{\delta}'$ is good for δ_0', $\vec{\rho}'^0$ w.r.t. $\vec{N}'$ inducing $\langle \vec{\rho}'^h \mid h \leq_T i+1 \rangle$, $\langle \gamma_h' \mid h \leq_T i+1 \rangle$

(e) $U' = U(\delta_0', \vec{N}', \vec{\rho}^0)$ is well founded and R has at least $\omega \cdot |p'| + |C(i+1, i+1)|$ many cutpoints, where $p' = p_{i+1}(\vec{\delta}')$.

(Note $p' < p_h(\vec{\delta}\,')$ in $\mathcal{U}'$, since h does not survive at $i+1$ (hence $p' = p_h{}^\frown \delta''$).)

(f) $S'' = N_{\gamma''}$, $\delta'' \upharpoonright \lambda_i = d(\xi)$, $\delta''(\lambda_i) > l(\xi)$

(Hence $\delta'' : P_{i+1} \xrightarrow[\Sigma^*]{} S''\ \min(\vec{p}\,''))$)

(g) Let $\theta' = \theta^R$. Then

$$V_{\theta_h}^{R_h} \models \varphi(\mu, G(x), \tilde{p}^i, \tilde{S}_i) \longleftrightarrow$$
$$\longleftrightarrow V_{\theta'}^{R} \models \varphi(\mu, \delta''(x), \vec{p}\,'', S'')$$

whenever $x \in P_{i+1}$, $\mu < \sup \tilde{\delta}_i{}'' \kappa_i$ and φ is a 1-st order formula.

proof.

We must show that there is an $X \in F_\beta$ whose points satisfy (a)–(g). One problem is that (g) cannot directly be formulated in N. Fix φ, x, μ as in (g). Then the set $X_{\varphi, x, \mu}$ of ξ which satisfy (a)–(f) and satisfy (g) for the specific triple $\langle \varphi, x, \mu \rangle$ is in N and, in fact, in W_2.

<u>Claim 1</u> $X_{\varphi, x, \mu} \in \hat{F}_\beta$.

Let $X' = X_{\varphi, x, \mu}$. Since $X' \in W_2$, it suffices to show:

<u>Claim 1.1</u> $\bar{\beta} \in X'$, where $\bar{\beta} = r(\beta)$.

We know that R_h has $\geq \omega \cdot |p_h|$ many cutoff points and that $|p'| < |p_h|$ in U_h. Let ξ be the $\omega \cdot |p'| + c(i+1, i+1)$-th cutoff pt. of R_h. Set:

$R' = \langle V_\xi^{R_h}, \in, \theta_h \rangle$. In R_h pick $Z \prec R'$ s.t. $\bar{\bar{Z}} < \tilde{\kappa}_i$, Z is ω-closed, and $V_{l(\bar{\beta})+2}^{R_h} \subset Z$, $\langle \delta_l^h \mid l \leq_T h \rangle \in Z$, $\langle \vec{\beta}^{h,l} \mid l \leq h \rangle \in Z$, $\langle \gamma_l^h \mid l \leq h \rangle \in Z$, and $\tilde{S}_i, \tilde{\beta}^i, G \in Z$. Let $\sigma : R \stackrel{\sim}{\longleftrightarrow} Z$, where R is transitive.

Note that $Q \in V_{l(\bar{\beta})}$, since $Q \in V_{\lambda_i^*}$, hence $\{\xi \mid Q \in V_{l(\xi)}\} \in F_\beta \cap W_2$. Hence $\vec{N}^{R_h} \in Z$ and we set: $\vec{N}' = \sigma^{-1}(\vec{N}^{R_h})$. Then $\vec{N}'$ has the appropriate definition from Q in R. We also set:

$S'' = \sigma^{-1}(\tilde{S}_i)$, $\delta'' = \sigma^{-1}(G)$, $\vec{\rho}\,'' = \sigma^{-1}(\tilde{\rho}^{\,i})$,
$\rho'^{\,l} = \sigma^{-1}(\vec{\rho}^{\,h,l})$ for $l \leq_T h$, $\vec{\rho}\,'^{\,i+1} = \vec{\rho}\,''$,
$\delta'_l = \sigma^{-1}(\delta^h_l)$ for $l \leq_T h$, $\delta'_{i+1} = \delta''$,
$\gamma'_l = \sigma(\gamma^h_l)$ for $l \leq_T h$, $\gamma'_{i+1} = \sigma^{-1}(\tilde{\gamma}_i)$. .

Note that $\mu < l(\bar{\beta})$, since $\mu < \lambda^*_i$ and hence $\{\xi \mid \mu < l(\xi)\} \in F_\beta \cap W_2$. The verifications of (a)–(e) and (g) are trivial. We verify (f).

(2) $\delta'' \restriction \lambda_i = r\delta^*_i \restriction \lambda_i$, since
$\delta''(\alpha) = \sigma^{-1}G(\alpha) = \sigma^{-1} r\delta^*_i \alpha)$, where
$r\delta^*_i(\alpha) < l(\bar{\beta})$, since $\delta^*_i(\alpha) < \pi(l)(\beta) = \lambda^*_i$
But $\sigma \restriction l(\bar{\beta}) = id$. Hence $\delta''(\alpha) = r\delta^*_i(\alpha)$.

(3) $\delta'' \restriction \lambda_i = d(\bar{\beta})$

prf. Let $\alpha < \lambda_i$, $\eta = \delta''(\alpha) = r\delta^*_i(\alpha)$.
Then $\langle \eta, \alpha \rangle \in d(\bar{\beta}) \longleftrightarrow \langle \delta^*_i(\alpha), \alpha \rangle \in \pi(d)(\beta)$
$\longleftrightarrow \delta^*_i(\alpha) = \delta^*_i(\alpha)$.

Then $\eta = \delta^*_i(\alpha)$ QED (3)

(4) $\delta''(\lambda_i) > l(\bar{\beta})$, since $\delta''(\lambda_i) = \sigma^{-1}(G(\lambda_i))$
and $G \restriction (l(\bar{\beta}) + 1) = id$, and
$G(\lambda_i) = G(\pi_{h,i+1}(\kappa_i)) = \tilde{\delta}_i(\kappa_i) = \tilde{\kappa}_i > l(\bar{\beta})$;
hence $\delta''(\lambda_i) = \sigma^{-1}(\tilde{\kappa}_i) > \sigma^{-1}(l(\bar{\beta})) = l(\bar{\beta})$.
QED (4)

This verifies (f) and completes the proof of Claim 1.

But since $\bar{\beta} \in X_{\varphi, x, \mu} \in W_1$ for all such $\langle \varphi, x, \mu \rangle$, it suffices to take $X = \bigcap \{ Y \in W_1 \mid \bar{\beta} \in Y \}$. Then $Y \in F_\beta$ for all $Y \in W_1$ s.t. $\bar{\beta} \in Y$, where $\overline{\overline{W_1}} < \tilde{\kappa}_i$ in N. Hence $X \in F_\beta$ and each $\xi \in X$ satisfies (a)–(g).

QED (Main Claim)

We then choose a function $F \in N$ s.t. $F(\xi) = \langle R(\xi), \vec{\delta}'(\xi), \vec{\rho}'^{0}(\xi) \rangle$ for $\xi \in X$ and set:

$\mathbb{F} \upharpoonright i+1 = \mathbb{E}$; $\mathbb{F}_{i+1} = \pi(F)(\beta)$.

$G = G_i : P_{i+1} \longrightarrow \tilde{S}_i$ was defined above.

The verifications are straightforward.

QED (Lemma 3.2)

<u>Lemma 3.3</u> Let $\mathbb{E}$ be a realization of $\mathbb{J}|(i+1)$ with room. Let $h=T(i+1)$ and suppose h survives at $i+1$. There is a realization $\mathbb{F}$ of $\mathbb{J}|(i+2)$ with room s.t.

(a) $\mathbb{F}\restriction h = \mathbb{E}\restriction h$

(b) $R^{\mathbb{F}}_{i+1} = R^{\mathbb{E}}_{h}$

(c) Let $l \leq_T h$. Then $(\delta_l^{i+1})^{\mathbb{F}} = (\delta_l^{h})^{\mathbb{E}}$, $(\vec{\rho}^{\,i+1,l})^{\mathbb{F}} = (\vec{\rho}^{\,h,l})^{\mathbb{E}}$, $(\gamma_l^{i+1})^{\mathbb{F}} = (\gamma_l^{h})^{\mathbb{E}}$.

proof.

We first note that $c(j,i) = c(h,i)$ for $h \leq j \leq i$. (Otherwise there would be $k \geq i$ s.t. $j = T(k+1)$, $h < j \leq i$ and j survives at $k+1$. $k \geq i$, since $T(k+1) \neq h = T(i+1)$. But then $\neg(h <_T j \leq_T i+1)$, hence $m(j) > m(h)$, since h survives at $i+1$. Hence $j < i+1 < k+1$ and $m(i+1) < m(j)$. Hence j does not survive at $k+1$. Contr!)

Clearly $|c(h,i)| \geq 1$, since h survives at $i+1$ and $h = T(i+1)$. Letting $c = |c(h,i)|$, it follows that $|c(j,i+1)| = c-1$ for $h \leq j \leq i$. Moreover, $|c(i+1,i+1)| \leq c$, since by the above reasoning there is at most one $j > i+1$

s.t. $i+1 = T(j)$ and $i+1$ survives at j.

For $h \le j \le i$ let ξ_j be the $\omega \cdot |p_j| + c$-th cutoff point of R_j. Let $Z_j \prec V_{\xi_j}^{R_j}$ in R_j s.t.

(a) $Z_j \prec V_{\xi_j}^{R_j}$, ${}^{\omega}Z_j \subset Z_j$

(b) $V_{\lambda_i^*+1} \subset Z_j$, $\overline{\overline{Z}}_j = V_{\lambda_i^*+1}$

(c) Z_j contains the following points:

$S_j, \vec{p}^{\,j}, \delta_j$,

$\langle \vec{p}^{\,j,l} \mid l \le_T j \rangle$, $\langle \gamma_l^j \mid l \le_T j \rangle$,

$\langle G_k \mid k \le i \wedge j = T(k+1) \rangle$.

Let $\varphi_j : R_j' \xrightarrow{\sim} Z_j$, where R_j' is transi

Set: $S_j', \vec{p}'^{\,j}, \delta_j' = \varphi_j^{-1}(S_j, \vec{p}^{\,j}, \delta_j)$

$\delta_l'^{\,j} = \varphi_j^{-1}(\delta_l^j)$ (similarly for $\vec{p}^{\,j,l}, \gamma^{j,l}$); $G_k' = \varphi_j^{-1}(G_k)$.

Note that $Q, Y \in V_{\lambda_i^*+1} \subset$ with $\varphi_j(Q, Y) = Q, Y$. We then get $\vec{N}^{R_j'}$ defined from Q in R_j'

the way $\vec{N}^{R_i}$ was defined from Q in R_i.
Clearly $\varphi_i(\vec{N}^{R_i'}) = \vec{N}^{R_i}$, $\varphi_i(\gamma^{R_i'}) = \gamma^{R_i}$.

<u>Def</u> $\mathbb{E}' = \langle \mathbb{E}_l' \mid l \leq i \rangle$ is defined by:
$\mathbb{E}' \upharpoonright h = \mathbb{E} \upharpoonright h$; For $h \leq j \leq i$ we set:
$\mathbb{E}_j' = \langle R_j', \vec{\delta}'^{j}, \rho'^{j,0} \rangle$.

It follows straightforwardly that:

(1) $\mathbb{E}'$ is a realization of $\mathcal{J} \mid i+1$

(2) Let $h \leq j \leq i$. R_j' has $\omega \cdot |p_j'|_{u_j'} + (c-1)$ many cutoff points, where p_j', u_j' are defined in R_j' from $\vec{\delta}'^{j}, \vec{\rho}'^{j,0}$ as p_j, u_j are defined in R_j from $\vec{\delta}^{j}, \vec{\rho}^{j,0}$.

We again let $F = F_{\nu_i}^{P_i}$, $F^* = E_{\gamma_i^*}^{S_i^*}$ and let $\langle N, \hat{F} \rangle$ be a background certificate for F^*.

(We again have $\mathcal{U}_{\kappa_i^*}^{R_i} = \mathcal{U}_{\kappa_i^*}^{R_h}$.)

Let $\pi : N \longrightarrow_{\hat{F}} N'$. Then $V_{\lambda_i^*+2}^{R_i} \subset N'$.

$\langle \mathbb{E}_j' \mid h \leq j \leq i \rangle$ is easily seen to be codable by a subset of λ_i^*+1 in R_i.

Hence:

(3) $\langle \mathbb{E}'_j \mid h \leq j \leq i \rangle \in N'$.

Pick $e \in N'$, $e : \kappa_i^* \longrightarrow N'$, $\beta < lh(\hat{F})$ s.t.

(4) $\langle \mathbb{E}'_j \mid h \leq j \leq i \rangle = \pi(e)(\beta)$

Define W_0, W_1 as in the proof of Lemma:
Set: W_2 = the set of $X \in \mathcal{P}(\kappa_i^*) \cap N$ which are N-definable in parameters from $V^{P_h}_{s+\omega} \cup \{e\} \cup \{y\}$, where $s =$
$= \sup \tilde{\delta}_i{}''\kappa_i$. Let $r : W_0 \longrightarrow \kappa_i^*$ fix $W_0 \cup W_1 \cup W_2$. Then:

(4) $\langle r(\vec{\alpha}) \rangle \in X \longleftrightarrow X \in F^*_{\langle \vec{\alpha} \rangle}$

for $\alpha_1, \dots, \alpha_n \in W_0$, $X \in W_1 \cup W_2$.

As before:

(5) $G : P_{i+1} \xrightarrow[\Sigma^*]{} \tilde{S}_i \min(\tilde{p}^i)$

where $G(\pi_{h,i+1}(f)(\alpha)) = \tilde{\delta}_i(f)(r\delta_i^*(\alpha))$

for $\alpha < \kappa_i$, $f \in \Gamma^*(P_i^*, \kappa_i)$.

Now set: $\mathbb{F}_j$

<u>Def</u> $\mathbb{F}_j = e(\bar{\beta})_j$ $(h \leq j \leq i)$, $\mathbb{F}_\ell = \mathbb{E}_\ell$ $(\ell < h)$

Then:

(6) $\mathbb{F}$ is a prerealization of $\mathcal{J}|(i+1)$.

proof.

We must show that $\mathbb{F}_j = \langle R_j'', \vec{\delta}''j, \vec{\rho}''j \rangle$ satisfies (i)–(iii) in the def. of prerealization for $j \le i$. For $j < h$ this is trivial. Let $h \le j \le i$. Let X be the set of $\xi < \kappa_i^*$ s.t. $e(\xi) : [h, i] \to V_{\kappa_i^*}^{R_i}$ and for all $j \in [h, i]$, $e(\xi)(j) = \langle R, \vec{\delta}, \vec{\rho} \rangle$ satisfying (i)–(iii) in the def. of prerealization. Then $X \in W_2$ and $X = \{\xi \mid \mathbb{E}\restriction h ^\frown e(\xi)$ is a prerealization of $\mathcal{J}|(i+1)\}$.

Clearly $\beta \in \pi(X)$, since $\pi(e)(\beta) = \langle \mathbb{E}_j' \mid h \le j \le i \rangle$. Hence $\bar{\beta} \in X$.

QED(6)

We define $\delta_j^{\mathbb{F}}, \delta_j^{*\mathbb{F}}, \tilde{\delta}_j^{\mathbb{F}}$ etc. as usual.

(7) $\delta_j^{\mathbb{F}} \restriction \lambda_j = \delta_j^{\mathbb{E}} \restriction \lambda_j$, $\delta_j^{*\mathbb{F}} \restriction \lambda_j = \delta_j^{*\mathbb{E}} \restriction \lambda_j$

$j < h$ is trivial. Let $h \le j \le i$ proof. Clearly $\delta_j^{\mathbb{E}} \restriction \lambda_j + 1 = \delta_j^{\mathbb{E}'} \restriction \lambda_j + 1 = \delta_j$

But $Y = \{\xi \in X \mid \delta_j^{e(\xi)}(\eta) = \gamma\} \in W_2$ for $\eta < \lambda_j$, $\gamma = \delta_j(\eta)$, where X is as above since $\delta_j(\eta) \le \delta_i(\eta)$. Hence $\beta \in \pi(Y)$ and $\bar{\beta} \in Y$. QED(7)

(8) $\mathbb{F}$ is a protorealization of $\mathcal{J}|(i+1)$.

prf.

Let $j < k \le i$. We claim:

(a) $\delta_k \upharpoonright \lambda_j = \delta_j^* \upharpoonright \lambda_j$ in $\mathbb{F}$

(b) $\lambda_j^* < \lambda_k^*$, $V_{\lambda_j^*+1}^{R_j} = V_{\lambda_j^*+1}^{R_k}$, $V_{\lambda_j^*+2}^{R_j} \subset R_k$ in $\mathbb{F}$:

(a) is immediate by (7): $\delta_k^{\mathbb{F}} \upharpoonright \lambda_j =$

$= \delta_k \upharpoonright \lambda_j = \delta_j^* \upharpoonright \lambda_j = \delta_j^{*\,\mathbb{F}} \upharpoonright \lambda_j$.

(b) is immediate for $k < h$. Now let $h \le j < k \le i$. Set:

$$Y = \{\xi \in X \mid \lambda_j^* < \lambda_k^* \wedge V_{\lambda_j^*+1}^{R_j} = V_{\lambda_j^*+1}^{R_k} \wedge V_{\lambda_j^*+2}^{R_j} \subset R_k \text{ in } e(\xi)\}$$

Then $\beta \in \pi(Y)$, Hence $\bar{\beta} \in Y$.

Now let $j < h \le k \le i$. Let $\lambda = \lambda_j^*$.

Let $u = V_{\lambda_j^*+1}^{R_j}$, $v = V_{\lambda_j^*+2}^{R_j}$. Then

$\lambda, u, v \in V_{\lambda+\omega}^{R_h}$ and we set:

$$Y = \{\xi \in X \mid \lambda < \lambda_k^*, u = V_{\lambda+1}^{R_k}, v \subset R_k \text{ in } \mathbb{F}\}.$$

Then $\beta \in \pi(Y)$ and $\bar{\beta} \in Y$. QED (8)

(9) $\mathbb{F}$ is a realization of $\mathcal{J}|(i+1)$.

proof.

We first prove (b) in the def. of realization. Let $j \le_T k \le i$, where π_{jk} is total. If $k < h$, the conclusion is trivial. Now let $h \le j < k \le i$. Let

Y = the set of $\xi \in X$ s.t. in $\mathbb{E}(\xi)$ we have:

$V_{\theta_j}^{R_j} \models \varphi(\mu, \delta_j(x), \vec{p}^{\,j}, s_j) \longleftrightarrow V_{\theta_k}^{R_k} \models \varphi(\mu, \delta_h \pi_{jk}(x), \vec{p}^{\,k}, s_h)$

for all $\mu < \sup \delta_j{}''\kappa_j$, $x \in P_j$ + all 1st order φ.

Then $Y \in W_2$ and $\beta \in \pi(Y)$. Hence $\bar{\beta} \in Y$.

Now let $j < h \le k \le i$. Set: T = the set of $\langle \varphi, \mu, x \rangle$ s.t. $\mu < \sup \delta_j{}''\kappa_j$, $x \in P_j$, φ is a 1-st order fmla, and $V_{\theta_j}^{R_j} \models \varphi(\mu, \delta_j(x), \vec{p}^{\,j}, s_j)$ in $\mathbb{E}$.

Then $T \in V_{\sup \delta_j{}''\kappa_j + \omega}^{R_j}$ where

$\sup \delta_j{}''\kappa_j \le \sup \delta_j^*{}''\kappa_j \le \sup \delta_h{}''\kappa_h = \lambda$,

since $\kappa_j < \lambda_j \le \kappa_i$ and $\delta_j^* \upharpoonright \lambda_j = \delta_h \upharpoonright \lambda_j$.

Hence $T \in V_{\lambda+\omega}^{R_h}$. Set:

Y = the set of $\xi \in X$ s.t. in $\mathbb{E}(\xi)$ we have:

$V_{\theta_k}^{R_h} \models \varphi(\mu, \delta_h \pi_{jk}(x), \vec{p}^{\,k}, s_h) \longleftrightarrow \langle \varphi, \mu, x \rangle \in T$.

Then $Y \in W_2$, $\beta \in \pi(Y)$ + hence $\bar{\beta} \in Y$.

QED(b)

We now prove (c) in the def. of "realization". Let $j = T(k+1)$ where $k < i$. We must find $G = G_{k+1} \in R_j^{\mathbb{F}}$ satisfying (i)–(iii) in $\mathbb{F}$. If $k+1 < h$, this is trivial.

Now let $h \le j \le k < i$. Let $Y =$

= the set of $\xi \in X$ s.t. in $\mathbb{C}(\xi)$ there is $G \in R_i$ satisfying (i)–(iii). Then $Y \in W_2$, $\beta \in \pi(Y)$; hence $\bar{\beta} \in Y$.

Now let $i < h \leq k+1$. Let $G = G_{k+1}$ verify (i)–(iii) in $\mathbb{E}$. We claim that G verifies (i)–(iii) in $\mathbb{F}$. (i), (iii) are trivial, so we must verify (ii) in $\mathbb{F}$. Let $T =$ the set of $\langle \varphi, x, \mu \rangle$ s.t. φ is a 1-st order fmla, $x \in P_{k+1}$, $\mu < \sup G''\kappa_k$ and $V_{\theta_i}^{R_i} \models \varphi(\mu, G(x), \tilde{\rho}^h, \tilde{s}^h)$ in $\mathbb{E}$.

Then $T \in V^{R_i}_{(\sup \tilde{\delta}_i''\kappa_h) + \omega} \subset V^{R_h}_{s+\omega}$ as before, since $\tilde{\delta}_i''\kappa_k = G''\kappa_k$ by (i).

Let $Y =$ the set of $\xi \in X$ s.t. in $\mathbb{C}(\xi)$:

$V_{\theta_{k+1}}^{R_{k+1}} \models \varphi(\mu, \delta_{k+1}(x), \vec{\rho}^{k+1}, s_{k+1}) \leftrightarrow \langle \varphi, x, \mu \rangle \in$

Then $Y \in W_2$, $\beta \in \pi(Y)$, $\bar{\beta} \in Y$.

QED (9)

(10) $\delta^h \frown G$ is a good sequence in $\mathbb{P}_h$

prf. trivial.

Def $\mathbb{F}' = \mathbb{F} \frown \langle R_h, \delta^h \frown G, \vec{\rho}^{h,0} \rangle$.

Then:

(11) $\mathbb{F}'$ is a prerealization of $\mathcal{I}(i+2)$.

prf. trivial.

(12) $\mathbb{F}'$ is a protorealization of $\mathcal{I}(i+2)$.

prf. We repeat some arguments in the prf. of Lemma 3.2.

(i) $\lambda_i^{*\mathbb{F}} < \lambda_{i+1}^{*\mathbb{F}'} = G(\lambda_{i+1})$, since

$$G(\lambda_i) = G(\pi_{h,i+1}(\kappa_i)) = \tilde{\delta}_i(\kappa_i) = \tilde{\kappa}_i ;$$
$$\lambda_i^{*\mathbb{F}} < \tilde{\kappa}_i < G(\lambda_{i+1}).$$

(ii) Let $\bar{\lambda} = \lambda_i^{*\mathbb{F}}$. Then

$$V_{\bar{\lambda}+1}^{\mathbb{R}_i^{\mathbb{F}}} = V_{\bar{\lambda}+1}^{\mathbb{R}_h^{\mathbb{E}}} = V_{\bar{\lambda}+1}^{\mathbb{R}_{i+1}^{\mathbb{F}'}}, \text{ since for}$$

$$Y = \{\xi \in X \mid V_{\bar{\lambda}+1}^{\mathbb{R}_i^{e(\xi)}} = V_{\bar{\lambda}+1}^{\mathbb{R}_h}\} \text{ (defined in } \mathbb{R}_h)$$

we have $Y \in W_2$, $\beta \in \pi(Y)$; hence $\bar{\beta} \in Y$.

(iii) $V_{\bar{\lambda}+2}^{\mathbb{R}_i^{\mathbb{F}}} \subset \mathbb{R}_i^{\mathbb{F}} = \mathbb{R}_i^{e(\bar{\beta})} \subset \mathbb{R}_h^{\mathbb{E}} = \mathbb{R}_{i+1}^{\mathbb{F}'}$

(iv) $\delta_{i+1}^{\mathbb{F}'} \upharpoonright \lambda_i = G \upharpoonright \lambda_i = \delta_i^{*\mathbb{F}} \upharpoonright \lambda_i$, since

for $\eta < \lambda_i$, we have: $G(\eta) = r\delta_i^*(\eta)$.

Let $Y = \{\langle \xi, \gamma \rangle \mid \xi \in X \wedge \gamma = \delta_i^{*e(\xi)}(\eta)\}$.

Then $\langle \beta, \delta_i^*(\eta) \rangle \in \pi(Y)$, where $Y \in W_2$. Hence $\langle \bar{\beta}, r\delta_i^*(\eta) \rangle \in Y$

Hence $G(\eta) = r\delta_i^*(\eta) = \delta_i^{*\mathbb{F}}(\eta)$.

QED(11)

It remains only to show that $\mathbb{F}'$ is a full realization of $\mathbb{I}|(i+1)$.

(12) $\mathbb{F}'$ satisfies (b) in the def. of "realization"

prf.

The only thing left to show is that if $\eta_i = h$

$$V^{R_h}_{\Theta_h} \models \varphi(\mu, \delta_h(x), \vec{\beta}^h, S_h) \longleftrightarrow$$

$$\longleftrightarrow V^{R_{i+1}}_{\Theta_{i+1}} \models \varphi(\mu, \delta_{i+1}\pi_{h,i+1}(x), \vec{\beta}^{i+1}, S_{i+1})$$

for $x \in P_h$, φ a 1-st order formula,

and $\mu < \sup \delta_h``\kappa_h$.

We note that $\delta_{i+1}\pi_{h,i+1}(x) = G\pi_{h,i+1}(x) = \tilde{\delta}_i^{\mathbb{E}}(x) = \delta_h^{\mathbb{E}}(x)$. We also note that $S_{i+1}^{\mathbb{F}'} = \tilde{S}_i^{\mathbb{E}} = S_h^{\mathbb{E}}$; $\vec{\beta}^{i+1\,\mathbb{F}'} = \tilde{\beta}^{i\,\mathbb{E}} = \vec{\beta}^{h\,\mathbb{E}}$.

Hence the assertion reduces to the statement that

$$V^{R_h}_{\Theta_h} \models \varphi(\mu, \delta_h(x), \vec{\beta}^h, S_h)$$

holds in $\mathbb{E}$ iff it holds in $\mathbb{F}$. Denote this statement by $\bar{\varphi}(\mu, x)$. Let

$Y = \{\xi \in X \mid \bar{\varphi}(\mu, x) \text{ holds in } \mathbb{C}(\xi)\}$.

Then $Y \in W_2$. Moreover $\bar{\varphi}(\mu, x)$ holds in $\mathbb{E}$ iff in $\mathbb{E}' = \pi(\mathbb{C})(\beta)$. Hence.

$\overline{\varphi}(\mu, x)$ holds in $\mathbb{E} \longleftrightarrow \beta \in \pi(Y)$

$\longleftrightarrow \bar{\beta} \in Y$

$\longleftrightarrow \overline{\varphi}(\mu, x)$ holds in $\mathbb{F}$

QED(12).

(13) $\mathbb{F}'$ is a realization of $\mathfrak{I}(i+2)$.

proof.

We must verify (c) for $h = T(i+1)$. It suffices to find $g : \lambda_i \longrightarrow \tilde{\kappa}_i$ s.t.

$$(*)\ V_{\theta_h}^{R_h} \models \varphi(\mu, g(\vec{\alpha}), \tilde{\delta}_i(x), \tilde{\beta}^i, \tilde{S}_i) \longleftrightarrow$$

$$\longleftrightarrow V_{\theta_{i+1}}^{R_{i+1}} \models \varphi(\mu, \delta_{i+1}(\vec{\alpha}), \delta_{i+1}\pi_{h,i+1}(x), \vec{\beta}^{i+1}, S_{i+1}$$

in $\mathbb{F}'$, whenever $\alpha_1, \ldots, \alpha_n < \lambda_i$, $x \in P_i^*$, $\mu < \sup \tilde{\delta}_i{}''\kappa_i$, and φ is a 1-st order fmla.

If we then set: $G(\pi_{h,i+1}(f)(\alpha)) = \tilde{\delta}_i(f)(g(\alpha))$ for $x = \pi_{h,i+1}(f)(\alpha) \in P_{i+1}$, $f \in \Gamma^*(P_i^*, \kappa_i)$, $\alpha < \lambda_i$, G will satisfy (i)–(iii). [Note, too, that $G \restriction \lambda_i$ satisfies (*) and G is defined this way from $G \restriction \lambda_i$.]

Let $\overline{\varphi}(\mu, \vec{\alpha}, \vec{x})$ be the statement:

$$\mathbb{V}^{\mathbb{R}_h}_{\Theta_h} \models \varphi(\mu, \vec{\alpha}, \tilde{\delta}_i(x), \tilde{\vec{\rho}}^{\,i}, \tilde{S}_i).$$

The right side of $(*)$ holds in $\mathbb{F}'$ iff $\overline{\varphi}(\mu, G(\vec{\alpha}), x)$ holds in $\mathbb{E}$, since $G = \delta^{\mathbb{F}'}_{i+1}$, $G\pi_{h,i+1}(x) = \tilde{\delta}_i(x)$, $\vec{\rho}^{\,i+1} = \tilde{\vec{\rho}}^{\,i}$, $S_{i+1} = \tilde{S}_i$. But $\overline{\varphi}(\mu, G(\vec{\alpha}), x)$ holds in $\mathbb{E}$ iff it holds in $\mathbb{E}'$.

Fix $\vec{\alpha}, x, \varphi$. Let $\lambda = \sup \tilde{\delta}_i{}'' \kappa_i$ and set:
$\Gamma = \Gamma_{\vec{\alpha}, x, \varphi} = \{\mu < \lambda \mid \overline{\varphi}(\mu, G(\vec{\alpha}), x) \text{ holds in } \mathbb{E}\}$.
Then $\Gamma \in \mathbb{R}_h$, since $\mathbb{V}^{\mathbb{R}_h}_{\Theta_h}$ satisfies the axiom of subsets. Let I_0 = the set of $\langle \vec{\alpha}, x, \varphi \rangle$ s.t. $\alpha_1, \ldots, \alpha_n < \lambda_i$, $x \in \tilde{P}_i$, and $\varphi = \varphi(w, a_1, \ldots, a_n, x, y, z)$ is a ZF-formula. Then I_0 is countable. Hence $\langle \Gamma_s \mid s \in I_0 \rangle \in \mathbb{R}_h$. If we set: I = the set of $\langle \mu, \vec{\alpha}, x, \varphi \rangle$ s.t. $\mu < \lambda \wedge \langle \vec{\alpha}, x, \varphi \rangle \in I_0$, and
$T = \{\langle \mu, \vec{\alpha}, x, \varphi \rangle \in I \mid \overline{\varphi}(\mu, \vec{\alpha}, x) \text{ in } \mathbb{E}\}$,
then $T \in \mathbb{R}_h$, since: $T =$
$= \{\langle \mu, \vec{\alpha}, x, \varphi \rangle \mid \langle \vec{\alpha}, x, \varphi \rangle \in I_0 \wedge \mu \in \Gamma_{\langle \vec{\alpha}, x, \varphi \rangle}\}$.

But $T \in V_{\lambda+\omega}^{R_h} = V_{\lambda+\omega}^{R_i}$ in $\mathbb{E}$.

We must prove:

<u>Claim</u> There is $g : \lambda_i \to \tilde{\pi}_i^{\mathbb{F}}$ s.t.
$\bar{\varphi}(g(\vec{\alpha}), \mu, x)$ holds in $\mathbb{F}$ whenever

$$\langle \varphi, \vec{\alpha}, \mu, x \rangle \in T.$$

Set: Y = the set of $\xi \in X$ s.t. in $e(\xi)$ the following holds:

$\tilde{T} \in R_h$ and there is $g : \lambda_i \to \tilde{\pi}_i$ in R_h s.t. $\bar{\varphi}(g(\vec{\alpha}), \mu, x)$ holds whenever

$$\langle \varphi, \vec{\alpha}, \mu, x \rangle \in T.$$

Then $Y \in W_2$ and $\beta \in \pi(Y)$, since $G \upharpoonright \lambda_i$ satisfies the condition in $\mathbb{E}' = \pi(e)(\beta)$. Hence $\bar{\beta} \in Y$, which proves the Claim.

QED (Lemma 3.3)