## § 3.1 Weasels

<u>Def</u> $W = J_\infty^E$ is a <u>weasel</u> iff iff $W|\alpha$ is a mouse for $\alpha < \infty$.

The notion of a <u>weasel iteration</u> (of length $\leq \infty$) is defined as before. Weasels are easily seen to be iterable. Comparison lemmas must be formulated with some care, however, since we are not allowed to go beyond $\infty$.

<u>Lemma 1.1</u> Let $M$ be a mouse, $W$ a weasel, let $\langle M_i \mid i < \theta \rangle$, $\langle W_i \mid i < \theta \rangle$ be the coiteration ($\theta \leq \infty$).

(a) If $\langle W_i \rangle$ is non simple, then $\theta = \delta + 1 < \infty$ and $M_\delta$ is an initial segment of $W_\delta$.

(b) If $\langle M_i \rangle$ is non simple, then $\theta = \infty$

(hence one side is simple).

prf. of Lemma 1.1

(a) is trivial, since from the point of truncation on the W-side, we are simply coiterating two mice. Using (a), it follows exactly as in the case of mice that both sides cannot be non simple. But if the M-side is non simple, $M_i$ is not sound from the point of truncation on. But then the iteration cannot terminate, since this would make $M_\delta$ a proper initial segment of $W_\delta$.

QED (Lemma 1.1)

<u>Lemma 1.2</u> Let $M, W$ be as above with coiteration indices $\langle \nu_i \rangle$. Let $E^M_{\nu_i}$ or $E^W_{\nu_i}$ be a measure on $\kappa_i$. Assume that $\Theta = \infty$. Then

(a) $\bigwedge \xi \bigvee i \;\; \pi_{W_0 W_i}(\xi) < \kappa_i$

(b) There is a cub class $C$ s.t. $\pi_{M_i M_j}(\kappa_i) = \kappa_j$ for $i, j \in C$.

prf. of Lemma 1.2

(a) Suppose not. Let $\kappa_i \le \pi_{WW_i}(\xi)$ for all $i$. Pick $\alpha > \xi$ which is regular in $W$. Set: $N_i = W_i | \pi_{WW_i}(\alpha)$. It is easily seen that $\langle M_i \rangle, \langle N_i \rangle$ is the coiteration of two mice. Hence $\theta < \infty$. Contradiction!

(b) For limit $\lambda$ set $f(\lambda) =$ the least $i < \lambda$ s.t. $\kappa_\lambda \in \mathrm{rng}(\pi_{M_i M_\lambda})$. Then $f(\lambda) = i_0$ on a stationary $C'$. But then there is $\tilde{\kappa} \in M_{i_0}$ s.t.
$C = \{ i \mid \pi_{M_{i_0} M_i}(\tilde{\kappa}) = \kappa_i \}$ is stationary. But then $\pi_{M_i M_j}(\kappa_i) = \kappa_j$ for $i, j \in C$; hence $\kappa_i = \mathrm{crit}(\pi_{M_i M_j})$. We claim that $C$ is cub. Let $\lambda$ be a limit pt. of $C$. Then $\kappa_\lambda \ge \kappa$ where
$\kappa = \sup\{\kappa_i \mid i \in \lambda \cap C\} = \pi_{M_{i_0} M_\lambda}(\tilde{\kappa})$.

But if $\kappa_\lambda > \kappa$, then $\pi_{M_\lambda M_i}(\kappa) = \kappa$ for all $i \ge \lambda$.

Hence $\pi_{i_0 i}(\tilde{\kappa}) = \kappa < \kappa_i$ for $i \geq \lambda$. Contr! QED (Lemma 1.2)

Note that since $\kappa_i = \mathrm{crit}(\pi_{M_i M_j})$ for $i<j$, $i,j \in C$, we have: $E^{M_i'}_{\nu_i}$ is normal on $\kappa_i$. If $X \in E^{M_j}_{\nu_j}$ and $X = \pi_{M_i M_j}(\bar{X})$, then $\kappa_i \in \pi_{M_i M_{i+1}}(\bar{X}) \subset X$.

Hence:

<u>Cor 1.3</u> Let $C$ be as in Lemma 1.2(b). Then $E^{M_i}_{\nu_i}$ is a measure on $\kappa_i$ in $M_i$ and $\{\kappa_h \mid h \in C \cap i\}$ is almost contained in each $X \in E^{M_i}_{\nu_i}$ for $i \in C$.

(Thus if $\lambda \in C$, $cf(\lambda) > \omega$, then $E^{M_\lambda}_{\nu_\lambda}$ is $\omega$-complete).

If a normal pweasel iteration $\langle W_i \mid i < \infty \rangle$ satisfies (a) of Lemma 1.2, it has a direct limit:

<u>Lemma 1.4</u> Let $\langle W_i \mid i < \infty \rangle$ be a simple normal pweasel iteration with indices $\langle \nu_i \mid i < \infty \rangle$. Let $E_{\nu_i}^{W_i}$ be a measure on $\kappa_i$ whenever $E_{\nu_i}^{W_i} \neq \emptyset$. Suppose that:

$(*)\ \bigwedge \xi \bigvee i \quad \pi_{W_0 W_i}(\xi) < \kappa_i$ .

Set: $W_\infty = \bigcup_i W_i | \kappa_i$ .

Define $\pi_{i\infty} : W_i \to W_\infty$ by:

$\pi_{i\infty}(x) = \pi_{ij}(x)$ where $j$ is large enough that $rn(\pi_{ij}(x)) < \kappa_j$.

Then $W_\infty, \langle \pi_{i\infty} \rangle = \lim\limits_{i \leq j} (W_i, \pi_{ij})$.

The proof is straightforward + will be left to the reader.

If we coiterate two search we get a result analogous to the foregoing:

<u>Lemma 1.5</u> Let $\langle W_i^h \mid i < \theta \rangle$ $(h = 0, 1)$ be the coiteration of the two search $W^0, W^1$ with indices $\nu_i$ . .

(a) One side is simple.

(b) If one side is not simple, then $\theta = \infty$.

(c) If $\theta = \infty$, there is one side with:

(*) $\bigwedge \xi \bigvee i \; \pi_{W_0^h W_i^h}(\xi) < \kappa_i$.

(d) If both sides satisfy (*), then $W_\infty^0 = W_\infty^1$.

(e) If $\theta = \infty$ and the $W^0$ side does not satisfy (*), then there is a cub $C$ s.t. $\pi_{W_i^0 W_j^0}(\kappa_i) = \kappa_j$ for $i, j \in C$, $i \leq j$. (Hence $E_{\nu_i}^{W_i^0}$ is a measure on $\kappa_i$ and each $X \in E_{\nu_i}^{W_i}$ almost contains $\{\kappa_\ell \mid \ell \in C \cap i\}$.)

The proof is left to the reader.

Suppose we are given a pair $M, N$ of premice whose coiteration exists. It follows as before that the coiteration terminates, yielding say $M', N'$, and that $M'$ is an initial segment of $N'$ if the iteration from $N$ to $N'$ is not simple. Now suppose that one of the pair is a mouse. The argument of §2.1 Lemma 1 shows that one side of the iteration must be simple. Using this, it follows that Lemmas 1.1 – 1.3 of this section hold for a pair $W, M$ s.t. $W$ is a weasel, $M$ is a premouse and $W, M$ are coiterable.

<u>Def</u> A weasel $W$ is <u>universal</u> iff the coiteration with any coiterable premouse terminates.

<u>Def</u> A weasel $W$ is <u>weakly universal</u> iff the coiteration with any mouse terminates.

We shall later describe a "minimally" chosen universal W which is called the <u>core</u> <u>model</u>. For the moment we content ourselves with demonstrating that a universal W exists. We shall construct this W in such a way that every $E_\nu^W$ which is a measure in W is ω-complete.

<u>Def</u> The <u>canonical</u> ω-<u>complete</u> <u>hierarchy</u> $W_\xi = \langle J_{\gamma_\xi}^{E^{W_\xi}}, E_{\gamma_\xi}^{W_\xi} \rangle \; (\gamma_\xi \leq \xi \leq \alpha$ is defined as follows: $W_0 = \langle \emptyset, \emptyset \rangle$. Now let $W_\xi$ be defined. If $W_\xi$ is not a mouse, $W_{\xi+1}$ is undefined. Otherwise let $\langle J_\gamma^{\bar{E}}, \bar{E}_\gamma \rangle = \mathrm{core}(W_\xi)$ and set $W_{\xi+1} = \langle J_{\gamma+1}^{\bar{E}}, \emptyset \rangle$.

Now let $W_\xi$ be defined for $\xi < \lambda$ where $\lim(\lambda)$ and $\lambda \leq \infty$. Set:

$\sigma_\xi = \sigma(\xi, \lambda) =$ the maximal $\sigma \leq \xi$ s.t. $J_\sigma^{E^{W_\xi}} = J_\sigma^{E^{W_\eta}}$ for $\xi \leq \eta < \lambda$ $(\xi < \lambda)$.

Then $\xi \leq \eta$ implies $\sigma_\xi \leq \sigma_\eta$ and $J_{\sigma_\xi}^{E^{W_\xi}} = J_{\sigma_\xi}^{E^{W_\eta}}$. We then ask whether

$$(*) \quad \bigwedge \xi < \lambda \bigvee \eta < \lambda \quad \sigma_\xi < \sigma_\eta .$$

If not, then $W_\lambda$ is undefined. If $(*)$ holds, let $J_{\tilde\lambda}^E = \bigcup_{\xi < \lambda} J_{\sigma_\xi}^{E^{W_\xi}}$.

If $\lambda = \infty$, set $W_\infty = J_{\tilde\lambda}^E$ (hence $\tilde\lambda = \infty$). Otherwise set:

$W_\lambda = \langle J_{\tilde\lambda}^E, F \rangle$, where $F$ is chosen - if possible - to be an $\omega$-complete measure s.t. $\langle J_{\tilde\lambda}^E, F\rangle$ is a mouse. If this is not possible, set $F = \emptyset$.

. . . . . .

Note that $W_\infty$, if defined, is a weasel. We will show it to be universal

<u>Lemma 2.1</u> $W_\xi$ is defined for $\xi \leq \infty$.

prf.

By ind. on $\xi$ we show:

(a) $W_\xi$ is defined

(b) $W_\xi$ is a mouse.

We first show (a). For $\xi = 0$ or $\xi$ a successor this is trivial. Now let $\xi = \lambda$, lim $(\lambda)$. We must show:

(*) $\bigwedge \xi < \lambda \bigvee \eta < \lambda \quad \sigma_\xi < \sigma_\eta$.

Suppose not. Let $\sigma_\xi = \sigma$ for $\xi_0 \leq \xi < \lambda$. Then

(1) $\qquad \omega\sigma \in W_\xi$ for $\xi_0 < \xi < \lambda$.

We prove this by ind. on $\xi$. If $\xi = \tau + 1$, then $W_\xi = \langle J^E_{\gamma+1}, \emptyset \rangle$ where $\langle J^E_\gamma, E_\gamma \rangle = \overline{W}_\tau = \text{core}(W_\tau)$. But $W_\tau$ is a simple normal iterate of $\overline{W}_\tau$; hence either $\overline{W}_\tau = W_\tau$ or the first measure applied in the iteration has the form $E^{\overline{W}_\tau}_\nu$ for a $\nu \geq \sigma$ (hence $\omega\sigma \subset \overline{W}_\tau$).

For limit $\xi$ note that if $\omega\sigma = \mathrm{On} \cap W_\xi$, then $\sigma = \sup\limits_{\eta < \sigma} \sigma(\eta, \xi)$

where $\sigma(\eta,\xi) < \sigma$ for $\eta < \xi$.
But for $\xi_0 \leq \eta < \xi$ we then have $\sigma = \sigma(\eta,\lambda) \leq \sigma(\eta,\xi) < \sigma$.
Contr! QED (1).

But for $\xi_0 < \xi < \lambda$, there must be $\eta > \xi$ s.t. $\eta < \lambda$ and $E_\sigma^{W_\eta} \neq E_\sigma^{W_\xi}$, since otherwise $\sigma + 1 \leq \sigma_\xi$.
Let $\eta$ be the least such. $\eta$ is easily seen to be a successor ordinal. Let $\eta = \tau + 1$. Then $W_\eta = \langle J^E_{\gamma+1}, \emptyset \rangle$ where $\langle J^E_\gamma, E_\gamma \rangle = \overline{W}_\tau = \mathrm{core}(W_\tau)$. $W_\tau$ is a simple normal iterate of $\overline{W}_\tau$; hence the first measure applied in the iteration must be $E_\sigma^{\overline{W}_\tau}$ (since $E_\sigma^{\overline{W}_\tau} = E_\sigma^{W_\xi} \neq E_\sigma^{W_\eta} = E_\sigma^{W_\tau}$). But then $E_\sigma^{W_\xi} \neq \emptyset$ and $E_\sigma^{W_\eta} = \emptyset$. Since this holds generally for $\xi \in (\xi_0, \lambda)$, we conclude that $E_\sigma^{W_\eta} \neq \emptyset$. Contr! QED (a)

We now prove (b). Note first that straightforward inductions on $\xi$ yield:

(2) If $\omega\alpha \in W_\xi$, then $W_\xi | \alpha$ is a mouse

(3) If $E_\nu^{W_\xi}$ is a measure in $W_\xi$, then $E_\nu^{W_\xi}$ is $\omega$-complete.

Suppose (b) to be false. Let $\langle N_i \mid i < \theta \rangle$ be an iteration of $W_\xi$ which cannot be continued. Let $\delta$ be regular s.t. $\langle N_i \rangle \in H_\delta$. Let $X \prec H_\delta$ be countable s.t. $\langle N_i \rangle \in X$. Let $F : \bar{H} \xrightarrow{\sim} X$ where $\bar{H}$ is transitive. Let $F(\langle \bar{N}_i \mid i < \bar{\theta} \rangle) = \langle N_i \mid i < \theta \rangle$. By absoluteness, $\langle \bar{N}_i \rangle$ is an iteration which cannot be continued. We consider two cases:

<u>Case 1</u> $\langle \bar{N}_i \rangle$ is simple.

Successively choose maps $\sigma_i$ s.t. $\sigma_i : \bar{N}_i \xrightarrow[\Sigma^*]{} W_\xi$ and $\sigma_i \pi_{\bar{N}_h \bar{N}_i} = \sigma_h$ for $h \le i$. Take $\sigma_0 = F \upharpoonright N_0$. $\sigma_{i+1}$ is then given by §1.4 Lemma 2.1, using (2). For limit $\lambda$ define $\sigma_\lambda$ by: $\sigma_\lambda \pi_{\bar{N}_i \bar{N}_\lambda} = \sigma_i \quad (i < \lambda)$.

We now derive a contradiction by showing that $\langle \bar{N}_i \rangle$ can be continued. First suppose $\lim(\bar{\theta})$. Let $\bar{N}, \langle \pi_i \rangle = \lim_{i \leq j < \bar{\theta}} (\bar{N}_i, \pi_{\bar{N}_i \bar{N}_j})$. We can imbed $\bar{N}$ in $W_\xi$ by $\sigma$, where $\sigma \pi_i = \sigma_i \quad (i < \bar{\theta})$. Hence $\bar{N}$ is well founded. Now let $\bar{\theta} = \tau + 1$. Let $\omega\nu, \omega\alpha \leq On \cap \bar{N}_\tau$ s.t. $E_\nu^{\bar{N}_\tau}$ is a measure in $N = \bar{N}_\tau | \alpha$. We claim that $\pi : N \xrightarrow[E_\nu]{*} N'$ exists. If $\omega\alpha = On \cap \bar{N}_\tau$, this follows by (2) and § 1.4 Lemma 1.2. Otherwise set $W = W_\xi | \sigma_\tau(\alpha)$. Then $(\sigma_\tau \upharpoonright N) : N \xrightarrow[\Sigma^*]{} W$, where $W$ is a mouse. QED (Case 1)

<u>Case 2</u> Case 1 fails.

There is then an $\eta$ s.t. $\langle \bar{N}_i | i \leq \eta \rangle$ is simple and $\omega\alpha_\eta \in \bar{N}_\eta$, where

$\langle \nu_i, \alpha_i \rangle$ are the indices of the iteration. By the argument of Case 1, there is $\sigma : \bar{N}_\eta \to_{\Sigma^*} W_\xi$. Set $N = \bar{N}_\eta | \alpha_\eta$, $W = W_\xi | \sigma(\alpha_\eta)$. Then $\sigma \upharpoonright N : N \to_{\Sigma^*} W$, where $W$ is a mouse. Hence $N$ is a mouse and the iteration is continuable. QED (Lemma 2.1)

As a corollary of the proof:

Cor 2.2 If $E_\nu^{W_\xi}$ is a measure in $W_\xi$, then it is $\omega$-complete.

It is not immediately clear that the $W_\xi$ are uniquely defined by our conditions, since we left some leeway for the choice of the measure in the limit case. We will now show that, in fact, the choice was illusory and the $W_\xi$ are unique. To this end we define:

<u>Def</u> A <u>bicephalus</u> is a structure $M = \langle J^E_\gamma, F, G \rangle$ s.t.

(a) $\langle J^E_\gamma, F \rangle$, $\langle J^E_\gamma, G \rangle$ are mice;

(b) $F, G$ are $\omega$-complete.

We wish to show that bicephali trivialize - i.e. $F = G$. As a first step we prove:

<u>Lemma 2.3.1</u> Let $M = \langle J^E_\gamma, F, G \rangle$ be a bicephalus s.t. each $E_\nu$ which is a measure in $M$ is $\omega$-complete. Then $F = G$.

proof.

The notion of an <u>iteration</u> of a bicephalus is defined as before except that in going from $M_i$ to $M_{i+1}$ via index $\langle \nu_i, \alpha_i \rangle$ we use any $E^i \in \mathcal{E}^{M_i}_{\nu_i}$, where $\mathcal{E}^{M_i}_\nu = \{E^{M_i}_\nu\}$ if $\omega\nu \in M$ and $\mathcal{E}_{\gamma_i} = \{F_i, G_i\}$ if $M_i = \langle J^E_{\gamma_i}, F_i, G_i \rangle$. The <u>coiteration</u> $\langle M_i \leq \theta \rangle$, $\langle N_i \leq \theta \rangle$ of two bicephali

M, N is defined as before except that we now take:

$\nu_i \simeq$ the least $\nu$ s.t. $\forall E \in \mathcal{E}_\nu^{M_i} \forall F \in \mathcal{E}_\nu^{N_i}\, E \neq F$ and $E^{i,M_i}$ = the least $E \in \mathcal{E}_\nu^{M_i}$ s.t. $E \neq F$ for an $F \in \mathcal{E}_\nu^{N_i}$ (letting F precede G in $\langle J_\gamma^E, F, G \rangle$) and $E^{i,N_i}$ = the least $F \in \mathcal{E}_\nu^{N_i}$ s.t. $F \neq E^{i,M_i}$.

The argument given above that each $W_\xi$ is a mouse can easily be adapted to show that each bicephalus satisfying the hypothesis of the lemma is iterable. It follows exactly as before that the coiteration of two bicephali terminates and that one side of the coiteration is simple. Now coiterate M with itself, getting N, N', where N is a simple iterate of M. Assume w.l.o.g. that N is an initial segment of N'. If N is a proper segment, then:

$F^N = G^N = E_\nu^{N'}$ for some $\nu$. If $N = N'$, then $F^N = G^N = F'^N = G'^N$. In either case $F = \pi_{MN}^{-1}{}''F^N = \pi_{MN}^{-1}{}''G^N = G$.

QED (Lemma 2.3.1).

We can now show:

<u>Lemma 2.3</u> If $M = \langle J_\gamma^E, F, G \rangle$ is a bicephalus, then $F = G$.

proof.

Let $\bar{M} = \langle J_\gamma^E, \emptyset \rangle$. Then $\bar{M}$ is a mouse with $\rho_{\bar{M}}^\omega = \gamma$, where $\kappa$ = the largest cardinal in $\bar{M}$. Thus $\bar{M}$ satisfies $ZF^-$ and $*$-iterations of $\bar{M}$ are the same as $\Sigma_0$-iterations (i.e. using only $f \in \bar{M}$). We define the <u>canonical</u> $\omega$-<u>complete iteration</u> $\langle \bar{M}_i \mid i \leq \theta \rangle$ by:

$\bar{M}_0 = M$; $\nu_i = \kappa_i^{+\bar{M}_i}$, where $\kappa_i$ = the least $\kappa$ s.t. $(E_{\kappa^+})^{M_i} \neq \emptyset$ and $(E_{\kappa^+})^{M_i}$ is not $\omega$-complete

Standard methods tell us that

This is a simple iteration which terminates at an $\theta < \infty$. Set:

$\pi = \pi_{\bar{M}\bar{M}_\theta}$, $\bar{M}' = \bar{M}_\theta$, $\kappa' = \pi(\kappa)$.

Then $\pi$ takes $\bar{M}$ cofinally to $\bar{M}'$ and $\kappa$ cofinally to $\kappa'$. Define $M' = \langle J^{E'}_{\gamma'}, F', G' \rangle$ by:

$\pi : M \xrightarrow[\Sigma_0]{} M'$.

<u>Claim</u> $M'$ is a bicephalus.

[This claim proves the theorem, since then $F' = G'$ by Lemma 2.3.1 + hence $F = \pi^{-1}{}''F' = \pi^{-1}{}''G' = G$.]

We know that $\langle J^{E'}_{\gamma'}, F' \rangle$, $\langle J^{E'}_{\gamma'}, G' \rangle$ are pms. Moreover $E'_\nu$ is $\omega$-complete whenever it is a measure in $M'$ $(\nu < \gamma')$. Hence the argument which showed each $W_\xi$ to be a mouse will show $\langle J^{E'}_{\gamma'}, F' \rangle$, $\langle J^{E'}_{\gamma'}, G' \rangle$ to be mice as soon as we know that:

<u>Claim</u> $F', G'$ are $\omega$-complete.

We show it for $F'$. Let $X_n \in F'$ $(n<\omega)$. Let $X_n \in J^{E'}_{\pi(\alpha_n)}$. Pick $Y_n \in F$ which is almost contained in each $Z \in F \cap J^E_{\alpha_n}$ (up to a difference $< \kappa$) for $n < \omega$. Then $\pi(Y_n)$ is almost contained in $X_n$. Pick $\delta_n < \kappa$ s.t. $\pi(Y_n \setminus \delta_n) \subset X_n$. Then

$$\emptyset \neq \pi'' \bigcap_n (Y_n \setminus \delta_n) \subset \bigcap_n X_n$$

QED (Lemma 2.3)

[ The proof of Lemma 2.3 shows:

<u>Cor 2.3.1</u> If $\langle J^E_\gamma, \emptyset \rangle$ is a mouse, $\langle J^E_\gamma, F \rangle$ a pm and $F$ is $\omega$-complete, then $\langle J^E_\gamma, F \rangle$ is a mouse. ]

<u>Cor 2.4</u> $W_\xi$ is unique.

pf. Ind on $\xi$, using Lemma 2.3

Thus the weasel $W_\infty$ is uniquely defined. We refer to it as the <u>canonical</u> $\omega$-<u>complete</u> <u>weasel</u>. We intend to give a simpler alternative characterization of $W_\infty$, but first we prove a technical lemma about the $W_\xi$ hierarchy.

<u>Def</u> Let $\xi < \eta \leq \infty$. $\sigma(\xi, \eta) =$ $=$ the maximal $\sigma$ s.t. $J_\sigma^{W_\xi} = J_\sigma^{W_\tau}$ for $\xi \leq \tau \leq \eta$.

[<u>Note</u> This does not conflict with the previous def. of $\sigma(\xi, \lambda)$ for limit $\lambda$.]

<u>Lemma 2.5</u> Let $\xi < \eta \leq \infty$, $\sigma = \sigma(\xi, \eta)$. Then $\mathbb{P}(\sigma) \cap W_\xi \subset W_\eta$.

prf. (By ind. on $\eta$).

<u>Case 1</u> $\eta = 0$ trivial

<u>Case 2</u> $\eta = \tau + 1$.

It suffices to prove it for $\xi = \tau$.

$W_\eta = \langle J^E_{\gamma+1}, \emptyset \rangle$ where $\langle J^E_\gamma, E_\gamma \rangle =$
$= \overline{W}_\tau = \mathrm{core}(W_\tau)$. $W_\tau$ is a simple normal iterate of $\overline{W}_\tau$. Thus either $W_\tau = \overline{W}_\tau$, or $E^{\overline{W}_\tau}_\sigma$ is the first measure applied in the iteration. In either case, $\mathcal{P}(\sigma) \cap W_\tau \subset \overline{W}_\tau \subset W_\eta$.

QED (Case 2)

<u>Case 3</u> $\eta = \lambda$, $\lim(\lambda)$

Set $\sigma_\mu = \sigma(\mu, \lambda)$ for $\mu < \lambda$. Then $W_\lambda = \langle J^E_\gamma, E_\gamma \rangle$, where $J^E_\gamma = \bigcup_{\eta < \lambda} J^{E^{W_\eta}}_{\sigma_\eta}$.

Let $\sigma = \sigma_\xi$ and pick $\eta > \xi$ s.t. $\sigma < \sigma_\eta$. It suffices to show:

<u>Claim</u> $\mathcal{P}(\sigma) \cap W_\xi \subset J^{E^{W_\eta}}_{\sigma_\eta}$.

To see this, pick the least $\mu > \eta$ s.t. $\sigma_\eta < \sigma_\mu$. Then $\mu = \tau + 1$, $\sigma_\tau = \sigma_\eta$, and by the argument of Case 2:
$\sigma_\tau = \sigma(\tau, \mu)$ is a cardinal in $W_\tau$.
By acceptability we then get:
$\mathcal{P}(\sigma) \cap W_\xi \subset J^{E^{W_\tau}}_{\sigma_\tau} = J^{E^{W_\eta}}_{\sigma_\eta}$, since
$\mathcal{P}(\sigma) \cap W_\xi \subset W_\tau$. QED (Lemma 2.5)

<u>Lemma 2.6</u>

(a) If $E_\nu$ is a measure in $W_\infty$, then $E_\nu$ is $\omega$-complete.

(b) If $\nu = \kappa^{+W_\infty}$ and there is an $\omega$-complete $F$ s.t. $\langle J_\nu^E, F\rangle$ is a mouse, then $F = E_\nu$.

[<u>Note</u> We shall later show that the conditions (a), (b) uniquely characterize $W_\infty$ ].

proof.

(a) is immediate by Lemma 2.2.

Now let $\langle J_\nu^E, F\rangle$ be as in (b). Let: $\xi$ = the least $\xi$ s.t. $\sigma_\xi = \sigma(\xi, \infty) \geq \nu$,

<u>Case 1</u> $\xi = \tau + 1$.

Then $W_\xi = \langle J_{\gamma+1}^{\bar{E}}, \emptyset\rangle$, $\langle J_\gamma^{\bar{E}}, \bar{E}_\gamma\rangle = \bar{W}_\tau$ = core(W

Since $\sigma(\tau, \xi) < \nu$, we must have: $\rho_{W_\tau}^\omega < \nu$. Hence $\bar{\bar{\nu}}^{W_\xi} < \nu$. Hence $\bar{\bar{\nu}}^{W_\infty} < \nu$ by Lemma 2.5. Contr!

<u>Case 2</u> $\xi = \lambda$, Lim$(\lambda)$

Clearly, $W_\lambda = \langle J_\nu^E, G\rangle$ where $\nu = \sup_{\eta<\lambda} \sigma_\eta$ and $\sigma_\eta = \sigma(\eta, \lambda) < \nu$

( ... $\eta < \lambda$

But $G$ is chosen to be $\omega$-complete. Hence $G = F$ by Lemma 2.3. But $\rho^{\omega}_{W\eta} \geq \nu$ for $\eta \geq \lambda$ by the argument of Case 1. Hence $F = E^{W_\infty}_{\nu}$.

QED (Lemma 2.6).

Using only Lemma 2.6 we now prove

<u>Lemma 2.7</u> $W_\infty$ is universal.

proof.

Suppose not. Let $N$ be a counterexample whose coiteration with $Q = W_\infty$ does not terminate. Let $C$ be the cub class of Lemma 1.2 (b) (i.e. $\pi_{N_i N_j}(\kappa_i) = \kappa_j$ for $i, j \in C$).

Consider the class $D$ of $\kappa$ s.t.

(a) $\kappa = \kappa_\kappa \in C$

(b) $\kappa$ is a limit cardinal

(c) $\mathrm{cf}(\kappa) > \omega$

(d) $\pi_{Q Q_\kappa}(\kappa) = \kappa$.

Note that (d) is satisfied if $\bigwedge \xi < \kappa \bigvee i < \kappa \ \pi_{Q Q_i}(\xi) < \kappa_i$.

Hence $D$ is closed under limits of cofinality $> \omega$.

Let $\kappa \in D$ and set $F = (E_{\kappa^+})^{N_\kappa | \alpha^N_\kappa}$.
Then $F$ is an $\omega$-complete measure on $\kappa$ in $N | \nu = \langle J^E_\nu, F \rangle$, where $\nu = \kappa^{+N_\kappa | \alpha^N_\kappa} = \kappa^{+Q_\kappa}$, and $J^E_\nu = J_\nu^{E^{Q_\kappa}} = J_\nu^{E^{N_\kappa}}$. Note that $\nu = \kappa^{+Q}$ and $\pi_{QQ_\kappa}(\nu) = \nu$.
Clearly $E^{Q_\kappa}_\nu \neq F$. From this we derive a contradiction.

<u>Case 1</u> $E^{Q_\kappa}_\nu \neq \emptyset$ for a $\kappa \in D$.
By the $\omega$-completeness of $E^Q_\nu$, it follows easily that $E^{Q_\kappa}_\nu$ is $\omega$-complete (cf. the proof of the final claim in Lemma 2.3). But then $\langle J^E_\nu, E^{Q_\kappa}_\nu, F \rangle$ is a bicephalus. Hence $E^{Q_\kappa}_\nu = F$. Contradiction!

<u>Case 2</u> Case 1 fails.
Let $\kappa \in D$ s.t. $D \cap \kappa$ is unbounded in $\kappa$. Then $\pi_{QQ_\kappa}(\tau) = \tau$ for

each $\tau \in D \cap \kappa$, by the failure of Case 1. Set:

$$\bar{F} = \{X \mid \pi_{QQ_\kappa}(X) \in F\}.$$

$\bar{F}$ is $\omega$-complete, since $D$ is almost contained in every $X \in \bar{F}$. We derive a contradiction by showing

<u>Claim</u> $\langle J_\nu^{E^Q}, \bar{F}\rangle$ is a mouse.

(1) $\langle J_\nu^{E^Q}, \bar{F}\rangle$ is amenable

proof.

Let $\xi < \nu$, $\bar{Z} = \bar{F} \cap J_\xi^{E^Q}$. <u>Claim</u> $\bar{Z} \in J_\nu^{E^Q}$

Set $Z = F \cap J^E_{\pi(\xi)}$. Then $Z =$

$= \pi(f)(\kappa_{i_0}, \ldots, \kappa_{i_n})$ and standard methods tell us that $\bar{Z} = \pi^{-1}{}''Z \in J_\nu^E$

QED(1)

(2) $\bar{F}$ is $\kappa$-complete in $\langle J_\nu^{E^Q}, \bar{F}\rangle$

prf.

Let $\langle X_i \mid i < \gamma\rangle \in J_\nu^{E^Q}$, $X_i \in \bar{F}$ for for $i < \gamma$, where $\gamma < \kappa$. Then

$$\pi(\bigcap_i X_i) = \bigcap_{j < \pi(\gamma)} \pi(\langle X_i\rangle)_j \in F.$$

Hence $\bigcap_i X_i \in \bar{F}$. QED(2)

(3) $\bar{F}$ is normal in $\langle J_\nu^{EQ}, \bar{F}\rangle$.

prf.

Let $f : X \to \kappa$ be regressive, where $X \in \bar{F}$, $f \in J_\nu^{EQ}$. Then $\pi(f)$ is regressive. Hence there is $\tau$ s.t. $\{\xi \mid \pi(f)(\xi) = \tau\} \in F$. Hence there is $\alpha < \kappa$ s.t. $\{\xi \mid \pi(f)(\xi) < \pi(\alpha)\} \in$ $\in F$. Hence $\{\xi \mid f(\xi) < \alpha\} \in \bar{F}$. The conclusion follows by (2).

QED (3)

(4) $\pi \upharpoonright J_\nu^{EQ} : \langle J_\nu^{EQ}, \bar{F}\rangle \to_{\Sigma_1} \langle J_\nu^E, F\rangle$

proof.

It suffices to show: $\pi(\bar{F} \cap u) = F \cap \pi(u)$ for $u \in J_\nu^{EQ}$. Assume w.l.o.g. that $u \subset \mathbb{P}(\kappa)$. By (3) there is $\bar{Y} \in \bar{F}$ which is almost contained in $X$ or $\kappa \setminus X$ for all $X \in u$. Let $Y = \pi(\bar{Y})$. Then $Y \in F$ is almost contained in $X$ or $\kappa \setminus X$ for $X \in \pi(u)$. Hence

$\pi(\bar{F} \cap u) = \pi(\{X \in u \mid \bar{Y} \subset\cdot X\}) =$

$= \{X \in \pi(u) \mid Y \subset\cdot X\} = F \cap \pi(u)$.

QED(4)

By (4), $\langle J_\nu^{E^Q}, \bar{F} \rangle$ is a

Iterability follows by the fact that $F$ is $\omega$-complete since $\pi_{Q,Q_n}(\tau) = \tau$ for $\tau \in D \cap n$ and all $E_\xi^Q$ which are measures in the structure are $\omega$-complete. QED (Claim)

This is a contradiction, since then $\bar{F} = E_\nu^Q$ and $F = E_\nu^{Q_n} \neq \emptyset$.

QED (Lemma 2.7)

<u>Corollary 2.8</u> $W_\infty$ is the unique weasel satisfying (a), (b) of Lemma 2.6

proof.

Let $W = W_\infty$ and let $W'$ be another weasel satisfying (a), (b). Then $W'$ is universal. Coiterate $W, W'$, getting $W_\nu, W'_\nu$ $(\nu < \theta \leq \infty)$. Both iterations must be simple, since e.g. if some $W'_\nu$ is a mouse, the iteration must terminate below $\infty$ by the universality of $W$, contradicting Lemma 1.1. But then if $\theta > 0$, it follows easily by the bicephalus lemma and the fact that $W, W'$ satisfy (a), (b) that $E^W_{\nu_0} = E^{W'}_{\nu_0}$. Contradiction!

QED (Cor 2.8)

<u>Cor 2.9</u> Let $W$ be a mouse. If $V = W$, then $W = W_\infty$.

pf. Suppose not.

$W$ satisfies (a). Hence there is a least $\nu = \kappa^+$ s.t. an $\omega$-compl. $F$ exists s.t. $\langle J^E_\nu, F\rangle$ is a mouse (where $W = J^E_\infty$). Hence $J^E_\nu = J^{E^{W_\infty}}_\nu$ and $E^{W_\infty}_\nu = F$. Hence $\kappa$ is the first pt. moved on the $W_\infty$-side & is not moved on the $W$-side in the coiteration of $W_\infty, W$. Hence letting $\sigma : W_\infty \to_E W'$, we have:

$\kappa^{++} = \kappa^{++W} = \kappa^{++W'} < \kappa^{++W_\infty} \leq \kappa^{++}$

Contr!

QED (Cor 2.9)

In the sequel we shall occasionally have to consider structures which are like mice but which we do not, initially, know to satisfy the initial segment condition ((c) in the definition of premouse).

<u>Def</u> A <u>quasi-premouse</u> (qpm) is a structure $M = \langle J_\alpha^E, E_\alpha \rangle$ s.t. $\langle J_\alpha^E, \emptyset \rangle$ is a premouse and $M$ satisfies conditions (a), (b) in the definition of premouse.

<u>Def</u> A <u>quasi mouse</u> (qm) is an iterable qpm.

We shall show that every quasi mouse is in fact a mouse. We first note that the proof that each $W_\xi$ is a mouse shows:

<u>Lemma 3.1</u> Let $M$ be a qpm (pm) in which each $E_\nu^M$ which is a measure in $M$ is $\omega$-complete. Then $M$ is a qm (mouse).

It is clear that everything through §2.2 goes thru for quasi mice, since we did not use the initial segment condition there. Moreover, if we coiterate two quasi mice, then one side of the coiteration must be simple, since we would otherwise be coiterating two mice from the point of truncation + hence would get a contradiction as before. Hence Lemmas 1.1 – 1.3 of this section go through if W is a weasel and M a quasi mouse. We now improve Lemma 2.6(b) to read:

<u>Lemma 3.2</u> Let $J^E_\infty = W_\infty$ be the canonical $\omega$-complete weasel. If $\nu = \kappa^{+W_\infty}$ and there is an $\omega$-complete $F$ s.t. $\langle J^E_\nu, F\rangle$ is a qpm, then $F = E_\nu$.

proof of Lemma 2.10. Suppose not. We first note that the proofs of Lemmas 2.5, 2.6 go thru with arbitrary $\mu$, $W_\mu$ in place of $\infty$, $W$. We use them in this generalized form

Let $\langle J_\nu^E, F\rangle$ be a counterexample. $\langle J_\nu^E, F\rangle$ is a qm by Lemma 3.1. Hence the initial segment condition fails. Let $\bar\nu < \nu$ be least s.t. $\langle J_{\bar\nu}^E, \bar F\rangle$ is a qpm, $\bar\nu > \kappa$ (where $\bar F$ is on $\kappa$) and $\bar F \neq E_{\bar\nu}$. Then $\langle J_{\bar\nu}^E, \bar F\rangle$ is a mouse.

Let $\xi$ be least s.t. $\sigma_\xi = \sigma(\xi, \infty) \geq \bar\nu$.

Case 1 $\xi = \tau + 1$.

Then $W_\xi = \langle J_{\gamma+1}^E, \emptyset\rangle$ where $\langle J_\gamma^E, E_\gamma\rangle = \bar W_\tau = \text{core}(W_\tau)$.

Since $\sigma_\tau < \bar\nu \leq \sigma_\xi$, $\sigma_\tau$ must be the first index in the iteration from $\bar W_\tau$ to $W_\tau$. But $\rho^\omega_{W_\tau} \geq \kappa$, since otherwise

$\varkappa$ would not be a cardinal in $W_\infty$.
Hence $\bar{\nu} = \varkappa^{+W_\tau} = \sigma_\tau$ and
$E_{\bar{\nu}}^{\bar{W}_\tau} \neq \emptyset$, $E_{\bar{\nu}}^{W_\tau} = \emptyset$. But $\bar{F} = E_{\bar{\nu}}^{W_\tau}$
by the generalized Lemma 2.6.
Contr! QED (Case 1).

<u>Case 2</u> Lim($\bar{\xi}$).
As in the proof of Lemma 2.6
we have: $W_{\xi} = \langle J_{\bar{\nu}}^{E}, G \rangle$.
But then $G = \bar{F}$ and hence
$E_{\bar{\nu}} = \bar{F}$, since $\sigma_{\xi} > \bar{\nu}$ as in
the proof of Lemma 2.6.
Contr! QED (Lemma 3.2).

But then the proof of Lemma 2.7
can be repeated verbatim to show:
<u>Lemma 3.3</u> Let N be a qpm which
is coiterable with $W_\infty$. Then
the coiteration terminates.

Hence:

Cor 3.4 Every quasi mouse is a mouse.

prf.

The coiteration of $N$ with $W$ so terminates. Hence $N$ has a simple iterate which is a mouse. Hence $N$ is a mouse. QED (Cor 3.4).

Note It follows easily that if we had defined the notion of mouse without using the initial segment condition (c), we would have characterized the same class of structures. Koepke has suggested a more elegant way of proving this: Our only use of (c) was in the proof of §2.3 Lemma 4.1. Koepke has found a slightly longer proof which makes no use of (c).

Hence everything goes thru without use of (c). We can then prove that all mice satisfy (c) by showing that the universal weasel $W_\infty$ satisfies (c) by virtue of its construction.