Appendix to "Measures of Order Zero"

Steel has found a more direct proof of the existence of K and of the theorems in § 3.7, avoiding the sequence lemma of § 3.5 and most of the material on beavers in § 3.6. However, his proof appears to make essential use of second order set theory (Morse-Kelley), whereas § 3.5 - 3.7 can be carried out in ZF. We give an account of Steel's proof.

Def J_α^E is incompressible iff there is a weasel W s.t.

(a) $J_\alpha^E = J_\alpha^{E^W}$

(b) Some $X \subset On$ is massive in W

(c) If X is massive in W, then $\alpha \subset h_W(X)$ (hence $J_\alpha^E \subset h_W(X)$).

It is clear that if W satisfies (a)-(c) and $W'|\alpha = W|\alpha$, then W' satisfies (a), (c). It is also clear that if $\alpha \leq \beta$ and J_β^E is incompressible, then so is J_α^E.

We also note:

<u>Lemma 1</u> For any α there is at most one incompressible J_α^E.

proof.

Let J_α^E, $J_{\alpha'}^{E'}$ be incompressible as witnessed by W, W' resp.

Let X, X' be massive in W, W' resp. Coiterate W, W' to $\hat{W}$ and set:

$$\hat{X} = \{\alpha \in X \cap X' \mid \alpha = \pi(\alpha) = \pi'(\alpha)\}$$

where $\pi = \pi_{W\hat{W}}$, $\pi' = \pi_{W'\hat{W}}$.

Then $\hat{X}$ is massive in $W, W', \hat{W}$.

Hence:

(1) $\pi''\alpha = \pi'{}''\alpha$ = the first α elements of $h_{\hat{W}}(\hat{X})$.

Let $\sigma : \overline{W} \overset{\sim}{\longleftrightarrow} h_{\hat{W}}(\hat{X})$, we

have: $\pi^{-1}\sigma : \overline{W} \to_{\Sigma_1} W$, $\pi^{-1}\sigma \upharpoonright \alpha = \mathrm{id}$.
Hence $J_\alpha^E = J_\alpha^{E^W} = J_\alpha^{E^{\overline{W}}}$. But
the same argument shows:
$J_\alpha^{E'} = J_\alpha^{E^{W'}} = J_\alpha^{E^{\overline{W}}}$. QED (Lemma 1)

<u>Lemma 2</u> For every α there is an incompressible J_α^E.

proof.

Let W be the canonical ω-complete weasel in which all measures are of cofinality ω_1. Then On is massive in W.

<u>Claim 1</u> There is an ω_2-closed unbounded class C of ordinals s.t. $\nu \in h_W(\nu \cup X)$ whenever $\nu \in C$ & X is massive in W.

proof.

C is defined (in V) to be the of cardinals ν s.t.
$cf(\nu) = \omega_1$ and $\omega_1^\omega < \nu$.

Then $\nu^+ = \nu^{+W}$ for $\nu \in C$.

Now let $\nu \in C$ be a counter-example, i.e. $\nu \notin h_W(\nu \cup X)$, where X is massive in W. Let $\sigma : \bar{W} \overset{\sim}{\longleftrightarrow} h_W(\nu \cup X)$. Then $\sigma : \bar{W} \xrightarrow[\Sigma_1]{} W$ and $\nu = \mathrm{crit}(\sigma)$. Thus $U = \{z \in \mathcal{P}(\nu) \cap \bar{W} \mid \nu \in \sigma(z)\}$ is a normal measure on $\bar{W}$ and $\langle J_\nu^{E^W}, U \rangle$ is easily seen to be amenable. Moreover U is ω-complete. Since $\neg 0^1$, $\langle J_\nu^{E^W}, U \rangle$ is easily seen to be a mouse. Hence $U = E^W_{\nu^+}$ by the def. of W. Now set: $\sigma' : \bar{W} \xrightarrow[U]{} \bar{W}'$. There is a map $k : \bar{W}' \xrightarrow[\Sigma_1]{} W$ defined by: $k(\sigma'(f)(\nu)) = \sigma(f)(\nu)$. But $k \restriction (\nu+1) = \mathrm{id}$ and $E^{\bar{W}'}_{\nu^+} = \emptyset$. Hence $E^W_{\nu^+} = \emptyset$. Contr!

QED (Claim 1)

We now inductively construct massive X_α in W s.t.

(a) $\alpha \leq \beta \to X_\beta \subset X_\alpha$

(b) If $\overline{W} \simeq h_W(X_\alpha)$, then $\overline{W}$ witnesses the incompressibility of $J_\alpha^{E^{\overline{W}}}$ (i.e. $\alpha \subset h_{\overline{W}}(X)$ for massive X,

(b) will follow from:

(c) If $X \subset h_W(X_\alpha)$ is massive in W, then $Z_\alpha \subset h_W(X)$, where Z_α = the first α elements of $h_W(X_\alpha$

(To see (c) $\to$ (b), coiterate $\overline{W}, W$ to $\hat{W}$ + let $\overline{X}$ be massive in $\overline{W}$. Let X be massive in W. Set:
$\hat{X} = \{\alpha \in X \cap \overline{X} \mid \alpha = \pi_{\overline{W}\hat{W}}(\alpha) = \pi_{W\hat{W}}(\alpha)\}$.
Then $\hat{X}$ is massive in $\overline{W}, W, \hat{W}$.
Hence $Z_\alpha \subset h_W(\hat{X}) \simeq h_{\overline{W}}(\hat{X})$.
It follows easily that
$\alpha \subset h_{\overline{W}}(\hat{X}) \subset h_{\overline{W}}(\overline{X})$).

Set: $X_0 = On$. For limit λ set:

$X_\lambda = \bigcap_{\nu < \lambda} X_\nu$. The verification of (a), (c) is straightforward. Now let X_α be given. We construct $X_{\alpha+1}$.

Claim 2 There is $\nu \supset \varkappa_\alpha$ s.t. $\nu \in h_W(X)$ for every massive X.

(Claim 2 proves the theorem, for we can take ν = the least such & set $X_{\alpha+1} = X_\alpha \cap \bigcap_{\xi \in [\alpha, \nu)} Y_\xi$, where Y_ξ is massive in W s.t. $\xi \notin h_W(Y_\xi)$).

To prove Claim 2, suppose not. For each $\nu \supset \varkappa_\alpha$ pick massive Y_ν in W s.t. $\nu \notin h_W(Y_\nu)$. Set:

$$\Gamma_\nu = \bigcap_{\varkappa_\alpha \leq \xi \leq \nu} Y_\xi \quad \text{for } \varkappa_\alpha < \nu.$$

Then Γ_ν is massive in W. Set:

β_ν = the least $\beta \in h_W(\Gamma_\nu)$ s.t. $\nu \leq \beta$.

Then $\nu < \beta_\nu$ for $\varkappa_\alpha < \nu$.

Fix $\xi \supset Z_\alpha$ s.t.

(a) $\xi \in h_W(X \cup \xi)$ for all massive X.

(b) $Z_\alpha \subset \nu < \xi \rightarrow \beta_\nu < \xi$.

(This is possible by Claim 1).

Pick $\vec{\eta} < \xi$, $\vec{\delta} \in \Gamma_\xi$ s.t. $\xi = t^W(\vec{\eta}, \vec{\delta})$.

Pick $\nu < \xi$ s.t. $\eta < \beta_\nu$. Then

(1) $\beta_\nu, \beta_\xi, \delta \in H = h(\Gamma_\nu)$.

(2) $W \models \bigvee \vec{\eta} < \beta_\nu \; t(\vec{\eta}, \vec{\delta}) \in (\beta_\nu, \beta_\xi)$.

Hence there is such $\vec{\eta}^* \in H \prec_{\Sigma_1} W$.

But $H \cap \beta_\nu \subset Z_\alpha$. Hence $\eta^* \in Z_\alpha$

and $t^W(\vec{\eta}^*, \vec{\delta}) \in (\beta_\nu, \beta_\xi) \cap h_W(\Gamma_\xi) = \emptyset$.

Contr! QED (Lemma 2)

Now set:

$$\hat{K} = J_\infty^{\hat{E}} = \bigcup \{ N \mid N \text{ incompressible} \}$$

$\hat{K}$ is a weasel by the foregoing lemmas. We claim that $\hat{K}$ satisfies the def. of K (hence $\hat{K} = K$).

In order to prove this we shall need a little bit of the material in §3.6, namely the definition of <u>beaver</u> and Lemmas 1–3. Using this we prove:

<u>Lemma 3</u> Let $\langle J_\alpha^{\hat{E}}, U\rangle$ be a beaver. Then $U = \hat{E}_\alpha$.

proof.

Let W witness the incompressibility of $J_{\alpha+1}^{\hat{E}}$. Let X be massive in W. Since $\langle J_\alpha^{\hat{E}}, U\rangle$ is a beaver, we can form the ultraproduct $\sigma : W \xrightarrow[U]{} W'$. Coiterate W, W' to P. Set:

$$\Gamma = \{\alpha \in X \mid \alpha = \pi_{W'P}\sigma(\alpha) = \pi_{QP}(\alpha)\}$$

Γ is massive in W, W', P by the remark at the end of §3.5. $\kappa = \mathrm{crit}(\pi_{W'P}\sigma)$; hence $\kappa \notin h_P(\kappa \cup \Gamma)$. But $\kappa \in h_W(\kappa \cup \Gamma)$ by incompossibility. Hence $\kappa = \mathrm{crit}(\pi_{Wp})$.

Since $\mathcal{P}(\kappa) \cap W \subset h_W(\Gamma)$,

we have $\pi_{W'p}\sigma \upharpoonright \mathcal{P}(\kappa) = \pi_{Wp} \upharpoonright \mathcal{P}(\kappa)$.

Hence:

$$\hat{E}_\alpha = \{X \subset \kappa \mid \kappa \in \pi_{Wp}(X)\}$$

$$= \{X \subset \kappa \mid \kappa \in \pi_{W'p}\sigma(X)\} = U.$$

QED (Lemma 3)

<u>Cor 4</u> $\hat{K}$ is universal

proof.

By Lemma 3, if $\langle J_\alpha^{\hat{E}}, U \rangle$ is a premouse

U is ω-complete, then $U = E_\alpha$.

The conclusion follows by the argument of §3.1. QED (Cor 4)

<u>Cor 5</u> $\hat{K} = K$

proof.

Let $\langle J_\alpha^{\hat{E}}, U \rangle$ be strong, $U \neq \emptyset$.

<u>Claim</u> $U = \hat{E}_\alpha$.

<u>Case 1</u> $\langle J_\alpha^{\hat{E}}, U \rangle = W \| \alpha$, where

W is univ + $\alpha = \kappa^{+W}$.

Then $\langle J_\alpha^{\hat{E}}, U \rangle$ is a beaver + the conclusion follows by Lemma 3.

Case 2 Case 1 fails.

Then there is a mouse N s.t. $N|\alpha = \langle J_\lambda^{\hat{E}}, U \rangle$ and $\rho_N^\omega \leq \kappa$ (where U is on κ). We take $On \cap N$ as minimal for this property. But there is an $M = \hat{K}|\beta$ s.t. $\beta \geq \alpha$ and $\rho_M^\omega \leq \kappa$, since otherwise the coiteration of $N, \hat{K}$ would not terminate. Let β be minimal for this property.

Coiterate M, N to M', N'.

Claim $M' = N'$ is a simple iterate of M, N.

proof.

Suppose e.g. that M' is a non simple iterate of M or N' is a proper segment of M'. Then

$A^n_N = A^n_{N'} \in M$ where $\omega\rho^n_N \leq \alpha$.
But α is a cardinal in M. Contr!

QED (Claim 1

But then $M = N$ and $\langle J^{\hat{E}}_\alpha, u \rangle =$
$= N|\alpha = \hat{K}|\alpha$. QED (Lemma 4)

The proofs of §3.7 Lemmas 1 – 2.1 then go through exactly as before (whereby the appeal to the fact that every beaver is strong is replaced by a direct use of Lemma 3.

Now let $E = \langle E_\nu \mid \nu < \omega\delta \rangle$ be s.t. $\langle J^E_\delta, \emptyset \rangle$ is a strong mouse. We wish to show that $K[E]$ exists. To this end we generalize the construction of $\hat{K}$ to obtain $\hat{K}[E]$ in the obvious sense!

Def J_α^E is incompressible wrt $\delta \leq \alpha$ iff there is a weasel W s.t.

(a) $J_\alpha^E = J_\alpha^{E^W}$

(b) Some X is massive in W

(c) If X is massive in W, then

$$\alpha \subset h_W(\delta \cup X).$$

Let $E' = \langle E_\nu \mid \nu < \omega\delta \rangle$ s.t. $\langle J_\delta^E, \emptyset \rangle$ is strong. Repeating the proofs of Lemmas 1, 2:

Lemma 5 For any $\alpha \geq \delta$ there is at most one incompressible J_α^E s.t. $J_\delta^E = J_\delta^{E'}$.

Lemma 6 For every $\alpha \geq \delta$ there is an incompressible J_α^E s.t. $J_\delta^E = J_\delta^{E'}$.

Set $\hat{K}[E'] = J_\infty^{\hat{E}}$ = the union of the incompressible J_α^E with $J_\delta^E = J_\delta^{E'}$. As before!

<u>Lemma 7</u> If $\langle J_\alpha^{\hat{E}}, U \rangle$ is a beaver and $\alpha \geq \delta$, then $U = \hat{E}_\alpha$

<u>Lemma 8</u> $\hat{K}[E'] = K[E']$.

The remaining lemmas of § 3.7 then go through as before.