## §10 The Model Construction

In the following assume that $\theta$ is a strongly inaccessible cardinal and that $V_\theta$ is closed under # (i.e. if $a \subset \gamma < \theta$, then $a^\#$ exists). Following Steel we construct a structure $N = J_\theta^E$ s.t. $N\|\alpha$ is a "weak mouse" for all $\alpha < \theta$ in the following sense:

(*) If $\sigma : Q \to_{\Sigma^*} N\|\alpha$ and $Q$ is a countable transitive structure, then $Q$ is a countably iterable basic premouse.

(Hence by the remark the end of the the appendix to §7, the conclusions of §7, §8 hold for weak mice in $V_\theta$, since $V_\theta$ is closed under #.)

The strategy is to construct $N$ as the "limit" of a sequence $M_\nu$ ($\nu < \theta$) of weak mice. The idea is that if we have $M_\nu = \langle J_\alpha^E, \emptyset \rangle$ and $F$ is an extender s.t. $\langle J_\alpha^E, F \rangle$ is a basic premouse and $F$ satisfies a sufficient "background condition", then $\langle J^E_\alpha, F \rangle$ is the next stage. Otherwise the next stage is $\langle J^E_{\alpha+1}, \emptyset \rangle$. In either case we call the next stage $N_{\nu+1}$ and, after verifying that $N_{\nu+1}$ is a weak mouse, set: $M_{\nu+1} = \mathrm{core}(N_{\nu+1})$. (This "coring down" process is necessary, since we do not know that $N_{\nu+1}$ is sound, but ultimately want each $N \| \alpha$ to be sound. Because of this, the $M_\nu$ do not form a linear hierarchy and we shall have to give some care to defining $M_\lambda$ at limit $\lambda$. Here, too, we shall first define $N_\lambda$ and set $M_\lambda = \mathrm{core}(N_\lambda)$.) We shall also have to verify that the choice of $F$ in $N_{\nu+1} = \langle J^E_\alpha, F \rangle$ is unique. The "background condition" can be varied. If $V_\Theta$ is sufficiently small (e.g. with no cardinal strong up to $\Theta$), it is enough to require that $F$ be $\omega$-complete. In general, however, it seems that stronger background conditions are needed. In Steel's original construction he assumed that $\Theta$ is Woodin and showed that in $L[N]$ some $\delta \leq \Theta$ is Woodin. His background condition was very strong indeed:

(**) There is an extender $F^*$ on $\kappa = \text{crit}(F)$ s.t. $F^*$ is $\lambda+2$-strong in $V$ (i.e. $V_{\lambda+2} \subset V'$ where $\pi : V \to_{F^*} V'$) and $F = (F^* | \lambda) \restriction M_\nu$ and $\lambda = lh(F)$).

However, Steel's extenders are shorter than ours and, as mentioned in the introduction, the requirement of strength up to $\lambda+2$ is less onerous than in our case. It is thus harder for us to admit extenders to the sequence $E$. For this reason, presumably, we were unable to prove that there is a Woodin cardinal in $L[N]$ (although we still think it very likely that it is provable).

In this chapter we carry through the construction of the model $N$

based on the background condition (**), verifying uniqueness and the weak mousehood condition (*). We closely follow Steel's construction. However Steel employed a weaker notion of iterability than we do, requiring the $Q$ in (*) to be only countably <u>normally</u> iterable. He in fact shows:

(***) If $\sigma : Q \longrightarrow_{\Sigma^*} N_\xi$, then $Q$ has a normal countable iteration strategy $S$. Moreover, if $Q'$ is an $S$-iterate of $Q$ with iteration map $\pi$, then there is a map $\sigma' : Q' \longrightarrow_{\Sigma_0} N_\gamma$ for a $\gamma \leq \xi$. If $\pi$ is a total map on $Q$, then $\gamma = \xi$ and $\sigma'\pi = \sigma$.

(Here we of course ignore the fact that Steel's literal statement involves the $n$-iterability of the $n$-core of $Q$ for each $n < \omega$.) We, on the other hand, require <u>good</u> iterability which implies e.g. that $Q'$ itself be countably normally iterable. The condition

$\sigma' : Q' \xrightarrow[\Sigma_0]{} N_\gamma$ is insufficient to conclude this. We resolve the problem by using the pseudo projecta developed in §9.

We prove:

(****) If $\sigma : Q \xrightarrow[\Sigma^*]{} N_\xi$ min $(\vec{\rho})$, then $Q$ has a normal countable iteration strategy $S$. Moreover, if $Q'$ is an $S$-iterate of $Q$ with iteration map $\pi$, then there is $\sigma' : Q' \xrightarrow[\Sigma^*]{} N_\gamma$ min $(\vec{\rho}\,')$, where $\gamma \leq \xi$. If $\pi$ is total, then $\gamma = \xi$, $\sigma'\pi = \sigma$ and $\rho'_i \leq \rho_i$ for $i < \omega$.

This enables us to prove (*). There remains, however, the problem of determining how 'big' $N$ is if $\theta$ is 'big'. Our results to date have been meager. (We can show: If there is no Woodin cardinal in $N$ but $\theta$ is Woodin and $V_\theta^\#$ exists, then $N$ is universal w.r.t. premice in $V_\theta$.) Happily, though, Steel has a weaker alternative to the background condition (**)

stated above - namely the existence of sufficiently large background certificates for F. (We explain this in §11. Essentially a background certificate is a partial extender on $\mathbb{R}(n)$ with the required strongness condition.) He denotes the resulting N as $K^c$. He adapts his proof of (***) to the $K^c$ construction. We believe - but have not checked - that our proof of (****) can be adapted in the same way, thus proving (*). In §11 we assume this and examine the size of $K^c = N$ if $V_\theta$ is large. We first repeat Steel's proof that if $\theta$ is measurable and there is no Woodin cardinal in N, then N satisfies the "cheap covering lemma". We then show that if $\theta$ is Woodin, then some $\delta \leq \theta$ is Woodin in $L[N]$. The fact that we can prove this for $K^c$ but not for the original N seems to reflect the relative ease with which one can add extenders in the $K^c$ construction. (Nonetheless our proof is more convoluted than the original Steel proof based on short extenders.)

We inductively define a sequence $M_\xi, N_\xi$ ($\xi < \bar{\theta} \leq \theta$), at each stage inductively verifying:

(a) $N_\xi$ is a basic premouse and $M_\xi = \mathrm{core}(N_\xi)$. Moreover, if $\bar{N}$ is countable and $\sigma : \bar{N} \xrightarrow[\Sigma^*]{} N_\xi$, then $\bar{N}$ is countably iterable.

(b) $M_\xi \| \mu_\alpha = M_\alpha \| \mu_\alpha$ for $\alpha < \xi$, where:

$$\kappa_\alpha = \kappa_{\alpha,\xi} =_{\mathrm{df}} \min \{ \omega\rho^\omega_{M_\nu} \mid \alpha < \nu \leq \xi \}$$

$$\mu_\alpha = \mu_{\alpha,\xi} = \kappa_\alpha^{+M_\alpha} =_{\mathrm{df}} \begin{cases} \mathrm{ht}(M_\alpha) \text{ if } \mathrm{ht}(M_\alpha) = \kappa_\alpha, \\ \tau \text{ if not, where} \\ \tau \leq \mathrm{ht}(M_\alpha) \text{ is maximal} \\ \text{s.t. } \kappa_\alpha = \text{the largest} \\ \text{cardinal in } J_\tau^{E^{M_\alpha}} \end{cases}$$

<u>Note</u> As shown in the appendices to §7-8, the second half of (a) is sufficient to give all the conclusions of those chapters for $N_\xi$. (This uses the closure of $V_\theta$ under # and the Fact proven at the end of the appendix to §7.)

We define $N_\xi$ by cases as follows;
We assume that $N_\nu$, $M_\nu = \mathrm{core}(N_\nu)$ are defined and that (a), (b) hold for $\nu < \xi$.

<u>Case 1</u> $\xi = 0$. $N_0 =_{\mathrm{df}} \langle J_1^\phi, \phi \rangle$

<u>Case 2</u> $\xi = \varsigma + 1$

<u>Case 2.1</u> $M_\varsigma = \langle J_\beta^E, \phi \rangle$ and there exists an extender $F^*$ on $V$ and an extender $F$ on $M_\varsigma$ s.t. for some $\kappa < \lambda < \beta$:

(i) $\kappa = \mathrm{crit}(F^*) = \mathrm{crit}(F)$

(ii) $\beta = \lambda^{+M_\varsigma}$; $lh(F^*) > lh(F) = \lambda$;

$F(X) = F^*(X) \cap \lambda$ for $X \in \mathcal{P}(\kappa) \cap M_\varsigma$;

$V_{\lambda+2} \subset \mathrm{Ult}(V, F^*)$

(iii) $\langle J_\beta^E, F \rangle$ is a basic pm.

We choose such $F^*$, $F$ and set: $N_\xi = \langle J_\beta^E, F \rangle$. (We shall later see that the choice of $F$ is unique, regardless of $F^*$.)

<u>Case 2.2</u> Case 2.1 fails and $M_\varsigma = \langle J_\beta^E, \phi \rangle$.
Set: $N_\xi = \langle J_{\beta+1}^E, \phi \rangle$.

Case 3 $\xi = \lambda$, Lim$(\lambda)$.

Then (a), (b) hold below $\lambda$. For $\alpha < \lambda$ set: $\tilde{\kappa}_\alpha = \tilde{\kappa}_{\alpha,\lambda} = \min\{\omega\rho^\omega_{M_\rho} \mid \alpha < \rho < \lambda\}$ $\tilde{\mu}_\alpha = \kappa_\alpha^{+M_\alpha}$ (in the same sense as before); $\tilde{\mu} = \sup_{\alpha<\lambda} \tilde{\mu}_\alpha$. Then $\alpha \leq \beta < \lambda \rightarrow \tilde{\mu}_\alpha \leq \tilde{\mu}_\beta$.

By (b) there is $E^\alpha$ s.t. $J^{E^\alpha}_{\tilde{\mu}_\alpha} = J^{E^{M_\rho}}_{\tilde{\mu}_\alpha}$ for all $\rho \in [\alpha, \lambda)$. Thus $J^{E^\alpha}_{\tilde{\mu}_\alpha} \subset J^{E^\beta}_{\tilde{\mu}_\beta}$ and we set: $J^E_{\tilde{\mu}} = \bigcup_{\alpha<\lambda} J^{E^\alpha}_{\tilde{\mu}_\alpha}$; $N_\lambda = \langle J^E_{\tilde{\mu}}, \emptyset \rangle$

$N_\lambda$ is obviously a basic pm.

Thus, at limit $\lambda$, $N_\lambda$ is always defined if $N_\mu$ is defined for $\mu < \lambda$. $N_{\xi+1}$ is defined if $N_\xi$ satisfies (a), (b). $N_\xi$ is a basic pm whenever defined.

It is clear from our definition that $\omega\rho^{\omega}_{M_\xi} \leq \omega\rho^{\omega}_{M_{\xi+1}}$. Moreover if $\xi+1$ satisfies Case 2.1, then $\omega\rho^{\omega}_{M_{\xi+1}} = \omega\rho^{\omega}_{N_{\xi+1}} < ht(M_\xi) = \omega\rho^{\omega}_{M_\xi}$ since $N_{\xi+1} = \langle M_\xi, F \rangle$ is a pm with $F \neq \emptyset$. Now suppose that $\omega\rho^{\omega}_{M_\xi} = \omega\rho^{\omega}_{M_{\xi+1}}$.

Then Case 2.2 holds. Set $\kappa = \omega\rho^{\omega}_{M_\xi}$. Then $\kappa^{+N_{\xi+1}} = ht(N_{\xi+1}) = ht(M_\xi)+1 > \kappa^{+M_\xi}$. But by §8 Lemma 5 we have: $\kappa^{+M_{\xi+1}} = \kappa^{+N_{\xi+1}}$. Thus:

(1) If $\kappa = \omega\rho^{\omega}_{M_\xi} = \omega\rho^{\omega}_{M_{\xi+1}}$, then

$$\kappa^{+M_{\xi+1}} = \kappa^{+N_{\xi+1}} > \kappa^{+M_\xi}.$$

($\kappa^{+M}$ is defined as above).

It follows that:

(2) $ht(M_\lambda)$ is a limit ordinal if $Lim(\lambda)$.

prf. It suffices to show that $ht(N_\lambda)$ is a limit ordinal.

Let $\tilde{\kappa}_\alpha = \tilde{\kappa}_{\alpha\lambda}$, $\tilde{\mu}_\alpha = \tilde{\mu}_{\alpha\lambda}$ be defined as before.

Case 1 $\sup_{\alpha<\lambda} \tilde{\kappa}_\alpha$ is a limit ordinal.

Let $\tilde{\kappa}_\alpha < \tilde{\kappa}_\beta$. Then $\tilde{\mu}_\alpha \overset{\text{by (b)}}{=} \tilde{\kappa}_\alpha^{+M_\alpha} \leq$

$\tilde{\kappa}_\alpha^{+M_\beta} \leq \tilde{\kappa}_\beta < \tilde{\kappa}_\beta + 1 \leq \tilde{\mu}_{\beta+1}$. Hence

$\bigwedge\alpha \bigvee\beta\ \tilde{\mu}_\alpha < \tilde{\mu}_\beta$. QED (Case 1)

Case 2 Case 1 fails. Let $\tilde{\kappa}_\alpha = \kappa$ for $\alpha_0 \leq \alpha < \lambda$. For $\alpha \in [\alpha_0, \lambda)$ pick $\gamma > \alpha$ s.t. $\omega p^{\omega}_{M_\gamma} = \kappa$. Then $\tilde{\mu}_\alpha = \kappa^{+M_\alpha} \leq$

$\leq \kappa^{+M_\gamma} < \kappa^{+M_{\gamma+1}} = \tilde{\mu}_{\gamma+1}$ by (b) and (1). Hence $\bigwedge\alpha \bigvee\gamma\ \tilde{\mu}_\alpha < \tilde{\mu}_\gamma$.

QED (2)

Now suppose $\bar{\theta} = \theta$. We define $\tilde{\kappa}_\alpha = \tilde{\kappa}_{\alpha,\theta}$, $\tilde{\mu}_\alpha = \tilde{\mu}_{\alpha,\theta} = \tilde{\kappa}_\alpha^{+M_\alpha}$ exactly as before and again get: $\bigwedge\alpha \bigvee\beta\ \tilde{\mu}_\alpha < \tilde{\mu}_\beta$. Hence $\sup_{\alpha<\theta} \tilde{\mu}_\alpha = \theta$, since $\theta$ is regular

We then set: $N = N_\theta = \bigcup_{\alpha<\theta} M_\alpha \| \tilde{\mu}_\alpha$.

(It is clear that $\sup_{\alpha<\theta} \tilde{\kappa}_\alpha = \theta$, since otherwise $N$ would have a largest cardinal. Hence $N = \bigcup M_\alpha \| \tilde{\kappa}_\alpha$.)

We now verify (a), (b). We assume that $N_\xi$ is defined and that (a), (b) hold below $\xi$. It suffices to prove (a), since (b) will follow using:

(1) $N_\xi \| \mu = M_\xi \| \mu$ if $\mu = (\rho^\omega_{N_\xi})^{+N_\xi}$

which is a consequence of §8 Lemma. Hence it remains to prove (a). By §9 we need only show that $\bar{N}$. is smoothly countably iterable. Clearly it suffices to produce an iteration strategy for direct smooth countable iterations (i.e. $\nu_i$ is defined everywhere)

Def Let $\mathcal{I} = \langle \langle M_i \rangle, \langle \nu_i \rangle, \dots, T \rangle$ be a direct normal iteration of limit length. A branch b in $\mathcal{I}$ is modest w.r.t. $\mathcal{I}$ iff b is a cofinal well founded branch and, letting $\delta = \sup_i \nu_i$, we have;
$E^{M_b}_\nu = \emptyset$ for $\delta \leq \nu \leq ht(M_b)$.

(Note Let $\lambda + 1 < lh(\mathcal{I})$, Lim($\lambda$). $b = \{i \mid i T \lambda\}$ cannot be modest, since $\nu_\lambda > \delta = \sup_{i<\lambda} \nu_i$.)

We shall prove:

<u>Lemma 1</u> Let $\sigma : Q \to_{\Sigma^*} N_\xi \min(\vec{\rho})$, where $Q$ is countable. Then $Q$ has a countable normal iteration strategy $S$. Moreover if $\mathcal{I} = \langle\langle Q_i\rangle, \langle \nu_i\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ is a countable normal $S$-iteration of length $\theta + 1$, Then:

(i) There is $\sigma' : Q_\theta \to_{\Sigma^*} N_\gamma \min(\vec{\rho}\,')$ for a $\gamma \le \xi$, where:

(ii) If $\pi_{0\theta}$ is not total then $\gamma < \xi$.

(iii) If $\pi_{0\theta}$ is total, then $\gamma = \xi$ and $\rho_i' \le \rho_i$ for $i <$

Moreover $\sigma' \pi_{0\theta} = \sigma$.

As a consequence:

<u>Corollary 2</u> Let $\sigma : Q \to_{\Sigma^*} N_\xi$ (where $Q$ is countable). Then $Q$ is countably smoothly iterable.

Proof.

We first define a strategy $S$. Let $\mathcal{I}$ be a smooth countable iteration of limit length, which resolves into a sequence $\langle \mathcal{I}_i \mid i < \theta\rangle$ of successive normal iterations. If $i+1 < \theta$, Then $\mathcal{I}_i = \langle\langle Q^i_j\rangle, \langle \nu^i_j\rangle, \langle \pi^i_{jh}\rangle, T^i\rangle$ is of length $\theta_i + 1$. $S$ is to give us, if

possible, a cofinal well founded branch in $\mathcal{T}$. If $\lim(\theta)$, this means that setting $\tilde{Q}_i = Q_0^i$ (hence $\tilde{Q}_{i+1} = Q_{\theta_i}^i$), and letting $\langle \tilde{\pi}_{ik} \mid i \leq k < \theta \rangle$ be the natural commutative system of partial maps s.t. $\tilde{\pi}_{i,i+1} = \pi_{0\theta_i}^i$, then $\{i \mid \tilde{\pi}_{i,i+1}$ is finite$\}$ and $\langle \tilde{Q}_i \rangle, \langle \tilde{\pi}_{ik} \rangle$ has a well founded direct limit.
If $\theta = \eta + 1$, this means simply that $\mathcal{T}_\eta$ is of limit length and has a cofinal well founded branch.
We first define $\sigma_i : \tilde{Q}_i \xrightarrow[\Sigma^*]{} N_\xi \min(\vec{\rho})$ for $i < \bar{\theta} \leq \theta$ and a sequence $\langle S_i \mid i < \bar{\theta} \rangle$ of normal iteration strategies as follows: $\sigma_0 = \sigma$. Let $\sigma_0 : Q \xrightarrow[\Sigma^*]{} N_\xi \min($(letting $\vec{\rho} = \min(\langle \rho^i_{N_\xi} \mid i < \omega \rangle)$). This gives $S_0$ for $\tilde{Q}_0 = Q$. Now let $\sigma_i, S_i$ be defined. If $\mathcal{T}_i$ is an $S_i$-iteration, $i+1 < \theta$, $\tilde{Q}_{i+1} = Q_{\theta_i}^i$ is a simple iterate of $\tilde{Q}_i$ in $\mathcal{T}_i$, then pick $\sigma_{i+1} : \tilde{Q}_{i+1} \xrightarrow[\Sigma^*]{} N_\xi \min(\vec{\rho})$ s.t. $\sigma_{i+1} \tilde{\pi}_{i,i+1} = \sigma_i$.

$\delta_{i+1}$ then gives $S_{i+1}$. Otherwise $\delta_{i+1}$ is undefined. Now let $\delta_i$ be defined for $i < \lambda < \theta$, where $Lim(\lambda)$. $\delta_\lambda : \tilde{Q}_\lambda \to N_\xi$ is defined by $\delta_\lambda \tilde{\pi}_{i\lambda} = \delta_i$. Since $\tilde{\pi}_{ij}$ is total for $i \leq j < \lambda$, we have $\tilde{\pi}_{i\lambda} : \tilde{Q}_i \xrightarrow[\Sigma^*]{} \tilde{Q}_\lambda$; hence if $\rho^n_{\tilde{Q}_0} = \rho^\omega_{\tilde{Q}_0}$, then $\rho^n_{\tilde{Q}_i} = \rho^\omega_{\tilde{Q}_i}$ for $i \leq \lambda$.

Since $\delta_i : \tilde{Q}_i \xrightarrow[\Sigma^*]{} N_\xi \bmod(\rho^i)$ it follows that $\rho^i_\omega = \rho^i_n$ for all $i$. But $\rho^i_h \leq \rho^j_h$ for $h = 0, \dots, n$, $i \leq j$. Hence there is $i_0$ s.t. $\vec{\rho}^{\,i} = \vec{\rho}^{\,i_0}$ for $i \geq i_0$. Since $\delta_i : \tilde{Q}_i \xrightarrow[\Sigma^*]{} N_\xi \min(\vec{\rho}^{\,i_0})$ it follows easily that $\delta_\lambda : \tilde{Q}_\lambda \xrightarrow[\Sigma^*]{} N_\xi \min(\vec{\rho}^{\,i_0})$ and we set: $\vec{\rho}^{\,\lambda} = \vec{\rho}^{\,i_0}$.

This defines $\langle \delta_i \mid i < \bar{\theta} \rangle$. We now define $S(\mathcal{Y})$, distinguishing several cases:

Case 1 $\bar{\theta} < \theta$. Then $\bar{\theta} = i+1$.

Case 1.1 $\mathcal{I}_i$ is an $S_i$ iteration. Since $\delta_{i+1}$ is undefined, $\tilde{Q}_{i+1}$ must be a non simple iterate of $\tilde{Q}_i$ in $\mathcal{I}_i$. Hence there is $\sigma : \tilde{Q}_{i+1} \underset{\Sigma^*}{\longrightarrow} N_\gamma \min(\vec{p}\,')$, where $\gamma < \xi$. But $\gamma$ then satisfies (a) by the ind. hyp. Hence there is a smooth countable iteration strategy $\bar{S}$ for $\tilde{Q}_{i+1}$. Let $\bar{\mathcal{I}}$ be the iteration of $\tilde{Q}_{i+1}$ which analyses into $\langle \mathcal{I}_j \mid \bar{\theta} \leq j < \theta \rangle$. Set: $S(\mathcal{I}) \simeq$ the branch determined by $\bar{S}(\bar{\mathcal{I}})$ in the obvious sense.

Case 1.2 Case 1.1 fails. $S(\mathcal{I})$ is undefined

Case 2 $\bar{\theta} = \theta$.

Case 2.1 $\theta = \eta + 1$.
$S(\mathcal{I}) \simeq$ the branch determined by $S_\eta(\mathcal{I}_\eta)$

Case 2.2 $\operatorname{Lim}(\theta)$.
Set: $\tilde{Q}, \langle \pi_i \rangle = \lim\limits_{i \leq j < \theta} (\tilde{Q}_i, \pi_{ij})$. $\tilde{Q}$ is well founded, since $\sigma : \tilde{Q} \to N_\xi$ is definable by $\sigma \pi_i = \sigma_i$. There is

This defines $S$. It is obvious that if $\mathcal{I}$ is an $S$-iteration, then $S(\mathcal{I})$ exists. Hence an $S$-iteration of limit length can be continued. We must still show that an $S$-iteration of successor length can be continued. In this case $\mathcal{I}$ analyzes into $\langle \mathcal{I}_i \mid i < \theta \rangle$ with $\theta = \eta + 1$, $\theta_\eta = \mu + 1$.

In Case 1 we observe that $\bar{\mathcal{I}}$ can be continued, since it is an $\bar{S}$-iteration. In Case 2 we either wish to continue $\mathcal{I}_\eta$ one more step, which is possible since $S_\eta$ is a normal iteration strategy for $\tilde{Q}_\eta$, or we want to apply some $E_\nu^{Q_\mu^\eta}$ to $Q_\mu^\eta$.

This is possible by Lemma 1, since there is $\sigma' : Q_\mu^\eta \xrightarrow[\Sigma^*]{} N_\gamma \min(\rho')$ for some $\gamma \leq \xi$ + hence there is a normal iteration strategy for $Q_\mu^\eta$. QED (Corollary 2)

We now turn to the proof of Lemma 1, which will closely follow Steel's original proof of normal iterability. We make use of the coarse iteration theory developed in the Martin - Steel paper [IT]. We recall some definitions from that paper:

**Def** Let $M = \langle M, \in, \delta \rangle$ be transitive, where $\delta \in On \cap M$. $M$ is a <u>coarse premouse</u> iff $M$ models the following axioms:

(a) nullset, pairing, union, infinity, powerset, choice (in the form $\bigwedge x \bigvee \alpha \; x \sim \alpha$), and full separation

(b) $\Sigma_2$ - collection;

$\bigwedge x \bigvee y \varphi \rightarrow \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v \varphi$

for $\Sigma_2$ formulae $\varphi$

(c) $V_\delta$ - collection;

$\bigwedge x \in V_\delta \bigvee y \varphi \rightarrow \bigvee v \bigwedge x \in V_\delta \bigvee y \in v \varphi$

for arbitrary formulae $\varphi$.

We write: $\delta^M = \delta$ for $M = \langle M, \in, \delta \rangle$.

Def $M, N$ agree thru $\eta$ iff $V_\eta^M = V_\eta^N$.

Def Let $M$ be a coarse pm. $\mathcal{I} = \langle \langle M_i \rangle, \langle E_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ is a coarse iteration of $M$ of length $\theta$ iff $T$ is an iteration tree and:

(a) $M_i$ is a coarse pm ($i < \theta$) and $M_0 = M$

(b) $E_i \in V_{\delta_i}^{M_i}$ ($\delta_i = \delta^{M_i}$) s.t. $M_i \models$ "$E_i$ is an $\omega$-complete extender" for $i+1 < \theta$

(c) If $i+1 < \theta$, $\xi = T(i+1)$, then $M_\xi$ agrees with $M_i$ through $\mathrm{crit}(E_i)+1$ and $\pi_{\xi, i+1} : M_\xi \xrightarrow[E_i]{} M_{i+1}$.

(d) $\pi_{ij} : M_i \to M_j$ ($i \leq_T j < \theta$) is a commutative system of embeddings which is continuous at limits.

(Note $\pi_{\xi, i} : M_\xi \prec M_{i+1}$ for $\xi = T(i+1)$. Hence $\pi_{ij} : M_i \prec M_j$ for $i \leq_T j$.)

<u>Def</u> An iteration $\mathcal{I} = \langle\langle M_i\rangle, \langle E_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ is a 2-<u>plus</u> iteration iff there are $\lambda_i$ $(i+1<\theta)$ s.t. $V^{U_i}_{\lambda_i+2} \subset U_{i+1}$ and $\kappa_j < \lambda_i$ whenever $T(j+1) \leq i < j$.

(<u>Note</u>: If $\lambda_i+2 \leq \lambda_j$ for $i<j$, then $V^{U_i}_{\lambda_i+2} = V^{U_j}_{\lambda_i+2}$ for $i \leq j$.)

Martin + Steel prove:

(MS): Let $\sigma: M \prec \langle V_\gamma, \in, \delta\rangle$ where $M$ is a countable coarse pm. Let $\mathcal{I} = \langle\langle M_i\rangle, \ldots\rangle$ be a countable normal iteration of $M$.

(a) If $lh(\mathcal{I}) = h+1$, then $\mathcal{I}$ can be continued.*) Moreover, there is $\sigma': M_h \prec \langle V_\gamma, \in, \delta\rangle$ s.t. $\sigma'\pi_{0h} = \sigma$.

(b) If $\theta = lh(\mathcal{I})$, $Lim(\theta)$, then $\mathcal{I}$ has a maximal well founded branch.**)

*) "$\mathcal{I}$ can be continued" means: If $E \in V^{U_h}_\delta$ s.t. $crit(E) \leq \lambda_i$ and $V^{U_i}_{\lambda_i+2} \subset U'$, where $\pi: U_i \xrightarrow[E]{} U'$, then $Ult(V_i, E)$ is well founded.)

<u>**</u>/ It does <u>not</u> follow that the branch given by (b) is cofinal in $\theta$, but merely that it is $\neq b_i = \{l \mid l \, T \, i\}$ for all $i < \theta$. It <u>will</u> be cofinal if for all $\lambda < \theta$ we have: $b_\lambda$ is the unique cofinal well founded branch in $\mathcal{T}|\lambda$.

Def Let $M$ be a premouse, $\nu \in M$, $E_\nu^M \neq \emptyset$. $\beta(M,\nu)$ = the maximal $\beta < ht(M)$ s.t. $\nu \le \beta$ and $\omega\rho^\omega_{M\|\beta} < \omega\rho^\omega_{M\|\xi}$ for $\nu \le \xi < \beta$.

(Note $\beta(M,\nu)$ exists and $\omega\rho^\omega_{M\|\beta} < \nu$, since $\omega\rho^1_{M\|\nu} \le \lambda < \nu$, where $\lambda$ = the largest cardinal in $J_\nu^{E^M}$.)

Def Let $\nu \le ht(M)$, $E_\nu^M \neq \emptyset$. Define $\bar\beta_m = \bar\beta_m(M,\nu)$ for $m \le p = p(M,\nu)$ as follows: $\beta_0 = ht(M)$. If $\bar\beta_m$ is defined and $\bar\beta_m > \nu$, set:
$\bar\beta_{m+1} = \beta(M\|\bar\beta_m, \nu)$. Otherwise $\bar\beta_{m+1}$ is undefined.

(Note $\beta_0 = ht(M)$ and $\beta_p = \nu$ )

Def Let $M, \nu$ be as above. Set:
$\beta^+(M,\nu)$ = the maximal $\beta \le ht(M)$ s.t. $\nu \le \beta$ and $\omega\rho^\omega_{M\|\beta} < \omega\rho^\omega_{M\|\xi}$ for $\nu \le \xi < \beta$

(Note $\beta(M,\nu) < \beta^+(M,\nu)$ is possible.)

<u>Lemma 3</u> Let $N_\xi$ be defined + let (a), (b) hold below $\xi$. Then

(i) Let $\nu < ht(N_\xi)$, $E_\nu^{N_\xi} \neq \emptyset$, $\beta = \beta(N_\xi, \nu)$. There is exactly one $\eta < \xi$ s.t. $N_\xi \| \beta = M_\eta$

(ii) Let (a) hold at $\xi$; $\nu \leq ht(M_\xi)$, $E_\nu^{M_\xi} \neq \emptyset$, $\beta = \beta^+(M_\xi, \nu)$. There is exactly one $\eta \leq \xi$ s.t. $M_\xi \| \beta = M_\eta$.

prf. Ind. on $\xi$.

We first prove (i).

<u>Case 1</u> $\xi = 0$ trivial

<u>Case 2</u> $\xi = \zeta + 1$

<u>Case 2.1</u> $N_\xi = \langle J_\alpha^E, F \rangle$ where $M_\zeta = \langle J_\alpha^E, \emptyset \rangle$. Then $\rho^\omega_{M_\zeta} = ht(M_\zeta)$ and $\beta = \beta^+(M_\zeta, \nu)$

<u>Case 2.2</u> $N_\xi = \langle J_{\alpha+1}^E, \emptyset \rangle$, $M_\zeta = \langle J_\alpha^E, E_\alpha$

Then $\beta = \beta^+(M_\zeta, \nu)$.

Case 3 $\xi = \lambda$, $Lim(\lambda)$.

Case 3.1 $\tilde{\kappa}_\alpha \geq \nu$ for some $\alpha < \lambda$.

Then $\tilde{\kappa}_\alpha \geq \nu$ for suff. large $\alpha < \lambda$; pick $\alpha$ s.t. $\tilde{\mu}_\alpha > \beta$. Then $\tilde{\kappa}_\alpha > \beta$, since otherwise $\omega\rho^\omega_{M_\alpha \| \beta} \leq \nu < \tilde{\kappa}_\alpha \leq \beta$ and $\tilde{\kappa}_\alpha$ is not a cardinal in $M_\alpha$. But for $\tilde{\mu}_\beta \leq \beta' \leq ht(M_\alpha)$, we have $\rho^\omega_{M_\alpha \| \beta'} \geq \tilde{\kappa}_\alpha > \beta$, since $\tilde{\kappa}_\alpha$ is a cardinal in $M_\alpha$ and $\omega\rho^\omega_{M_\alpha} \geq \tilde{\kappa}$ by definition. Hence $\beta = \beta(M_\alpha \| \tilde{\mu}_\alpha, \nu) = \beta^+(M_\alpha, \nu)$. QED (3.1)

Case 3.2 Case 3.1 fails.

Then $\tilde{\kappa}_\alpha < \nu$ for all $\alpha$. Hence $\tilde{\kappa}_\alpha = \kappa =$ = the largest cardinal in $N_\xi$ for sufficiently large $\alpha$, since $\tilde{\kappa}_\alpha < \tilde{\kappa}_\beta \to \tilde{\mu}_\alpha \leq \tilde{\mu}_\beta$ + we would otherwise have:

$\sup \tilde{\kappa}_\alpha = \sup \tilde{\mu}_\alpha > \nu$.

Pick $\alpha$ with $\tilde{\kappa}_\alpha = \kappa$, $\tilde{\mu}_\alpha > \beta$. Clearly $\omega\rho^\omega_{N_\xi\|\beta} = \omega\rho^\omega_{M_\alpha\|\beta} = \kappa$. But $\omega\rho^\omega_{M_\alpha\|\beta'} \geq \kappa$ for $\tilde{\mu}_\alpha \leq \beta' \leq ht(M_\alpha)$, since $\kappa = \tilde{\kappa}_\alpha$ is a cardinal in $M_\alpha$ and $\omega\rho^\omega_{M_\alpha} \geq \tilde{\kappa}_\alpha$. Hence $\beta = \beta(N_\xi\|\tilde{\mu}_\alpha, \nu) = \beta^+(M_\alpha, \nu)$,

QED (Case 3)

This proves (i). To prove (ii), let $\beta = \beta^+(M_\xi, \nu)$. If $\beta = ht(M_\xi)$, there is nothing to prove. Let $\beta < ht(M_\xi)$. Let $\sigma : M_\xi \to N_\xi$ be the core map. Then $p^\omega_{M_\xi} = \rho^\omega_{M_\xi} \geq \beta$ and $\sigma \upharpoonright \omega\rho^\omega_{M_\xi} = id$. Hence $\beta = \beta(N_\xi, \nu)$.

QED (Lemma 3)

<u>Def</u> Let $\nu \leq On \cap N_\xi$, $E_\nu^{N_\xi} \neq \emptyset$, where, as before, $N_\xi$ is defined and (a), (b) hold below $\xi$. The <u>trace</u> of $\nu$ in $N_\xi$ is defined to be:

$S(\nu,\xi) = \langle\langle \eta_1, \beta_1, \sigma_1 \rangle, \ldots, \langle \eta_{\tilde{p}}, \beta_{\tilde{p}}, \sigma_{\tilde{p}} \rangle\rangle$

(with $\tilde{p} = \tilde{p}(\xi,\nu) < \omega$ and $S(\nu,\xi) = \emptyset$ if $\tilde{p} = 0$). $S(\nu,\xi)$ is defined by induction on $\xi$ as follows:

<u>Case 1</u> $\nu = ht(N_\xi)$. $S(\nu,\xi) = \emptyset$

<u>Case 2</u> $\nu < ht(N_\xi)$. $\langle \eta_1, \beta_1, \sigma_1 \rangle$ is defined by: $\beta_1 = \beta(N_\xi, \nu)$,

$\eta_1 =$ that $\eta < \xi$ s.t. $N_\xi \| \beta_1 = M_\eta$,

$\sigma_1 =$ the core map $\sigma : M_{\eta_1} \to N_{\eta_1}$.

$S(\nu,\xi) =_{Df} \langle \eta_1, \beta_1, \sigma_1 \rangle ^\frown S(\sigma_1(\nu), \eta_1)$

[Here $\sigma_1(\nu) =_{Df} ht(N_{\eta_1})$ if $\nu = ht(M_{\eta_1})$]

<u>Def</u> We write: $\eta_h[\nu,\xi]$ for $\eta_h$; similarly for $\beta_h[\nu,\xi]$, $\sigma_h[\nu,\xi]$. We also set:

$\eta_0 = \eta_0[\nu,\xi] = \xi$; $\beta_0 = ht(N_\xi)$; $\sigma_0 = id \restriction N_\xi$.

(Note $S(\nu, \xi)$ traces the "history" of $E_\nu^{N_\xi}$ back to a top extender $E_{\sigma_{\tilde{p}}(\nu)}^{N_{\eta_{\tilde{p}}}}$ of $N_{\eta_{\tilde{p}}}$, which was introduced at the $\eta_{\tilde{p}}$-th stage. For this reason Steel calls $S(\nu, \xi)$ the "resurrection sequence".)

We then set: $\sigma^{(m)} = \sigma^{(m)}[\nu, \xi] =_{\mathrm{Df}}$ $=_{\mathrm{Df}} \sigma_m \circ \ldots \circ \sigma_0$ $(m \leq \tilde{p})$. It follows that:

(1) $S(\nu, \xi) = \langle\langle \eta_1, \beta_1, \sigma_1 \rangle, \ldots, \langle \eta_m, \beta_m, \sigma_m \rangle\rangle$
$\frown S(\sigma^{(m)}(\nu), \eta_m)$

for $m \leq \tilde{p}$.

Hence:

(2) $\langle \eta_{m+h}, \beta_{m+h}, \sigma_{m+h} \rangle =$
$= \langle \eta_h, \beta_h, \sigma_h \rangle [\sigma^{(m)}(\nu), \eta_m]$.

An easy induction shows:

(3) $\beta_{h+1} \simeq \sigma^{(h)}(\bar{\beta}_{h+1})$; hence:

(4) $\tilde{p} = p(N_\xi, \nu)$.

Obviously:

(5) $\bar{\beta}_p = \nu$, $\beta_p = \sigma^{(p)}(\nu) = ht(N_{\eta_p})$.

Def $\sigma^* = \sigma^*[\xi, \nu] =_{\text{Df}} \sigma^{(p)}$; $\eta^* \simeq \eta^*[\nu, \xi] = \eta_p$ where $p = p(\xi, \nu)$. Then:

(6) $\sigma^* : N_\xi \| \nu \longrightarrow_{\Sigma^*} N_{\eta^*}$.

We note the following facts:

Fact 1 Let $\lambda < \nu$ be a cardinal in $N_\xi$, $\sigma^{(m)} = \sigma^{(m)}[\nu, \xi]$. Then $\sigma^{(m)} \upharpoonright \lambda = id$

proof. (Induction on $m$).

$m = 0$ is trivial. Let $m = n+1$. Then $\sigma_m : N_{\eta_m} \| \beta_m \longrightarrow N_{\eta_m}$ is the core map.

But $\tilde{\lambda} = \sigma^{(n)}(\lambda)$ is a cardinal in $N_{\eta_n}$ and $\beta_m \in N_{\eta_\beta}$. Hence

$\omega\rho^\omega_{N_{\eta_m} \| \beta_m} \geq \tilde{\lambda}$ and $\sigma_m \upharpoonright \tilde{\lambda} = id$. But

$\sigma^{(m)} = \sigma_m \sigma^{(n)}$. QED (Fact 1).

Fact 2 Let $\lambda < \nu$ be a successor cardinal in $N_\xi$, $\sigma^{(m)} = \sigma^{(m)}[\nu, \xi]$

Then $\sigma^{(m)} \upharpoonright (\lambda + 1) = id$.

(The proof of Fact 2 is as before, observing that $\sigma_m : N_{\eta_m} \| \beta_m \longrightarrow N_{\eta_m}$ is a core map and $\tilde{\lambda}$ is a successor cardinal, where $\omega\rho^\omega_{N_{\eta_m}} = \omega\rho^\omega_{N_{\eta_m}\|\beta_m} \geq \tilde{\lambda}$.

Hence $\sigma_m(\tilde{\lambda}) = \tilde{\lambda}$. )

* * * *

Having developed this machinery, we turn to the proof of Lemma 1 (which, as we have seen, proves the properties (a), (b) for $N_\xi$). From now on let:

(7) $\delta : Q \xrightarrow[\Sigma^*]{} N_\xi \; \min(\rho^\omega)$, where $Q$ is countable.

Let $U = \langle V_\nu, \in, \lambda \rangle$ be a coarse premouse (in the Martin–Steel sense), where $\nu < \theta$ and $\langle N_\eta \mid \eta \leq \xi \rangle \in V_\lambda$. We must produce a countable normal iteration strategy for $Q$. From now on let $\mathcal{J} = \langle \langle Q_i \rangle, \langle \nu_i \rangle, \langle \eta_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a countable normal iteration of $Q$ of length $\Gamma$.

We first define a coarse structural iteration $\mathcal{I}^* = \langle\langle U_i\rangle, \langle F_i^*\rangle, \langle \tilde{\pi}_{ij}\rangle, T'\rangle$ of $U$ of length $\bar{\Gamma} \leq \Gamma$, where $T' = T \cap \bar{\Gamma}^2$. Simultaneously we define maps $\delta_i$ s.t.

(a) $\delta_i : Q_i \underset{\Sigma^*}{\longrightarrow} \hat{Q}_i \ \min(\vec{\rho}^{\,i})$ for $i < \bar{\Gamma}$,
where $\hat{Q}_i = \tilde{\pi}_{0i}(\vec{N})_{\gamma_i}$ for a $\gamma_i \leq \tilde{\pi}_{0i}(\xi)$.

(b) $\gamma_i \leq \tilde{\pi}_{ji}(\gamma_j)$ for $j \leq_T i$. Moreover, $\gamma_i = \tilde{\pi}_{ji}(\gamma_j)$ iff $\pi_{ij}$ is total, in which case $\tilde{\pi}_{ji}{}''\rho^j_h \subset \rho^i_h \leq \tilde{\pi}_{ji}(\rho^j_h)$ for $h < \omega$.

(c) Let $j < i$. Then $\delta_i \restriction \lambda_j = \sigma_j^* \delta_j \restriction \lambda_j$,
where $\sigma_j^* = \sigma^*[\delta_j(\gamma_j), \gamma_j]^{U_i}$.

<u>Note</u> By (c) we have:

(d) Let $k < j < i$. Then $\delta_j \restriction \lambda_k = \delta_i \restriction \lambda_k$.

proof. $\delta_j(\lambda_k)$ is a cardinal in $\hat{Q}_j$.
Hence $\sigma_j^* \restriction \delta_j(\lambda_k) = \mathrm{id}$. QED

<u>Note</u> Since $\tilde{\pi}_{ji}$ is elementary, we have:

(e) Let $\gamma_i = \tilde{\pi}_{ji}(\gamma_j)$ and $\tilde{\pi}_{ji}(\rho^j_h) = \rho^i_h$ for all $h < \omega$. Then:

$$\tilde{\pi}_{ji} \restriction \hat{Q}_j : \hat{Q}_j \underset{\Sigma^*}{\longrightarrow} \hat{Q}_i \ \mathrm{mod}(\vec{\rho}^{\,j}, \vec{\rho}^{\,i}).$$

In addition to (a)–(c) we shall have:

(f) Set $\lambda_i^* = \sigma_i^* \delta_i(\lambda_i)$, $\kappa_i^* = \sigma_i^* \delta_i(\kappa_i)$. Then $\kappa_h^* = \mathrm{crit}(F_h^*)$ and

$$V^{U_h}_{\lambda_h^*+2} \subset U_i \quad \text{for } i = h+1.$$

<u>Note</u> Falls $i \leq j$ and $T(j+1) \leq i$, then $\kappa_j < \lambda_i$, hence $\kappa_j^* < \lambda_i^*$. Hence $\mathcal{I}$ is a 2-plus iteration and (MS) holds. By (c) it follows easily that $h < i \longrightarrow \lambda_h^* + 2 < \lambda_i^*$. Hence $V^{U_h}_{\lambda_h^*+2} = V^{U_i}_{\lambda_h^*+2}$ for all $h < i$.

We define $\delta_\ell$, $\tilde{Q}_\ell$, $U_\ell$, $\langle \tilde{\pi}_{i\ell} \mid i T \ell \rangle$ by induction on $i$ as follows. Simultaneously we verify (a), (b), (c).

<u>Case 1</u> $\ell = 0$. $\tilde{Q}_0 = N_\xi$, $\delta_0 = \delta$, $U_0 = U$.

<u>Case 2</u> $\ell = \lambda$, $\lim(\lambda)$.

Then $\mathcal{J}^* | \lambda$ is given. Set $b = \{i \mid i T \lambda\}$. Then $b$ is a cofinal branch in $\mathcal{J}^* | \lambda$. If $U_b$ is ~~not~~ well founded, then $U_\lambda$ is undefined. Otherwise set:

$U_\lambda = U_b$, $\tilde{\pi}_{i\lambda} = \tilde{\pi}_{ib}$, where:

$U_b, \langle \tilde{\pi}_{ib} \mid i < \lambda \rangle$ = the transitive direct limit of $\langle U_i \rangle, \langle \tilde{\pi}_{ij} \rangle$.

Since $\tilde{\pi}_{i\lambda}(\gamma_i) \leq \tilde{\pi}_{j\lambda}(\gamma_j)$ for $j \leq i$ in $b$, there must be $i_0 \in b$, $\gamma \leq \tilde{\pi}_{0\lambda}(\xi)$ s.t. $\tilde{\pi}_{i\lambda}(\gamma_i) = \gamma$ for $i \geq i_0$ in $b$. Set:

$\gamma_\lambda = \gamma$. Then $\pi_{ji}$ is total for $i_0 \leq_T j \leq_T$ [illegible]

Hence $\pi_{i_0\lambda}$ is total and we can define $\delta_\lambda : Q_\lambda \to \tilde{Q}_\lambda$ by:

$\delta_\lambda \pi_{i\lambda} = \tilde{\pi}_{i\lambda} \delta_i$ for $i_0 \leq i < \lambda$.

(where, of course, $\tilde{Q}_\lambda = \tilde{\pi}_{0\lambda}(\vec{N})_\gamma =$
$= \tilde{\pi}_{i\lambda}(\tilde{Q}_i)$ for $i_0 \le i < \lambda$ in $T$.)

We define $\vec{\rho}^{\,\lambda}$ as follows. Let $h < \omega$. Then $\tilde{\pi}_{ji}(\rho_h^j) \ge \rho_h^i$ for $i_0 \le j \le i < \lambda$ in $T$. Hence $\tilde{\pi}_{ji}(\rho_h^j) = \rho_h^i$ for $j \ge \beta_h$, for a $\beta_h \le_T \lambda$. Set: $\rho_h^\lambda = \tilde{\pi}_{\beta_h \lambda}(\rho_h^{\beta_h})$.

(as in the limit case of §9 Lemma 5.)

It follows easily that $\vec{\rho}^{\,\lambda}$ is good for $\tilde{Q}_\lambda$ and $\sigma_\lambda : Q_\lambda \xrightarrow{\Sigma^*} \tilde{Q}_\lambda \bmod (\vec{\rho}$. The other verifications are straightforward.

<u>Case 3</u> $l = i+1$. Let $k = T(l)$.

~~<u>Case 3.1</u> $\eta_i^{\mathcal{J}} = ht(Q_k)$.~~

~~Let $F = E^{Q}_{\nu_i}$, $\tilde{F} = E^{\tilde{Q}}_{\tilde{\nu}_i}$ where $\tilde{\nu} = \sigma_i(\nu_i)$. Let $\tilde{F}$ is an extender on $\tilde{\kappa} = \sigma_i(\kappa_i)$. Then~~

~~$\eta^* = \eta^*[\tilde{\nu}, \gamma_i]^{U_i}$.~~

~~(8) $\sigma_i^* : \tilde{Q}_i \| \tilde{\nu} \xrightarrow{\Sigma^*} \tilde{N}_{\eta^*}$ in $U_i$~~

~~where $\langle \tilde{N}_\xi \mid \xi \le \tilde{\pi}_0(\xi) \rangle = \tilde{\pi}_{0i}(\vec{N})$.~~

Case 3 $l = i+1$.

Set: $\bar\eta = \eta_i^{\mathcal{I}}$. Let $k = T(l)$. Set: $Q = Q_k \| \bar\eta$

There is then an $m \le p = p(Q_h, \nu)$ s.t.

(1) $\bar\eta = \bar\beta_m(Q_k, \nu_k)$

prf. $wp^{\omega}_{Q_h \| \bar\eta} \le \varkappa_i$, where $\nu_k \le \bar\eta \le ht(Q_k)$

and $wp^{\omega}_{Q_i \| \xi} \ge \tau_i$ for $\xi < \bar\eta$. QED(1)

Set: $\tilde\varkappa_i, \tilde\tau_i, \tilde\nu_i = \delta(\varkappa_i, \tau_i, \nu_i)$.

Set: $\tilde\varkappa, \tilde\tau = \delta_k(\varkappa_i, \tau_i)$; $\tilde\nu_k = \delta_k(\nu_k)$.

Set: $\tilde\eta = \delta_h(\bar\eta)$. Then

(2) $\tilde\eta = \bar\beta_m(\tilde Q_k, \tilde\nu_k)$.

Set: $\sigma_i^* = \sigma^*[\gamma_i, \tilde\nu_i]^{U}$ and

$\sigma_k^* = \sigma^*[\gamma_h, \tilde\nu_k]^{U_k}$. Set:

$\sigma_k^{(m)} = \sigma^{(m)}[\gamma_h, \tilde\nu_h]^{U_k}$. Then

(3) $\sigma_k^{(m)} : \tilde Q_k \| \tilde\eta \xrightarrow[\Sigma^*]{} Q^*$, where $Q^* =$

$= \pi_{0,k}(N_{\gamma^*})$ for a $\gamma^* \le \gamma_k$,

where $\gamma^* = \eta_m[\gamma_k, \tilde\nu_k]^{U_k}$.

Set: $\sigma'_k = \sigma^*[Q^*, \nu^*]^{U_k}$. Then

(4) $\sigma^*_k = \sigma'_k \sigma^{(m)}_k$.

Since $\tau^* < \nu^*$ is a successor cardinal in $Q^*$, we have:

(5) $\sigma'_k \upharpoonright (\tau^* + 1) = \mathrm{id}$.

(6) $\sigma^{(m)}_k \delta_k \upharpoonright (\tau_i + 1) = \sigma^*_k \delta_k \upharpoonright (\tau_i + 1) = \sigma^*_i \delta_i \upharpoonright (\tau_i +$

proof. The first equation follows by (5). The second is trivial if $k = i$. Now let $k < i$. Then $\tilde{\tau}_i < \tilde{\nu}_i$ is a successor cardinal in $\tilde{Q}_i$, since $\tilde{\tau}_i < \delta_i(\lambda_k)$, where $\delta_i(\lambda_k)$ is a limit cardinal in $\tilde{Q}_i$. Hence $\sigma^*_i \upharpoonright \tilde{\tau}_i + 1 = \mathrm{id}$ and $\sigma^*_i \delta_i \upharpoonright (\tau_i + 1) = \delta_i \upharpoonright (\tau_i + 1) = \sigma^*_k \delta_k \upharpoonright (\tau_i + 1)$. QED (6).

Now let $\sigma^*_i : Q_i \| \tilde{\nu}_i \xrightarrow[\Sigma^*]{} Q^*_i$, where $Q^*_i = \tilde{\pi}_{0i}(\vec{N})_{\gamma^*_i}$, $\gamma^*_i = \eta^*[\gamma_i, \tilde{\nu}_i]$.

Set: $\kappa^*_i, \tau^*_i = \sigma^*_i(\tilde{\kappa}_i, \tilde{\tau}_i)$.

(7) $J^{E^{Q_i^*}}_{\tau_i^*} = J^{E^{Q^*}}_{\tau^*}$ (hence $\tau_i^* = \tau^*$, $\kappa_i^* = \kappa^*$, and $\mathcal{P}(\kappa^*) \cap Q_i^* = \mathcal{P}(\kappa^*) \cap Q^*$).

proof. By (6):

$$J^{E^{Q_i^*}}_{\tau_i^*} = \sigma_i^* \delta_i (J^{E^{Q_i}}_{\tau_i}) = \sigma_k^{(m)} \delta_k (J^{E^{Q_k}}_{\tau_k}) = J^{E^{Q}}_{\tau^*}$$

QED (7)

Now let $F'$ be the top extender in $Q_i^* = \tilde{\pi}_{0i}(\vec{N})_{\gamma_i^*}$. Then $Q_i^*$ is obtained by Case 2.1 in the def. of $\vec{N}$ from $\tilde{\pi}_{0i}(\vec{N})_{\gamma_i^*-1}$. Let $F'$ be derived from $F^*$ as in that case. $F^*$ is an extender on $\kappa$ in $U_i$ and hence in $U_k$, since $V^{U_k}_{\lambda_k^*} = V^{U_k}_{\lambda_k^*}$ and $\kappa^* < \lambda_k^*$. $U_{i+1}$, $\tilde{Q}_{i+1}$ will only be defined if:

(*) $\mathrm{Ult}(U_k, F^*)$ is well founded.

Assume (*). We define:

$$\tilde{\pi}_{k,i+1}: U_k \longrightarrow_{F^*} U_{i+1}\ ;\ \tilde{Q}_{i+1} = \tilde{\pi}_{0,i+1}(Q^*).$$

Then:

(8) $\langle \sigma_k^{(m)}\delta_k , \sigma_i^*\delta_i \upharpoonright \lambda_i \rangle : \langle \bar{Q}, F \rangle \longrightarrow \langle Q^*, F' \rangle$.

prf. Let $\alpha_1, ..., \alpha_m < \lambda_i$, $X \in \mathbb{P}(\kappa_i) \cap \bar{Q}$,

$\vec{\alpha} \in F(X) \longleftrightarrow \delta_i(\vec{\alpha}) \in \tilde{F}(\delta_i(X))$ $\quad (F = E^{\tilde{Q}_i}_{\tilde{\nu}_i})$

$\longleftrightarrow \sigma_i^*\delta_i(\vec{\alpha}) \in F'(\sigma_i^*\delta_i(X))$,

where $\sigma_i^*\delta_i(X) = \sigma_k^{(m)}\delta_k(X)$ by (6). QED

Now set;

$$\vec{\rho}^* = \begin{cases} \vec{\rho}^k \text{ if } \bar{\eta}_i = ht(Q_k); \\ \min(Q^*, \sigma_k^{(m)}\delta_k, \langle \rho^n_{Q^*} \mid n < \omega \rangle) \\ \text{if not.} \end{cases}$$

It is obvious that we can replace $Q^*$ by $Q^* | \rho_0^*$ in (8). $\delta_{i+1}$, $\vec{\rho}^{i+1}$ will remain undefined unless:

(**) $\langle \sigma_k^{(m)}\delta_k , \sigma_i^*\delta_i \upharpoonright \lambda_i \rangle : \langle \bar{Q}, F \rangle \xrightarrow{**} \langle Q^* | \rho_0^*, F' \rangle$

Assume (**). §9 Lemma 4 then gives us $\delta_l$, $\vec{\rho}^l$ ($l = i+1$) s.t.

(9) $\delta_l : Q_l \underset{\Sigma^*}{\longrightarrow} \tilde{Q}_l \min(\vec{\rho}^l)$, where $\delta_l$ is defined by: $\delta_l(\pi_{kl}(f)(\alpha)) =$
$= \tilde{\pi}_{kl}\delta_k(f)(\sigma_i^*\delta_i(\alpha))$ for $\alpha < \lambda$, $f \in \Gamma^*(\kappa_i, \bar{Q})$, (where $\delta_k(f)$ has the same functionally absolute def. mod $(\vec{\rho}^*)$).

Also:

(10) $\tilde{\pi}_{kl}{}''\rho^*_n \subset \rho^l_n \leq \tilde{\pi}_{kl}(\rho^*_n)$ $(n < \omega)$.

§9 Lemmas 4.5 – 4.7 also apply. In particular:

(11) If $\tilde{\pi}_{kl}(\rho^*_n) = \rho^l_n$ for $n < \omega$,

then $\tilde{\pi}_{kl} : Q^* \longrightarrow_{\Sigma^*} \tilde{Q}_l \bmod(\vec{\rho}^*, \vec{\rho}^l)$.

(12) Let $\omega\rho^1_{\bar{Q}} \leq \kappa_i$. Then $\rho^l_1 = \sup \tilde{\pi}_{kl}{}''\rho^*_1$

and whenever $A \subset \tau_i$ is $\Sigma_1(Q_l)$ in $p$ and $\tilde{A} \subset \tau^*$ is $\Sigma_1(\tilde{Q}_l, \vec{f}^l)$ in $\tilde{p} = \delta_l(p)$ by the same definition, then $A$ is $\Sigma_1(\bar{Q})$ in some $q$ and $\tilde{A}$ is $\Sigma_1(Q^*, \vec{\rho}^*)$ in $\tilde{q} = \sigma^{(m)}_k \delta_k(q)$ by the same definition.

. . . .

This completes the construction in Case 3. We now verify (a), (b), (c), (f) at $l$. (a), (b), (f) are immediate. We verify (c). For $\alpha < \lambda_i$ we have:

$\delta_\ell(\alpha) = \delta_\ell(\pi_{h\ell}(id)(\alpha)) = \tilde{\pi}_{h\ell}\delta_h(id)(\sigma_i^*\delta_i(\alpha)) =$
$= \sigma_i^*\delta_i(\alpha)$. Hence $\delta_\ell \upharpoonright \lambda_i = \sigma_i^*\delta_i \upharpoonright \lambda_i$. Now let $h < i$. Then $\delta_\ell \upharpoonright \lambda_h = \sigma_i^*\delta_i \upharpoonright \lambda_h$
$= \delta_i \upharpoonright \lambda_h = \sigma_h^*\delta_h \upharpoonright \lambda_h$ since $\delta_i(\lambda_h) < \tilde{\ddot{\nu}}_h$ is a cardinal in $\tilde{Q}_h$. QED (Case 3)

This completes the construction of $\mathcal{Y}^*$, $\langle \delta_i \mid i < \bar{\Gamma} \rangle$, $\langle \vec{\rho}^{\,i} \mid i < \bar{\Gamma} \rangle$.

There are three conditions under which $\delta$ can be undefined;

(A) Lim($\ell$) and $\{j \mid j T \ell\}$ is not a well founded branch in $T^* | i$

(B) $\ell = i+1$ and (*) fails

(C) $\ell = i+1$ and (**) fails.

(A), (B) are failures of well foundedness. We show that (C) cannot occur:

<u>Lemma 4</u> Let $\delta_i$ be defined. Let $k = T(i+1)$ and let $\sigma_k^{(m)}$, $\sigma_i^*$ be as in Case 3. Then

(a) $\langle \sigma_k^{(m)} \delta_k, \sigma_i^* \delta_i \upharpoonright \lambda_i \rangle : \langle \bar{Q}, F \rangle \xrightarrow{**} \langle Q^* | \rho_0^*, F' \rangle$

(b) $\langle$ " " $\rangle : \langle \bar{Q}, F \rangle \xrightarrow{*} \langle Q^*, F' \rangle$

The proof will closely follow that of §9 Lemma 5.1. The case $\nu_i < ht(Q_i)$ is again trivial, so we may assume that $F = E_{\nu_i}^{Q_i}$ is the top extender. The main auxiliary lemmas are again proven by induction on the <u>possibility</u> of using the top extender. We define:

<u>Def</u> Let $l < lh(\mathcal{J}^*)$ s.t. $E_{ht}^{Q_l} \neq \emptyset$. Set:

$\kappa' = \kappa'_l = \text{crit}(E_{ht}^{Q_l})$, $\tau' = \tau'_l = \kappa'^{+Q_l}$.

$\mu = \mu_l$ = the least $\mu$ s.t. $\mu = i$ or $\kappa' < \lambda_\mu$.

$\beta = \beta_l$ = the maximal $\beta \leq ht(Q_\mu)$ s.t. $\tau'$ is a cardinal in $Q_\mu \| \beta$.

Exactly as in Case 3: $\beta = \bar{\beta}_n(Q_\mu, \nu_\mu)$ for an $n \leq p' = p(Q_\mu, \nu_\mu)$. Let $\delta_\mu$ be defined and set: $\tilde{\beta} = \tilde{\beta}_l = \delta_\mu(\beta)$.

Then $\tilde{\beta} = \bar{\beta}_n(\bar{Q}_\mu, \tilde{\nu}_\mu)$. Let $\sigma^{(n)} = \sigma_\mu^{(n)} = \sigma^{(n)}[\gamma_\mu, \tilde{\nu}_\mu]^{U_\mu}$. Let

$\sigma^{(n)} : \tilde{Q}_\mu \| \tilde{\beta} \longrightarrow_{\Sigma^*} \hat{Q}$, where

$\tilde{Q} = \tilde{\pi}_{0\mu}(\vec{N})_{\gamma'}$, $\gamma' = \eta_m[\tilde{Q}_\mu, \tilde{\nu}_\mu]^{U_\mu}$. Set:
$\tilde{\kappa}', \tilde{\tau}' = \sigma_\mu^{(m)} \delta_\mu(\kappa', \tau')$, $\tilde{\nu} = \sigma^{(m)}(\tilde{\nu}_\mu)$,
$\sigma'' = \sigma^*[\gamma', \tilde{\nu}]^{U_\mu}$. Then just as in
Case 3: $\sigma_\mu^* = \sigma'' \sigma^{(m)}$; $\sigma'' \restriction (\tilde{\tau}'+1) = id$.
For $\alpha = ht(\tilde{Q}_\ell)$ we of course have:
$\sigma^*[\gamma_\ell, \alpha]^{U_\ell} = id$. Hence as in Case 3
we get: $\delta_\ell \restriction (\tau'+1) = \sigma_\mu^{(m)} \delta_\mu \restriction (\tau'+1)$;
$\tilde{\tau}' = \tilde{\tau}_\ell$, $\tilde{\kappa}' = \tilde{\kappa}_\ell$, $J^{E^{\tilde{Q}}}_{\tilde{\tau}'} = J^{E^{\tilde{Q}_\ell}}_{\tilde{\tau}_\ell}$.
Set: $Q' = Q_\ell \| \beta$. Define:

$$\vec{p}' = \begin{cases} \vec{p}^{\,\ell} & \text{if } Q' = Q_\ell \text{; otherwise} \\ \min(\tilde{Q}, \sigma_\mu^{(m)} \delta_\mu, \langle p_m^{\tilde{Q}} \mid m < \omega \rangle . \end{cases}$$

<u>Lemma 4.1</u> Let $\mu_\ell, \delta_\ell$ be defined s.t. (a), (b) hold below $\ell$. Then
(+) Let $A \subset \tau'$ be $\Delta_1(Q_\ell)$ in $p$ and $\tilde{A} \subset \tilde{\tau}'$ be $\Delta_1(\tilde{Q}_\ell)$ in $p' = \delta_\ell(p)$ by the same definition. Then $A$ is $\Delta_1(Q')$ in some $q$ and $\tilde{A}$ is $\Delta_1(\tilde{Q})$ in $q' = \sigma_\mu^{(m)} \delta_\mu(q)$ by the same definition.

(We closely imitate §9 Lemma 5.1.1)

Proof of Lemma 4.1.

Suppose not. Let $l$ be the least counterexample. Then $\mu < l = i+1$ for some $i$. Let $k = T(l)$. Let $\eta = \eta_i^{\nu}$, $\overline{Q} = Q_k \| \overline{\eta}$. As in Case 3,

$\eta = \overline{\beta}_m (Q_k, \nu_k)$ where $m \leq p = p(Q_k, \nu_k)$.

Set: $\tilde{\eta} = \sigma_k(\eta) = \overline{\beta}_m(\tilde{Q}_k, \tilde{\nu}_k)$ and let $\sigma_k^{(m)} : \tilde{Q}_k \| \tilde{\eta} \xrightarrow[\Sigma^*]{} Q^*$, where

$Q^* = \tilde{\pi}_{0k}(\vec{N})_{\gamma^*}$, $\gamma^* = \eta_m[\tilde{Q}_k, \tilde{\nu}_k]^{u_k}$.

As before, set: $\tilde{\kappa}, \tilde{\tau} = \sigma_k(\kappa_i, \tau_i)$, $\kappa^*, \tau^* = \sigma_k^{(m)}(\tilde{\kappa}, \tilde{\tau})$. We use all the facts established in Case 3.

(1) $\kappa' < \kappa_i$ (hence $\pi_{kl} \upharpoonright \tau' + \overline{Q} = \mathrm{id}$)

proof. Like (1) in §9 Lemma 5.1.1.

(2) $\mu \leq k$ since $\kappa' < \kappa_k < \lambda_k$.

(3) $\omega\rho^1_{Q_l} \leq \tau'$.

The proof is an almost literal repetition of (3) in §9 Lemma 5.1.1.

We let $A \subset \tau'$ be $\Delta_1(Q_\ell)$ in $p$ and $\tilde{A}$ be $\Delta_1(\tilde{Q}_\ell)$ in $\tilde{p} = \sigma_\ell(p)$ by the same def. We show that $\tilde{A}$ is $\Delta_1(\tilde{Q}_\ell | \tilde{p}_0^\ell)$ by the same def. + then conclude as before that $A \in \bar{Q}$ and $\tilde{A} = \sigma_k^{(m)} \delta_k(A)$, thus verifying (+). Contr! QED (3)

(4) $\rho^1_{\bar{Q}} \leq \tau'$ (since $\pi_{k\ell} : Q' \xrightarrow[\Sigma^*]{} Q_\ell$)

(5) Let $A \subset u_i$ be $\Sigma_1(Q_\ell)$ in $p$ and $\tilde{A} \subset \tilde{u}_i$ be $\Sigma_1(\tilde{Q}_\ell)$ in $\tilde{p} = \delta_\ell(p)$ by the same def. Then $A$ is $\Sigma_1(\bar{Q})$ in some $q$ and $\tilde{A}$ is $\Sigma_1(Q^*)$ in $q^* = \sigma_k^{(m)} \delta_k(q)$ by the same definition.

(The proof is an exact repetition of (5) in §9 Lemma 5.1.1, using again that $\tilde{\pi}_{k\ell} : Q^* \xrightarrow[\Sigma_0]{} Q_\ell$ cofinally.)

(6) $k > \mu$.

We again imitate (6) in §9 Lemma 5.1.

Suppose not, Then $k = \mu$ and $\eta_i^\nu = \eta \leq \beta$, since $\tau' < \kappa_i$. If $\eta = \beta$, then $\bar{Q} = Q'$, $\tilde{Q} = Q^*$, $m = n$, and it is immediate from (5) that (+) holds. Now let $\eta < \beta$. Hence $n < m$. Let $A, \tilde{A}$ be as in (5). Set $\sigma = \sigma_{m-n}[\gamma; \tilde{\nu}]^{U_k}$.

Then $\sigma\sigma_k^{(n)} = \sigma_k^{(m)}$ and

$\sigma : \tilde{Q}' | \beta' \xrightarrow[\Sigma^*]{} Q^*$, where $\beta' =$
$= \bar{\beta}_{m-n}(\tilde{Q}', \tilde{\nu}')$. Moreover, $\sigma \upharpoonright \tilde{\tau} + 1 =$
$= \mathrm{id}$. Then $A$ is $\Sigma_1(\bar{Q})$ in $q$ and $\tilde{A}$ is $\Sigma_1(Q^*)$ in $q^* = \sigma\sigma_k^{(n)}(q)$

by the same def.

$\tilde{A}$ is $\overrightarrow{\Sigma}_1(\tilde{Q}' \| \beta')$ in $q' = \sigma_k^{(n)}(q)$

by the same definition. But

$\eta = \eta_i^\nu = \bar{\beta}_{m-1}(Q', \nu_k) + \bar{Q} = Q' \| \eta$.

But $\sigma_k^{(n)} \delta_k(\eta) = \beta'$. Thus

$\sigma_k^{(n)} \delta_k(\bar{Q}) = \tilde{Q}' \| \beta'$ and

$\sigma_k^{(n)} \delta_k(A) = \tilde{A}$. This verifies (+).

Contr! QED (6)

(7) $\bar{Q} = Q_k$ (i.e. $\eta_i = ht(Q_k)$).

prf. As in §9 Lemma 5.1.1, if not, then $\tau^{+Q_k} > \omega\eta_i = On \cap \bar{Q}$ by (4). But $\tau < \lambda_\mu$, where $\lambda_\mu < \lambda_k$ is a limit cardinal in $Q_k$. Contr!

QED (7).

Thus $\pi_{kl} : Q_k \longrightarrow_{\Sigma^*} Q_l$, $\pi_{kl}(\kappa') = \kappa'$. Hence $\kappa' = \kappa'_k$, $\tau' = \tau'_k$, $\mu = \mu_k$, $\beta = \beta_k$. Since (+) holds at k, it follows by (5) that (+) holds at $l$. Contr!

QED (Lemma 4.1)

<u>Def</u> $l$ is <u>bold</u> iff $\mu_l$ is defined and whenever $A \subset \tau'_l$ is $\Delta_1(Q_l)$ in p and $A' \subset \tilde{\tau}'_l$ is $\Delta_1(\tilde{Q}_l)$ in $p' = \sigma_l(p)$ by the same def, then $A \in Q'$ and $A' = \sigma_\mu(A)$.

Just as in §9 we prove a pendant to Lemma 4.1:

Lemma 4.2 Let $\mu_l, \delta_l$ be defined s.t. (a), (b) hold below $l$ and $l$ is not bold, (++) Let $A \subset \tau'$ be $\Sigma_1(Q_l)$ in $p$ and $\tilde{A} \subset \tilde{\tau}'$ be $\Sigma_1(\tilde{Q}_l | \rho_0^l)$ in $p' = \delta_l(p)$ by the same def. Then $A$ is $\Sigma_1(Q')$ in some $q$ and $\tilde{A}$ is $\Sigma_1(\tilde{Q} | \rho_0')$ in $q' = \sigma_\mu^{(m)} \delta_\mu(q)$ by the same def.

prf.

Let $l$ be the least counterexample.

Then $\mu < l = i+1$. Let $k = \tau(l)$.

Let $\eta = \eta_i^{\gamma}$, $\bar{Q} = Q_k \| \bar{\eta}$ etc. Let $\tilde{\eta}$, $\sigma_k^{(m)}$, $Q^*$, $\delta^*$ be defined as before. Similarly for $\tilde{\kappa}, \tilde{\tau}$, $\kappa^*, \tau^*$.

(1) $\kappa' < \kappa_i$ (as before)

(2) $\mu \leq k$ (as before)

(3) $\omega\rho^1_{Q_l} \leq \tau'$ (as before, but somewhat easier)

(4) $\rho^1_{\bar{Q}} \leq \tau'$ (as before)

(5) is formulated exactly as before & has exactly the same proof.

By (12) of Case 3 we then have:

(5.1) Let $A \subset u_i$ be $\Sigma_1(Q_\ell)$ in $p$ and $\tilde{A} \subset \tilde{u}_i$ be $\Sigma_1(\tilde{Q}_\ell | \rho_0^\ell)$ in $\tilde{p} = \sigma_\ell(p)$ by the same def. Then $A$ is $\Sigma_1(\bar{Q})$ in some $q$ and $\tilde{A}$ is $\Sigma_1(Q^*)$ in $q^* = \sigma_k^{(m)} \sigma_k(q)$ by the same def.

(6) $k > \mu$ is proven as before by contradiction. We use (5.1) to contradict $\eta = \beta$. We use (5) and the non boldness of $\ell$ to contradict $\eta < \beta$.

(7) $\bar{Q} = Q_k$ is exactly as before.

We then get a contradiction exactly as before, using (5.1) instead of (5) to show that (++) holds at $\ell$.

QED (Lemma 4.2).

We are now ready to prove Lemma 4. We proceed by induction on $i$. Let $F = E^{Q_i}_{\nu_i}$, $\tilde{F} = E^{\tilde{Q}_i}_{\tilde{\nu}_i}$. Let $\bar{\alpha} < \lambda_i$, $\alpha = \sigma_i(\bar{\alpha})$.

If $k=i$, the conclusion is trivial, so assume $k<i$.

<u>Case 1</u> $\bar{F} \in Q_i$. Then $F_\alpha = \sigma_i(\bar{F}_{\bar{\alpha}})$. But $\bar{F}_{\bar{\alpha}} \in J_{\lambda_k}^{E^{Q_i}} = J_{\lambda_k}^{E^{\bar{Q}}} \subset \bar{Q}$. Hence $\sigma_k^{(m)} \sigma_k(\bar{F}_{\bar{\alpha}}) = F_\alpha$. This verifies (b). But then $F_\alpha \in Q^* | \rho_0^*$. Hence it verifies:

$$\langle \sigma_k^{(m)} \sigma_k , \sigma_i^* \sigma_i \restriction \lambda_i \rangle : \langle \bar{Q}, \bar{F} \rangle \xrightarrow{*} \langle Q^* | \rho_0^*, F \rangle,$$

Hence (a) holds.

<u>Case 2</u> Case 1 fails. Then $\bar{F}$ is the top extender and $k=\mu_i$, $\bar{Q}=Q_i'$, $\kappa_i=\kappa_i'$, $\tau_i=\tau_i'$ etc. $\bar{F}_{\bar{\alpha}}$ is $\Delta_1(Q_i)$ in $\bar{\alpha}$ by:

$$(*) \quad X \in \bar{F}_{\bar{\alpha}} \longleftrightarrow \bigvee_{\beta \in Q_i} \bigvee Y \in J_\beta^{E^{Q_i}} (\bar{\alpha} \in Y = F(X))$$

$$X \notin \bar{F}_{\bar{\alpha}} \longleftrightarrow \quad \text{"} \quad \text{"} \quad (\bar{\alpha} \notin Y = F(X)).$$

$F_\alpha$ is $\Delta_1(\tilde{Q}_i)$ in $\alpha$ by the same def. Hence by Lemma 4.1 $\bar{F}_{\bar{\alpha}}$ is $\Delta_1(\bar{Q})$ in some $\bar{q}$ and $F_\alpha$ is $\Delta_1(Q^*)$ in $q = \sigma_k^{(m)} \sigma_k(\bar{q})$ by the same def.

This verifies (b). We verify (a)

<u>Case 2.1</u> $i$ is bold.

Then $\bar{F}_{\bar{\alpha}} \in \bar{Q}$ and $F_\alpha = \sigma_k^{(m)} \sigma_k(\bar{F}_{\bar{\alpha}})$.

The conclusion follows as in Case 1.

Case 2.2 Case 2.1 fails.

Set $\bar{G} = \bar{F}_{\bar{\alpha}}$ and let $G$ be $\Sigma_1(\tilde{Q}_i|\rho_0^i)$ in $\alpha$ by the same $\Sigma_1$ definition (*).

Set: $\bar{H} = {}^{\kappa_i}\mathbb{P}(\kappa_i) \cap Q_i$. $\bar{H}$ is definable by:

$$X \in \bar{H} \longleftrightarrow \bigvee \beta \in Q_i \bigwedge j < \kappa_i \bigvee Y \in \mathcal{J}_\beta^{E^{Q_i}} \; Y = F(X_j)$$

for $X : \kappa_i \to \mathbb{P}(\kappa_i)$. Let $H$ have the same $\Sigma_1$ def. over $\tilde{Q}_i|\rho_0^i$.

Clearly:

$$X \in H \to \bigwedge j < \tilde{\kappa}_i \; (X_j \text{ or } \tilde{\kappa}_i \setminus X_j \in G),$$

But by Lemma 4.2 $\bar{G}, \bar{H}$ are $\Sigma_1(\bar{Q}|$ in some $\bar{q}$ and $G, H$ are $\Sigma_1(Q^*|\rho_i^*)$ in $q = \sigma_h^{(m)} \sigma_k(\bar{q})$. Hence $\bar{G}, G, \bar{H}, H$ verify (a). QED (Lemma 4)

We can now prove Lemma 1. Let $\delta : Q \longrightarrow_{\Sigma^*} N_\xi \min(\vec\rho)$, where $Q$ is countable. Let $U = \langle V_\nu, \in, \lambda \rangle$ be a coarse premouse s.t. $\nu < \Theta$ and $\langle N_\eta \mid \eta \leq \xi \rangle \in V_\lambda$. Let $\sigma : U' \prec U$, where $\overline{U}$ is countable and $Q, \delta, \vec N, \vec\rho \in \mathrm{rng}(\sigma)$. Then $\sigma(Q) = Q$. Let $\sigma(\delta', \vec N', \vec\rho') = \delta, \vec N,$. Then $\delta' : Q \longrightarrow_{\Sigma^*} N'_{\xi'} \min(\vec\rho')$, where $\sigma(\xi') = \xi$. Let $\mathcal{Y} = \langle\langle Q_i\rangle, \langle \nu_i \rangle, \langle \eta_i \rangle, \langle \pi_{ij} \rangle; T\rangle$ be a countable normal iteration of limit length $\Gamma$ of $Q$. We attempt to define a strategy $S$ which gives a cofinal branch $S(\mathcal{Y})$. We repeat the above construction using $U'$, $\vec N', \delta', \vec\rho'$ in place of $U, \vec N, \delta, \vec\rho$, getting $\mathcal{Y}' = \langle\langle U'_i\rangle, \langle E^*_i \rangle, \langle \tilde\pi'_{ij} \rangle, T'\rangle$ (Simultaneously form $\tilde Q_i = \tilde\pi_{0i}(\vec N) \mid \delta_i$ and $\delta_i : Q_i \longrightarrow_{\Sigma^*}^{\tilde W} \tilde Q_i \min(\vec\rho^{\,i})$) of length $\overline\Gamma \leq \Gamma$. If $\overline\Gamma < \Gamma$, then $S(\mathcal{Y})$ is undefined. If $\overline\Gamma = \Gamma$ choose, if possible, a cofinal well founde branch b in $\mathcal{J}'$. It follows easily that b is a cofinal well founded branch in $\mathcal{J}$ and that there is $\sigma'_b : Q_b \xrightarrow[\Sigma^*]{} \tilde{Q}_b$ min(;) where for sufficiently large $i_0 \in b$ we have: $\pi_{i_0 i}$ is total for $i_0 \leq_T i$ in b, $\tilde{\pi}_{i_0 i}(\vec{\rho}^{\,i_0}) = \vec{\rho}^{\,i}$, $\tilde{Q}_b = \tilde{\pi}_{i_0 b}(\tilde{Q}_{i_0}) = \tilde{\pi}_{0 b}(\vec{N}')_{\tilde{\pi}_{i_0 b}(\gamma_{i_0})}$. [$\sigma'_b$ is defined by $\sigma'_b \tilde{\pi}_{ib} = \tilde{\pi}_{ib} \sigma'_i$ for $i \geq i_0$] (The verifications are just like Case 2 in our construction) There may be many such b's available so we make our selection in such a way that b is a <u>modest</u> branch in $\mathcal{J}$ if possible. We set: $S(\mathcal{J}) = b$

<u>Lemma 5</u> S is a countable normal iteration strategy for Q.

proof.

Let $\mathcal{J}$ be a countable normal iteration (of length $\Gamma$). Form $\mathcal{J}' = \langle \langle U_i \rangle, \langle F_i^* \rangle, \langle \tilde{\pi}_{ij} \rangle, T' \rangle$ as above of length $\bar{\Gamma}$.

<u>Claim 1</u> $\bar{\Gamma} = \Gamma$

proof of Claim 1. We prove: $U_i, \delta_i$ is defined by ind. on $i$. $i=0$ is trivial. Now let $U_i, \delta_i$ be defined. We must verify (*), (**) in Case 3 of the construction. (*) holds by MS(a). (**) holds by Lemma 4. Hence $U_{i+1}, \delta_{i+1}$ are defined. Now let $U_i, \delta_i$ be defined for $i < \lambda < \Gamma$, where $\mathrm{Lim}(\lambda)$. Then $\{i \mid i \leq_T \lambda\}$ was chosen to be a cofinal well founded branch in $\mathcal{T}' | \lambda$. Hence $U_\lambda, \delta_\lambda$ are defined.

QED (Claim 1)

<u>Claim 2</u> $\mathcal{T}$ can be continued

prf. By cases as follows:

<u>Case 1</u> $\Gamma = i+1$.

We pick $\nu = \nu_{i+1}$ s.t. $\nu > \nu_i$ and $E_\nu^{Q_i} \neq \emptyset$. Let $k$ = the least $k$ s.t. $k = i$ or $\kappa < \lambda_k$, where $\kappa = \mathrm{crit}(E_\nu$

Let $\eta$ = the max. $\eta$ s.t. $\tau = \kappa^{+ \mathcal{J}^{E^{Q_i}}_{\lambda_i}}$ is a cardinal in $Q_k \| \eta$.

Let $\mathbb{D} = \langle D, =_D, \in_D, E_D \rangle$ be the term model representation of $\mathrm{Ult}^*(Q_k \| \eta, F)$ where $F = E_\nu^{Q_i}$. [That is, $D$ = the set of $\langle \alpha, f \rangle$ s.t. $f \in \Gamma^*(Q_k \| \eta, \kappa)$ and $\alpha < \lambda = F(\kappa)$, $\langle \alpha, f \rangle \in_D \langle \beta, g \rangle$ iff iff $\{\langle \xi, \zeta \rangle < \kappa \mid f(\xi) \in g(\zeta)\} \in F_{\langle \alpha, \beta \rangle}$.]

<u>Claim</u> $\in_D$ is well founded.

By MS(a), $\tilde{\pi}_{k,i+1} : U_k' \xrightarrow[F^*]{} U_{i+1}'$ exists, where $F^*$ is chosen as in Case 3 of the construction. But we can then define $\delta : \mathbb{D} \xrightarrow[\Sigma_0]{} \tilde{Q}_{i+1} = \tilde{\pi}_{k,i+1}(Q^*)$ by: $\delta(\langle \alpha, f \rangle) = \sigma_k^{(m)} \delta_k(f)(\sigma_i^* \delta_i(\alpha))$. Hence $\mathbb{D}$ is well founded. QED (Case 1)

<u>Case 2</u> $\mathrm{Lim}(\Gamma)$.

We must show that $\mathcal{J}'$ has a well founded cofinal branch. We know that $\mathcal{J}'$ has a well founded

<u>maximal</u> branch $b$ by $MS(b)$. We must show that $b$ is cofinal. Suppose not. Let $\lambda = \sup b < \theta$. Then $b \neq b_\lambda$ by maximality. $b_\lambda$ is not modest, since otherwise $\mathcal{J}|\lambda+1$ could not be continued. Hence $b$ is not modest, since otherwise $b_\lambda = S(\mathcal{J}|\lambda)$ would have been chosen as modest. Let $\delta = \sup_{i<\lambda} \nu_i$. By §6 $\delta$ is Woodin in $Q_b \cap Q_{b_\lambda}$, hence in $N = \langle J_\alpha^E, \emptyset \rangle$, where $\alpha = \min(lh(Q_b), lh(Q_{b_\lambda}))$ and $E = E^{Q_b}|\delta = E^{Q_{b_\lambda}}|\delta$. But by §6 $N = Q_b$ or $Q_{b_\lambda}$, since $Q$ is basic. Hence $b$ or $b_\lambda$ is modest. Contr!

QED (Lemma 5)

To finish the proof, let $\mathcal{J}$ ~~be~~ an $S$-iteration of length $\theta+1$ and let $\mathcal{J}'$, $\langle \delta_i \mid i \leq \theta \rangle$ etc. be as above. Let $\sigma' : U'_\theta \longrightarrow U$ s.t. $\sigma' \tilde{\pi}_{0\theta} = \sigma$. Then $\sigma'(\tilde{\pi}_{0\theta}(\vec{N}')) = \vec{N}$.

Hence $\sigma'(\tilde{Q}_\theta) = N_\gamma$ for a $\gamma \leq \xi$. If $\pi_{0\theta}$ is not total, then $\tilde{Q}_\theta = \tilde{\pi}_{0\theta}(\vec{N}')_{\gamma_\theta}$ where $\gamma_\theta < \tilde{\pi}_{0\theta}(\xi')$. Hence $\gamma = \sigma'(\gamma_\theta) < \xi$. If $\pi_{0\theta}$ is total, then $\gamma_0 = \xi' = \sigma^{-1}(\xi)$ and $\gamma = \sigma'(\gamma_\theta) = \xi$.

Set: $\delta'' = \sigma'\delta'_\theta$. Since $\delta'_\theta : Q_\theta \xrightarrow[\Sigma^*]{} \tilde{Q}_\theta \min(\vec{\rho}^\theta)$ it follows that $\delta'' : Q_\theta \xrightarrow[\Sigma^*]{} N_\gamma \bmod (\vec{\rho}'')$, where $\vec{\rho}'' = \sigma'(\vec{\rho}^\theta)$. Set: $\vec{\rho}' = \min(N_\gamma, \delta'', \rho'')$. Then $\delta'' : Q_\theta \to N_\gamma \min(\rho')$ and $\rho'_n \leq \rho''_n$ for $n$ $\omega$. But $\rho^\theta_n \leq \tilde{\pi}_{0\theta}(\rho^0_n)$ & hence $\rho'_n \leq \rho''_n = \sigma_\theta(\rho^\theta_n) \leq \rho_n = \sigma_\theta \tilde{\pi}_{0\theta}(\rho^0_n)$ (da $\sigma_0(\rho^0_n) = \rho_n$). QED (Lemma 1).

We have thus succeeded in constructing a sequence $\langle N_\xi \mid \xi < \theta \rangle$, $\langle M_\xi \mid \xi < \theta \rangle$ satisfying (a), (b) as stated at the outset. - We now verify the uniqueness of the construction: In Case 2.1 we set $N_\xi = \langle J^E_\beta, F \rangle$ where $N_\xi$ is a pm., $M_{\xi-1} = \langle J^E_\beta, \emptyset \rangle$, and there is an extender $F^*$ on $V$ s.t. $F$ is the restriction of $F^*$ to $J^E_\beta$ (i.e. $F = (F^* | \lambda) \cap J^E_\beta$, where $\lambda$ = the largest cardinal in $J^E_\beta = F(\kappa)$, where $\kappa = \text{crit}(F^*)$); and $F^*$ is $\lambda+2$-strong (i.e. $V_{\lambda+2} \subset U$, where $\pi : V \longrightarrow_{F^*} U$).

We now show that $F$ is independent of the choice of $F^*$. Our main tool is the concept of <u>bicephalus</u>.

<u>Def</u> A <u>prebicephalus</u> (pb) is a structure $\langle J^E_\alpha, F, G \rangle$ s.t. $\langle J^E_\alpha, F \rangle$, $\langle J^E_\alpha, G \rangle$ are premice and $F, G \neq \emptyset$.

Def Let $M = \langle J^E_\alpha, F, G\rangle$ be a prebicephalu[s]. Let $\nu \leq \alpha = ht(M)$. $E^M_{\nu h} = E_\nu$ for $\nu < \alpha$, $h < 2$. $E_{\nu 0} = F$, $E_{\nu 1} = G$. If $M$ is a mouse, $\nu \leq ht(M)$, set: $E^M_{\nu h} = E^M_\nu$ $(h < 2)$.

Def Let $M$ be a pm or pb. A generalized $\Sigma_0$ - iteration of $M$,

$\mathcal{I} = \langle\langle M_i \mid i < \theta\rangle, \langle\langle \nu_i, h_i\rangle \mid i \in D\rangle, \langle \eta_i \mid i+1 < \theta\rangle, \langle \pi_{ij} \mid i \leq_T j\rangle$

defined exactly as in §4, except that for $i \in D$, $\xi = T(i+1)$, we have:

$\pi_{\xi i} :$ $\quad$ $M_{i+1}$ if $i+1$ is simple an[d]

$\pi_{\xi i} : M_\xi \| \eta_i \longrightarrow^*_{E_{\nu_i h_i}} M_{i+1}$ if not.

The notions direct, standard, normal are then defined exactly as before, as is the notion of iteration strategy. $M$ is again called countably normally iterable iff there is a strategy $S$ s.t. every countable normal iteration $\mathcal{I}$ of $M$ can be continued and $S(\mathcal{I})$ is defined.
(Similarly for normally iterable).

Def Let $M^h$ be a pm or pb $(h=0,1)$. The <u>coiteration</u> $\langle \mathcal{J}^0, \mathcal{J}^1 \rangle$ of $M^0, M^1$ is the pair of normal $\Sigma_0$ iterations:

$\mathcal{J}^h = \langle \langle M_i^h \rangle, \langle \langle \nu_i, l_i^h \rangle \rangle, \langle \eta_i^h \rangle, \langle \pi_{ji}^h \rangle, T^h \rangle$

defined by: $M_0^h = M_h$;

$\nu_i \simeq$ the least $\nu$ s.t. $\bigvee l, l'$ $E_{\nu l}^{M_i^0} \neq E_{\nu l'}^{M_i^0}$.

$i \in D^h \longleftrightarrow E_{\nu 0}^{M^h} \neq \emptyset$; If $i \notin D^{1-h}$, set: $l_i^h = 0$. If $i \in D^h \cap D^{1-h}$, let $\langle l_i^0, l_i^1 \rangle$ be lexicographically least s.t. $E_{\nu_i l_i^0}^{M^0} \neq E_{\nu_i l_i^1}^{M^1}$.

Just as in §4: If $M^0, M^1$ are normally $\Sigma_0$-iterable, then the coiteration terminates. (If $M^0, M^1$ are countable, then countable $\Sigma_0$ iterability is enough, since the coiteration must terminate in $< \omega_1$ many steps).

Exactly as in §7 we then get: Let $M^0, M^1$ be presolid (i.e. $M^h \| \alpha$ is solid for $\alpha < ht(M^h)$) and let the coiteration terminate in $N^0, N^1$. Then

(a) One side of the coiteration is simple on the main branch

(b) If the coiteration of $M^h$ to $N^h$ is nonsimple, then $N^{1-h}$ is a segment of $N^h$.

<u>Def</u> A <u>bicephalus</u> is a presolid pb $M$ s.t. whenever $\sigma : Q \xrightarrow{\Sigma_1} M$ and $Q$ is countable, then $Q$ is countably normally $\Sigma_0$-iterable.

The main lemma on bicephali says that they trivialize:

<u>Lemma 6.1</u> Let $M = \langle J_\alpha^E, F, G \rangle$ be a bicephalus. Then $F = G$.

prf.

By Löwenheim-Skolem it suffices to prove it for countable $M$.

Coiterate $M$ against itself, getting $N, N'$. Assume w.l.o.g. that $N$ is a simple iterate of $M$ and a segment of $N'$. Let $N = \langle J_{\tilde{\alpha}}^{\tilde{E}}, \tilde{F}, \tilde{G} \rangle$

Then $\tilde{F} = \tilde{G} = E^{N'}_{\tilde{\alpha}, l}$ $(l = 0, 1)$. Hence $F = G$, since $\pi_{0,\tilde{\alpha}} : M \to_{\Sigma_1} N$.

QED (6.1)

We can now prove uniqueness:

<u>Lemma 6.2</u> Let $\xi$ be as in Case 2.1 in the construction of $\vec{N}$. Let $M_{\xi-1} = \langle J^E_\alpha, \emptyset \rangle$ and let $F, F^*$ be as in Case 2.1. Let $G, G^*$ be another such pair. Then $G = F$.

proof (sketch).

It suffices to show that $N' = \langle J^E_\alpha, F, G \rangle$ is a bicephalus. $N'$ is obviously a pb and is presolid. Let $\delta : Q \to_{\Sigma_1} N'$, where $Q$ is countable. We must show that $Q$ is countably normally $\Sigma_0$-iterable. Let $\langle V_\nu, \in, \lambda \rangle$ be a coarse premouse with $N' \in V_\lambda$. Let $\mathcal{Y}$ be a countable normal

$\Sigma_0$ iteration of $Q$ of length $\Gamma$.. with $\mathcal{I} = \langle\langle Q_i\rangle, \langle\langle \nu_i, l_i\rangle\rangle, \dots, \langle \pi_{ij}\rangle, T\rangle$ We first construct a coarse iteration $\mathcal{I}' = \langle\langle U_i\rangle, \langle F_i^*\rangle, \langle \tilde{\pi}_{ij}\rangle, T'\rangle$ of $U = \langle V_\nu, \epsilon,$ of length $\bar{\Gamma} \leq \Gamma$ with $T' = T \mid \bar{\Gamma}$, Simultaneously we construct maps $\sigma_i : Q_i \longrightarrow \tilde{Q}_i = \tilde{\pi}_{0i}(\vec{N})_{\gamma_i}$ (here $\vec{N} = \langle N_i \mid i \leq \xi\rangle$ with $N_\xi =_{\mathrm{Df}} N'$). $\sigma_i$ is $\Sigma_0$ preserving if $i$ is simple in $\mathcal{I}$. Otherwise $\sigma_i : Q_i \xrightarrow[\Sigma^*]{} \tilde{Q}_i \min(\vec{p}^i$ (Hence $\vec{p}^i$ is only defined when $i$ is non simple.) The construction is a straightforward modification of our previous one. The details are left to the reader. If $\sigma_i, U_i$ is defined for $i < \lambda$, $\lim(\lambda)$, then $\sigma_\lambda, U_\lambda$ will be defined iff $\{j \mid j T \lambda\}$ is a cofinal well founded branch in $\mathcal{J}'|\lambda$. If $\delta_i, U_i$ are defined, we need:

(*) $\mathrm{Ult}(U_i, F_i^*)$ is well founded

to define $\delta_{i+1}, U_{i+1}$. [If $i$ is non simple in $\mathcal{J}$, we also need (**) (as defined earlier), but it follows exactly as before that (**) will hold.] Using this, we define a strategy for Q as before. Let $\sigma : U' \prec U$, $\sigma(\vec{N}') = \vec{N}$. Let $\mathcal{J}$ be a countable normal $\Sigma_0$ iteration of Q. Set $\delta' = \sigma^{-1}\delta : Q \to N'_\xi$, and form $\mathcal{J}'$, $\langle \delta_i' \mid i < \Gamma' \rangle$ as before. $S(\mathcal{J})$ is defined iff $\Gamma' = \Gamma$ and $\mathcal{J}'$ has a cofinal well founded branch b. In this case we choose b – if possible – to be modest in $\mathcal{J}$ and set: $S(\mathcal{J}) = b$. It follows as before that S is a strategy for Q.*)

QED (Lemma 6.2)

*) We use the obvious fact that §6 Lemmas 1-3 hold for normal $\Sigma_0$ iterations of ph's