§1 Remarks on the gap two problem

Consider a first order language $\mathcal{L}$ with predicates $\dot{\in}$, A, B and axioms:
ZFC$^-$ + A is an infinite cardinal +
+ $B = A^+$ is the largest cardinal.

By a (τ, τ^{++})-model of $\mathcal{L}$ we understand a model
$$\mathfrak{A} = \langle |\mathfrak{A}|, A_{\mathfrak{A}}, B_{\mathfrak{A}}, \in_{\mathfrak{A}}, \ldots \rangle$$
s.t. $\overline{\overline{On_{\mathfrak{A}}}} = \tau^{++}$ and $\overline{\overline{A_{\mathfrak{A}}}} = \tau$.

Note The usual notion of (τ, τ^{++})-model requires only that $\overline{\overline{|\mathfrak{A}|}} = \tau^{++}$ and $\overline{\overline{A}}_{\mathfrak{A}} = \tau$. If we added a predicate F and the axiom $F : On \leftrightarrow V$, then the two notions would become equivalent for models of this theory.

Note If $\mathfrak{A}$ is a (τ, τ^{++})-model, then $\overline{\overline{B}}_{\mathfrak{A}} = \tau^+$. To see this note that, letting $<_{\mathfrak{A}}$ be the natural ordering of $On_{\mathfrak{A}}$ in $\mathfrak{A}$, then for all $x \in B$, $\overline{\overline{\{z \mid z <_{\mathfrak{A}} x\}}} \leq \tau$. (This is because either $\{z \mid z <_{\mathfrak{A}} x\} = \emptyset$ or else

$\mathfrak{A} \models \bigvee f \; f : A \xrightarrow{\text{onto}} x$. Let $\mathfrak{A} \models f : A \xrightarrow{\text{onto}} x$. Set $\tilde{f} = \{\langle z, y\rangle \mid \mathfrak{A} \models z = f(y)\}$. Then $\tilde{f} : A \xrightarrow{\text{onto}} \{z \mid z <_{\mathfrak{A}} x\}$.] Hence $B = \bigcup_{x \in B} \{z \mid z <_{\mathfrak{A}} x\}$ has cardinality $\leq \tau^+$. Suppose $\overline{\overline{B}} \leq \tau$. Then, by the above argument, for each $x \in On_{\mathfrak{A}}$ we have $\overline{\overline{\{z \mid z <_{\mathfrak{A}} x\}}} \leq \tau$. But $On_{\mathfrak{A}} = \bigcup_{x \in On_{\mathfrak{A}}} \{z \mid z <_{\mathfrak{A}} x\}$. Hence $\overline{\overline{On_{\mathfrak{A}}}} \leq \tau^+ < \tau^{++}$. Contr!

. . . .

<u>Lemma 1</u> Let $\mathfrak{A}$ be a (τ, τ^{++}) model. Let b be an initial segment of $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}\rangle$ s.t. $cf(b) = \tau^+$. Then b has a supremum in $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}\rangle$. (i.e. there is $z \in On_{\mathfrak{A}}$ s.t.

$$z \leq_{\mathfrak{A}} u \longleftrightarrow \bigwedge x \in b \; x \leq_{\mathfrak{A}} u$$

for all $u \in On_{\mathfrak{A}}$).

Pf. of Lemma 1. Suppose not.

Since $\overline{\overline{\{z \mid z <_{\mathfrak{A}} x\}}} \leq \tau^+$ for $x \in On_{\mathfrak{A}}$ and $\overline{\overline{On_{\mathfrak{A}}}} = \tau^{++}$, we have:

$cf(On_{\mathfrak{A}}) = \tau^{++}$ in $<_{\mathfrak{A}}$. Hence b is a proper segment of $On_{\mathfrak{A}}$. Let $x \in On_{\mathfrak{A}} \setminus b$. Let $f \in \mathfrak{A}$ s.t.

$\mathfrak{A} \models f : B \xrightarrow{\text{onto}} x$. Then

$\tilde{f} = \{\langle u, v\rangle \mid \mathfrak{A} \models u = f(v)\}$ is a map of B onto $\tilde{x} = \{z \mid z <_{\mathfrak{A}} x\}$. For $z \in B$ let $\mathfrak{A} \models f_z = f \upharpoonright z$ + let $\tilde{f}_z$, $\tilde{z}$ have the obvious definitions.

Then $\tilde{f}_z : \tilde{z} \xrightarrow{1-1} \tilde{x}$. Since $x \neq \sup b$ in $<_{\mathfrak{A}}$, there is $u \in \tilde{x} \setminus b$. Hence there is $z \in B$ s.t. $u \in rng(\tilde{f}_z)$.

However:

(1) $rng(\tilde{f}_z) \cap b$ is bounded in b,

since $\overline{\overline{rng(\tilde{f}_z)}} = \overline{\overline{\tilde{z}}} \leq \tau$ and $cf(b) = \tau^+$ in $<_{\mathfrak{A}}$.

Pick $d \in b$ s.t. $rng(\tilde{f}_z) \cap b \subset \tilde{d}$.

Then there is a unique $q_z \in On_{\mathfrak{A}}$ s.t.

(2) $\mathfrak{A} \models q_z = \min\{v \in rng(f_z) \mid v > d\}$.

It follows immediately that

(3) $q_z = \min(rng(\tilde{f}_z) \setminus b)$ in $<_{\mathfrak{A}}$.

(Note that def. of q_z does not depend on d; it would be the same for any $d' \in b$ s.t. $rng(\tilde{f}_z) \cap b \subset \tilde{d}'$.)

Obviously:

(4) $z <_{\mathfrak{A}} z' \in B \to q_{z'} \leq_{\mathfrak{A}} q_z'$

We now define $\langle n_u \mid u \in B \setminus \tilde{z} \rangle$ s.t. $\mathfrak{A} \models n_u \in \omega$ as follows:

Pick a $d = d_u \in b$ s.t. $b \cap rng(f_u) \subset \tilde{d}$.

Working in $\mathfrak{A}$ define a map

$q_u = q : [z, u] \to On$ by:

$q(w) = \min\{v \in rng(f_w) \mid v > d\}$.

Then in fact $\mathfrak{A} \models q_w = q(w)$ for

$z \leq_{\mathfrak{A}} w \leq_{\mathfrak{A}} u$.

In $\mathfrak{A}$ let $a = a_u = \{q(w) \mid z \leq w \leq u\}$.

Then $\mathfrak{A} \models a$ is finite, since

$q(w) \leq q(w')$ for $w' \leq w$.

Let $\mathfrak{A} \models n_u = \bar{\bar{a}}$; Then

(5) $\mathfrak{A} \models n_u \in \omega$; hence $n_u \in A$.

(6) $z \leq_{\mathfrak{A}} u \leq_{\mathfrak{A}} u' \in B \rightarrow$

$\rightarrow n_u \leq_{\mathfrak{A}} n_{u'}$,

since $\mathfrak{A} \models q_u = q_{u'} \upharpoonright [z, u]$ + hence

$\mathfrak{A} \models a_u \subset a_{u'}$.

(7) Let $z \leq_{\mathfrak{A}} u \in B$. There is $u' \in B$

s.t. $\mathfrak{A} \models a_u \subsetneqq a_{u'}$

(hence $n_u <_{\mathfrak{A}} n_{u'}$).

prf.

$q_u = q(u) = \min a$ in $\mathfrak{A}$.

Let $p \in On_{\mathfrak{A}} \setminus b$ s.t. $p <_{\mathfrak{A}} q_u$.

(This must exist, since otherwise $q_u = \sup b$ in $<_{\mathfrak{A}}$.)

Then $\mathfrak{A} \models \bigvee u' \in B \cdot p \in \operatorname{rng} f \upharpoonright u'$.

Let $p \in \operatorname{rng} f_{u'}$, where $u \leq_{\mathfrak{A}} u' \in B$.

Then $q_{u'} \leq_{\mathfrak{A}} p < q_u$. Hence

$\mathfrak{A} \models a_{u'} = \operatorname{rng}(q_{u'}) \supsetneqq \operatorname{rng}(q_u) = a_{u'}$

Hence $n_{u'} >_{\mathfrak{A}} n_u$. QED(7)

Now select $\langle z_\xi \mid \xi < \tau^+ \rangle$ s.t.

$z_\xi \in B$ by: $z_0 = z$;

$z_{\xi+1} >_\Omega z_\xi$ s.t. $n_{z_{\xi+1}} > n_{z_\xi}$;

z_λ s.t. $z_\lambda > z_\xi$ for all $\xi < \lambda$

($\mathrm{Lim}(\lambda)$).

[z_λ exists since $\mathrm{cf}(B) = \tau^+$ in $<_\Omega$]

Then $n_{z_\xi} < n_{z_{\xi'}}$ for $\xi < \xi'$.

Hence $\langle n_{z_\xi} \mid \xi < \tau^+ \rangle$ injects

τ^+ into A, where $\bar{\bar{A}} = \tau < \tau^+$.

Contradiction! QED (Lemma 1)

Now let $\mathcal{L}^*$ be $\mathcal{L}$ together with a new predicate C and the additional axiom:

$C = \langle C_\lambda \mid B < \lambda \wedge \mathrm{Lim}(\lambda) \rangle$ is a $\Box_B$ - sequence.

We shall give a model of set theory satisfying $GCH + \Diamond^+$ s.t. $\mathcal{L}^*$ has no (ω, ω_2) - model. (Moreover, $\Box_\tau$ will hold for $\tau > \omega_1$. Hence $(\kappa, \kappa^{++}) \not\to (\omega, \omega_2)$ for $\kappa > \omega_1$.) Since $\Diamond^+$ holds, there will be a Kurepa tree in the model. This will show that the gap 2 conjecture can fail at (ω, ω_2) even ~~in~~ the presence of a Kurepa tree.

The absence of a (ω, ω_2) - model for $\mathcal{L}^*$ means, of course, that $\Box$ fails. Since $\Box$ holds whenever ω_2 is not Mahlo in L, we shall force over a ground model containing a Mahlo cardinal κ. The forcing has two stages. In the first stage we do ordinary collapsing to turn κ into ω_2. The resulting generic extension has neither a Kurepa tree nor an (ω, ω_2)-model for $\mathcal{L}^*$. We then force to reinstate the principle $\diamondsuit^+$, but without adding a (ω, ω_2)-model of $\mathcal{L}^*$.

In the following let N be a countable transitive model of ZFC + GCH + $\diamondsuit^+$ + there is a Mahlo cardinal.
Let κ be a Mahlo cardinal in N.
Let $S = \langle S_\alpha \mid \alpha < \omega_1 \rangle$ be a fixed $\diamondsuit$-sequence in N. In the first stage of our forcing we use the normal condition for collapsing to make κ become ω_2:

<u>Def</u> Let $\omega_1 \leq \tau < \mu$.
$\mathbb{C}^\mu_\tau =_{Df}$ the set of maps p with $\mathrm{dom}(p) \subset \omega \times [\tau, \mu)$, $\bar{\bar{p}} \leq \omega$, and $p(i, \nu) < \nu$ for $\langle i, \nu \rangle \in \mathrm{dom}(p)$

$p \leq q \underset{Df}{\longleftrightarrow} p \supset q$ for $p, q \in \mathbb{P}^\mu_\tau$.

We also set: $\mathbb{C}^\mu = \mathbb{C}^\mu_{\omega_1}$.

The properties of this forcing are well known:

(a) $\mathbb{C}_\tau^\mu$ is ω_1 - distributive

(b) (Assume GCH) If $\mu > \omega_1$ is regular, then $\mathbb{C}_\tau^\mu$ satisfies the μ - cc (i.e. every antichain has cardinality $< \mu$),

(c) $\mathbb{C}_\tau^\mu \Vdash \bigwedge \xi < \check{\mu} \;\; \bar{\bar{\xi}} \leq \check{\omega}_1 \qquad (\tau < \mu)$

It follows that if $\mu > \omega_1^N$ is regular in N and G is $\mathbb{C}_\tau^\mu$ - generic over N, then $\omega_1^{N[G]} = \omega_1^N$ and $\omega_2^{N[G]} = \mu$.

We also note that (a), (b) are satisfied by $\mathbb{C}_\tau^\mu \times \mathbb{C}_\tau^{\mu'}$, and that $\mathbb{C}_\tau^\mu =$
$= \mathbb{C}_\alpha^\mu \times \mathbb{C}_\tau^\alpha$ for $\omega_1 \leq \tau < \alpha < \mu$.

We force with $\mathbb{C}^\kappa$, where κ is a Mahlo cardinal in N. It is known that $\square$ then becomes false in the resulting model. We improve this to:

<u>Lemma 2</u> Let κ be Mahlo in N and let G be $\mathbb{C}^\kappa$ - generic over N. Then $\mathcal{L}^*$ has no (ω, ω_2) - model in N[G].

prf. of Lemma 2.

Suppose not. Let $\mathfrak{A}$ be an (ω, ω_2)-model of $\mathcal{L}^*$ in $N[G]$. Let

$$\mathfrak{A} = \langle |\mathfrak{A}|, \in_{\mathfrak{A}}, A_{\mathfrak{A}}, B_{\mathfrak{A}}, C_{\mathfrak{A}}, \ldots \rangle.$$

We can assume w.l.o.g. that $\overline{\overline{\mathfrak{A}}} = \kappa$ and hence that $|\mathfrak{A}| \subset \kappa$ in $N[G]$.

Let $\mathfrak{A} = \mathring{\mathfrak{A}}^G$ and let $\theta > \kappa$ be regular in N s.t. $\mathring{\mathfrak{A}} \in H_\theta$ in N. Set $H = H_\theta$. Since G is $\mathbb{C}^\kappa$-generic over the ZFC$^-$ model H and the above properties of $\mathfrak{A}$ are absolute in $H[G]$, there is $p \in G$ s.t.

(1) $p \Vdash^H (\mathring{\mathfrak{A}}$ is an (ω, ω_2)-model of $\mathcal{L}^*$ $\wedge\ |\mathring{\mathfrak{A}}| \subset \check{\kappa})$

Fix Skolem functions for H and set:

$X_\alpha =$ the smallest $X \prec H$ s.t. $\alpha \cup \{p, \mathring{\mathfrak{A}}, \kappa\} \in X$

for $\alpha < \kappa$. Set:

$$C = \{\alpha < \kappa \mid \kappa \cap X_\alpha = \alpha\}.$$

Then C is cub in κ. By Mahloness there is a regular $\tau \in C$. Let $X = X_\tau$ and set: $\sigma : \overline{H} \stackrel{\sim}{\longleftrightarrow} X$, where $\overline{H}$ is transitive.

Then $\sigma : \bar{H} \prec H$, $\tau = \mathrm{crit}(\sigma)$, $\sigma(\tau) = \kappa$, and $\sigma(\mathbb{C}^{\tau}) = \mathbb{C}^{\kappa}$. Clearly:

(2) $q \Vdash^{\bar{H}}_{\mathbb{C}^{\tau}} \varphi(t_1, \dots, t_n) \longleftrightarrow$

$\longleftrightarrow q \Vdash^{H}_{\mathbb{C}^{\kappa}} \varphi(t_1, \dots, t_n)$,

for $q \in \mathbb{C}^{\tau}$, $t_1, \dots, t_n \in \bar{H}$, since $\sigma(q) = q$.

Let $\bar{G} = G \cap \mathbb{C}^{\tau}$. By (2):

(3) There is a unique $\tilde{\sigma} : \bar{H}[\bar{G}] \prec H[G]$ defined by $\tilde{\sigma}(t^{\bar{G}}) = t^{G}$.

Hence:

(4) $\tilde{\sigma} \upharpoonright \bar{H} = \sigma$, since

$\tilde{\sigma}(x) = \tilde{\sigma}(\check{x}^{\bar{G}}) = \sigma(\check{x})^{G} = \sigma(x)$ for $x \in \bar{H}$.

In particular,

(5) $\tilde{\sigma} \upharpoonright \tau = \mathrm{id}$.

By (3) we have $\tilde{\sigma} \upharpoonright \bar{\mathfrak{U}} : \bar{\mathfrak{U}} \prec \mathfrak{U}$, but $\tilde{\sigma} \upharpoonright \bar{\mathfrak{U}} = \mathrm{id}$ by (5). Hence

(6) $\bar{\mathfrak{U}} \prec \mathfrak{U}$.

Let $\bar{\mathfrak{U}} = \langle |\bar{\mathfrak{U}}|, \in_{\bar{\mathfrak{U}}}, A_{\bar{\mathfrak{U}}}, B_{\bar{\mathfrak{U}}}, C_{\bar{\mathfrak{U}}}, \dots \rangle$

(7) $\in_{\mathfrak{U}}$ is an end extension of $\in_{\bar{\mathfrak{U}}}$

prf.

Let $x \in \bar{\mathfrak{U}}$. <u>Claim</u> $\in_{\mathfrak{U}}``\{x\} = \in_{\bar{\mathfrak{U}}}``\{x\}$.

The case $\bar{\mathfrak{U}} \models x = \emptyset$ is trivial by (6).

So assume $\bar{\mathfrak{U}} \models x \neq \emptyset$. There is

$f \in \bar{\mathfrak{v}}$ s.t. $\bar{\mathfrak{v}} \models f : B \xrightarrow{\text{onto}} x$. Thus
$\tilde{f} : B_{\bar{\mathfrak{v}}} \xrightarrow{\text{onto}} \in_{\bar{\mathfrak{v}}} {}''\{x\}$, where
$\bar{\bar{B}}_{\bar{\mathfrak{v}}} = \gamma = \omega_1^N$ in $\bar{H}$. Let $g \in \bar{H}$ s.t.
$g : \gamma \xrightarrow{\text{onto}} \in_{\bar{\mathfrak{v}}} {}''\{x\}$. Then
$\tilde{\sigma}(g) : \gamma \xrightarrow{\text{onto}} \in_{\mathfrak{v}} {}''\{x\}$. But
$\tilde{\sigma}(g)(\nu) = \tilde{\sigma}(g(\nu)) = g(\nu)$ by (5). Hence
$\tilde{\sigma}(g) = g$ and $\in_{\mathfrak{v}} {}''\{x\} = \operatorname{rng}(g) = \in_{\bar{\mathfrak{v}}} {}''\{x\}$.

QED(7)

The same proof shows:
(8) $A_{\bar{\mathfrak{v}}} = A_{\mathfrak{v}}$; $B_{\bar{\mathfrak{v}}} = B_{\mathfrak{v}}$; $<_{\mathfrak{v}}$ is
an end extension of $<_{\bar{\mathfrak{v}}}$ (where
$<_{\mathfrak{v}} = \in_{\mathfrak{v}} \cap On_{\mathfrak{v}}^2$).

Thus $On_{\bar{\mathfrak{v}}}$ is an initial segment
of $On_{\mathfrak{v}}$ in $N[G]$. Moreover:
(9) $cf(On_{\bar{\mathfrak{v}}}) = cf(\tau) = \omega_1$ in $N[G]$.
Hence there is $u \in On_{\mathfrak{v}}$ s.t.
(9.1) $u = \sup On_{\bar{\mathfrak{v}}}$ in $<_{\mathfrak{v}}$.

Note W.l.o.g. we take the axiom:
$C = \langle C_\lambda \mid B < \lambda \wedge Lim(\lambda)\rangle$ is a $\square_B$-sequence.
as meaning:
(a) C_λ is cub in λ and $otp(C_\lambda) \leq B$
(b) If $\eta \in C_\lambda$ and $Lim(\eta)$, then $C_\eta = \eta \cap C_\lambda$.
(This is the usual formulation of $\square_B$ with the additional condition that, whenever $\xi \in C_\lambda$ is a successor point in C_λ, then ξ is a successor ordinal. It is trivial that if there is a $\square_B$-sequence, then there is one with the additional condition.)

Def For $x \in On_{\mathfrak{U}}$ s.t. $\mathfrak{U} \models Lim(x)$ set:
$\check{C}_x = \{y \mid \mathfrak{U} \models y \in C_x\}$. ~~In~~ particular set: $D = C_u$, where $u = \sup On_{\bar{\mathfrak{U}}}$,
Set $\Gamma = \{x \in On_{\bar{\mathfrak{U}}} \mid \bar{\mathfrak{U}} \models Lim(x)\}$
and $\tilde{x} = <_{\bar{\mathfrak{U}}} {}''\{x\}$ for $x \in On_{\bar{\mathfrak{U}}}$.
Then D has the properties:
(10) (a) $D \cap \Gamma$ is cofinal in $On_{\bar{\mathfrak{U}}}$
(b) If $x \in D \cap \Gamma$, then $D \cap \tilde{x} = C_x$.

Def Any $P \subset On_{\bar{\mathfrak{U}}}$ satisfying (10) is called devilish.

We have shown that there is a devilish set in $N[G]$. But $N[G] = N[\bar{G}][G']$ where $G' = G \cap \mathbb{C}^{\kappa}_{\tau}$ is $\mathbb{C}^{\kappa}_{\tau}$-generic over $N[\bar{G}]$. Hence there is $q \in G'$ s.t.

(11) $q \Vdash^{N[\bar{G}]}_{\mathbb{C}^{\kappa}_{\tau}}$ (there is a devilish set).

Now let $G_0 \times G_1$ be $\mathbb{C}^{\kappa}_{\tau} \times \mathbb{C}^{\kappa}_{\tau}$ - generic over $N[\bar{G}]$ with $q \in G_i$ $(i = 0,1)$. We know that $\mathbb{C}^{\kappa}_{\tau} \times \mathbb{C}^{\kappa}_{\tau}$ is ω_1-distributive. Hence ω_1 is absolute in $N[\bar{G}][G_0 \times G_1] =$ $= N[\bar{G}][G_0][G_1]$. Let $D_i \in N[\bar{G}][G_i]$ be devilish. We derive a contradiction.

We first note:

(12) $\Gamma \cap D_0 \cap D_1$ is bounded in $On_{\bar{\mathfrak{U}}}$.

proof.

Suppose not. Then

$$D_0 = D_1 = \bigcup_{x \in \Gamma \cap D_0 \cap D_1} C_x . \quad \text{Hence}$$

$$D = D_i \in N[\bar{G}][G_i] \quad \text{for } i = 0,1.$$

By the product lemma, $D \in N[\bar{G}]$.

But $\bar{\mathfrak{U}} \models \mathrm{otp}(C_x) < B$, hence $\bar{\mathfrak{U}} \models \overline{\overline{C_x}} \leq A$, for $x \in \Gamma \cap D$.

By this we get:

Claim $cf(D) \leq \omega_1$ in $<_{\bar{\mathfrak{A}}}$ (in $N[\bar{G}]$)

prf. Suppose not. Let $\langle x_i \mid i < \omega_2 \rangle$ be a monotone sequence in D. Then $\langle x_i \mid i < \omega_1 \rangle$ is bounded in $On_{\bar{\mathfrak{A}}}$ and there is $w \in On_{\bar{\mathfrak{A}}}$ s.t. $w = \sup_{i < \omega_1} x_i$ in $\leq_{\bar{\mathfrak{A}}}$ by Lemma 1. But then $x_i \in D \cap \tilde{w} = C_w$ for $i < \omega_1$. Hence $\bar{\bar{C}}_w \geq \omega_1$, where $w \in \Gamma \cap D$. Contr! QED

Since D is unbounded in $On_{\bar{\mathfrak{A}}}$ it follows that: $cf(On_{\bar{\mathfrak{A}}}) = \omega_1$ in $<_{\bar{\mathfrak{A}}}$. But in $N[\bar{G}]$ we have: $cf(On_{\bar{\mathfrak{A}}}) = \tau = \omega_2$. Contr! QED (12)

Now let $x_0 \in On_{\bar{\mathfrak{A}}}$ s.t. $(D_0 \cap D_1 \cap \Gamma) \setminus \tilde{x}_0 = \emptyset$. In $\mathfrak{A}$ we define for each $z \in D_0 \cap \Gamma$ an m_z s.t. $\bar{\mathfrak{A}} \models m_z \in \omega$, (hence $m_z \in A$). We do this as follows: Pick a $z' \in D_1 \cap \Gamma$ s.t. $z <_{\bar{\mathfrak{A}}} z'$. Arguing in $\bar{\mathfrak{A}}$, there is a sequence $\langle r^z(i) \mid i < m_z \rangle$ defined by:

$r^z(0) \simeq$ the least $r \in C_z$ s.t. $r > x_0$

$r^z(i+1) \simeq$ the least $r \in C_z$ s.t.

$\bigvee s\,(s \in C_{z'} \wedge r^z(i) < s < r)$.

Then $\bar{\mathfrak{A}} \models m_z < \omega$, since otherwise

$\bar{\sigma} \models n_z = \omega$ and there is υ s.t.
$\bar{\sigma} \models \upsilon = \sup_{i<\omega} \sigma^z(i)$. Then $\upsilon \in \Gamma \cap D_0 \cap D_{11}$
where $\upsilon > x_0$. Contr!

It is obvious that the definition of n_z does not depend on the $C_z{}'$ chosen, since $C_z{}' = \tilde{z}' \cap D_1$.

Also:

(13) $z < z'$ in $(\Gamma \cap D_0) \rightarrow n_z \leq_{\bar{\sigma}} n_{z'}$.

Moreover:

(14) If $z \in (\Gamma \cap D_0)$, there is $z' \geq_{\bar{\sigma}} z$ in $\Gamma \cap D_0$ s.t. $n_z < n_{z'}$.

prf.

Choose $w \in D_1 \cap \Gamma$ s.t. $w > x_0, z$.
Choose $z' \in D_0 \cap \Gamma$ s.t. $z' > w$.
Then $\bar{\sigma} \models \sigma^z \subsetneq \sigma^{z'}$. Hence
$\bar{\sigma} \models n_z < n_{z'}$. QED(14)

Now choose $z_\xi \in D_0 \cap \Gamma$ $(\xi < \omega_1)$ s.t. $z_\xi <_{\bar{\sigma}} z_{\xi+1}$ and $n_{z_\xi} <_{\bar{\sigma}} n_{z_{\xi+1}}$ and $z_\lambda \geq_{\bar{\sigma}} z_\xi$ for all $\xi < \lambda$ $(\lim(\lambda))$. Then $n_{z_\xi} < n_{z_{\xi'}}$ for $\xi < \xi' < \omega_1$. Hence
$\langle n_{z_\xi} \mid \xi < \omega_1 \rangle$ injects ω_1

into A_{ϖ}, where $\overline{\overline{A_{\varpi}}} = \omega$. Contr!

QED (Lemma 2)

Our main result is:

<u>Lemma 3</u> Let G be $\mathbb{C}^{\kappa}$-generic over N. There is a generic extension $N[G][H]$ of $N[G]$ s.t. in $N[G][H]$ (a) $\kappa = \omega_2$; (b) $\diamondsuit^{+}$ holds; (c) $\mathcal{L}^{*}$ has no (ω, ω_2)-model.

The proof depends on the construction of forcing conditions $\mathbb{P} \in N[G]$ s.t. (a)-(c) are forced. The actual construction of $\mathbb{P}$ will be given in §2. Here we list the salient properties of $\mathbb{P}$ and derive Lemma 3 from them. Since GCH holds in N, there is an $A_0 \in N$ s.t. $A_0 \subset \kappa$ and $L_{\tau}[A_0] = H_{\tau}$ for all cardinals τ s.t. $\omega \leq \tau \leq \kappa$; ~~Let~~ G be $\mathbb{C}$-generic over N. Set:

$A_1 = \{\langle \mu, \eta, i \rangle \mid \bigvee p \in G\ p(\eta, i) = \mu\}$.

$A = \{\langle \mu, i \rangle \mid (i = 0 \wedge \mu \in A_0) \vee (i = 1 \wedge \mu \in A_1)\}$.

Let τ be regular in N s.t. $\omega_1 \leq \tau \leq \kappa$. ~~Let~~ $\overline{G} = G \cap \mathbb{C}^{\tau}$. Then $\overline{G}$ is $\mathbb{C}^{\tau}$-generic over N and

and $N[\bar{G}] = N[A_1 \cap \tau]$. Moreover $H_\tau = L_\tau[A]$ in $N[\bar{G}]$. In particular $L_\kappa[A] = H_\kappa$ in $N[G]$.
In §2 we shall define a set of conditions $\mathbb{P} \in N[G]$ with the following properties: *)

(A) $\mathbb{P} \subset L_\kappa[A]$ is $L_\kappa[A]$-definable.

(B) Each $p \in \mathbb{P}$ is a function s.t. $\mathrm{dom}(p)$ is a countable subset of κ. Moreover, $q \leq p$ in $\mathbb{P} \longleftrightarrow q \supset p$ for $p, q \in \mathbb{P}$.

(C) $\mathbb{P}$ is ω_1-distributive

Def Let $\tau < \omega_1$. Set $\mathbb{P}^\tau = \{p \restriction \tau \mid p \in \mathbb{P}\}$ with: $p \leq q$ in $\mathbb{P}^\tau \longleftrightarrow p \supset q$.

(D) $\mathbb{P}^\tau \subset \mathbb{P} \in N$

(E) Let $p \in \mathbb{P}$, $\tau < \kappa$, $q \in \mathbb{P}^\tau$ s.t. $q \leq p \restriction \tau$ in $\mathbb{P}^\tau$. Then $q \cup p \in \mathbb{P}$. (Hence $q \cup p \leq p$ in $\mathbb{P}$.)

(F) Let $\langle L_\tau[A], A \cap \tau\rangle \prec \langle L_\kappa[A], A\rangle$ s.t. $\mathrm{cf}(\tau) = \omega_1$. Then $\mathbb{P}^\tau \subset L_\tau[A]$.

*) Note. The actual definition of $\mathbb{P}$ given in §2 refers to a $\diamondsuit$-sequence $S = \langle S_\alpha \mid \alpha < \omega_1\rangle$. We have assumed that $\diamondsuit$ holds in N. It is known that any $\diamondsuit$-sequence in N remains a $\diamondsuit$-sequence in $N[G]$. Hence we may take: $S =$ the $L_\kappa[A_0]$-least $\diamondsuit$-sequence.

By (D), (E) a standard proof gives:

(1) $\mathbb{P}$ satisfies the ω_2-CC.

Prf. of (1). Let X be a maximal antichain in $\mathbb{P}$. Define $\langle \tau_i \mid i \leq \omega_1 \rangle$ by: $\tau_0 = 0$. Given τ_i, select for each $p \in \mathbb{P}^{\tau_i}$ a $q_p \in X$ s.t. q_p is compatible with p. Let τ_{i+1} be the least $\tau \geq \tau_i$ s.t. $q_p \in \mathbb{P}^{\tau}$ for all $p \in \mathbb{P}^{\tau_i}$. For limit $\lambda \leq \omega_1$ set: $\tau_\lambda = \sup_{i<\lambda} \tau_i$.

<u>Claim</u> $X \subset \mathbb{P}^{\tau_{\omega_1}}$ (hence $\overline{\overline{X}} \leq \omega_1$)

Suppose not. Let $p \in X \setminus \mathbb{P}^{\tau_{\omega_1}}$. Then $p \restriction \tau_{\omega_1} \in \mathbb{P}^{\tau_i}$ for some $i < \omega_1$, since dom(p) is countable. By our construction there is $q \in \mathbb{P}^{\tau_{i+1}}$ s.t. $q \in X$ and q is compatible with $p \restriction \tau_{\omega_1}$. But then there is $q' \in \mathbb{P}^{\tau_{i+1}}$ s.t. $q' \leq q, p \restriction \tau_{\omega_1}$. Set $p' = p \cup q'$. Then $p' \in \mathbb{P}$ and $p' \leq q', q$. Contr! QED(1)

A further property of $\mathbb{P}$ is:

(G) $\mathbb{P} \Vdash \Diamond^+$.

Now let $\tau \leq \kappa$ and let $\bar{H}$ be $\mathbb{P}^{\tau}$-generic over $N[G]$. Define $\mathbb{P}_{\bar{H}}$ by: $\mathbb{P}_{\bar{H}} = \{ p \in \mathbb{P} \mid p \upharpoonright \tau \in \bar{H} \}$.

By (E), (D) a standard proof gives:

(2) H is $\mathbb{P}$-generic over $N[G]$ iff iff $\bar{H} = H \cap \mathbb{P}^{\tau}$ is $\mathbb{P}^{\tau}$-generic over $N[G]$ and H is $\mathbb{P}_{\bar{H}}$-generic over $N[G][\bar{H}]$ $\quad (\tau \leq \kappa)$.

proof.

($\rightarrow$) $\bar{H}$ is $\mathbb{P}^{\tau}$-generic, since if $\Delta \in N[G]$ is dense in $\mathbb{P}^{\tau}$, then $\Delta^* = \{ p \in \mathbb{P} \mid p \upharpoonright \tau \in \Delta \}$ is dense

by (E) in $\mathbb{P}$. Now let $\Delta \in N[G][\bar{H}]$ be dense in $\mathbb{P}_{\bar{H}}$. Claim $H \cap \Delta \neq \emptyset$.

Let $\Delta = \mathring{\Delta}^{\bar{H}}$, where $\mathring{\Delta} \in N[G]$. Suppose w.l.o.g. that $\mathbb{P}^{\tau} \Vdash \mathring{\Delta}$ is dense in $\mathbb{P}_{\mathring{\bar{H}}}$, where $\mathring{\bar{H}}$ is the canonical term for $\bar{H}$.

(Hence $\mathbb{P}^{\tau} \Vdash \mathring{\bar{H}}$ is $\check{\mathbb{P}}^{\tau}$-generic) and $q \Vdash \check{q} \in \mathring{\bar{H}}$ for $q \in \mathbb{P}^{\tau}$.) Set:

$\tilde{\Delta} = \{ p \in \mathbb{P} \mid p \upharpoonright \tau \Vdash \check{p} \in \mathring{\Delta} \}$. It suffices to show:

<u>Claim</u> $\tilde{\Delta}$ is dense in $\mathbb{P}$.

prf. of Claim. Let $p \in \mathbb{P}$. Let $\bar{H}$ be $\mathbb{P}^{\tau}$-generic s.t. $p \restriction \tau \in \bar{H}$. Then $p \in \mathbb{P}_{\bar{H}}$. Let $p' \leq p$ s.t. $p' \in \Delta = \mathring{\Delta}^{\bar{H}}$. Pick $q \in \bar{H}$ s.t. $q \leq p \restriction \tau, p' \restriction \tau$ and $q \Vdash_{\mathbb{P}^{\tau}} \check{p}' \in \mathring{\Delta}$. Set $p'' = q \cup p'$. Then $p'' \leq p$, $p'' \in \tilde{\Delta}$. QED($\rightarrow$)

($\leftarrow$) Let Δ be dense in $\mathbb{P}$.

Claim $\Delta \cap H \neq \emptyset$.

It suffices to show: $\Delta \cap \mathbb{P}_{\bar{H}}$ is dense in $\mathbb{P}_{\bar{H}}$. Let $p \in \mathbb{P}_{\bar{H}}$. Then $\Delta^* = \{p' \restriction \tau \mid p' \leq p \wedge p' \in \Delta\}$ is dense in $\{q \in \mathbb{P}^{\tau} \mid q \leq p \restriction \tau\}$ by (E). Hence there is $q \in \Delta^* \cap \bar{H}$. Hence there is $p' \leq p$, $p' \in \Delta$ s.t. $p' \restriction \tau = q$. Hence $p' \in \Delta \cap H$. QED(2)

We also need:

(H) Let $\bar{H}$ be $\mathbb{P}^{\tau}$-generic over $N[G]$. Then $\mathbb{P}_{\bar{H}} \times \mathbb{P}_{\bar{H}}$ is ω_1-distributive in $N[G][\bar{H}]$.

It suffices to prove:

Sublemma 3.1 Let H be $\mathbb{P}$-generic over $N' = N[G]$. There is no (ω, ω_2)-model for $\mathcal{L}^*$ in $N'[H]$.

proof. Suppose not.

By a good model of $\mathcal{L}^*$, let us understand an (ω, ω_2)-model whose elements are ordinals $< \omega_2$. We can assume w.l.o.g. that there is a good model in $N'[H]$.

But then there is a good model in $H_\theta^{N'[H]} = H_\theta^{N'}[H] = H_\theta^N[G][H]$, where $\theta > \kappa$ is regular in N.

G is then $\mathbb{C}^\kappa$-generic over $M = H_\theta^N$ and H is $\mathbb{P}$-generic over $M' = M[G]$.

Thus there is a $q \in H$ s.t.

$q \Vdash_{\mathbb{P}}^{M'}$ (there is a good model).

Hence there is $p \in G$ s.t.

(3) $p \Vdash_{\mathbb{C}^\kappa}^{M}$ "$\check{q} \Vdash_{\dot{\mathbb{P}}}^{\check{M}[\dot{G}]}$ (there is a good model)"

where $\dot{G}$ is the canonical term for G (in particular $\Vdash_{\mathbb{C}^\kappa}$ $\dot{G}$ is $\check{\mathbb{C}}^\kappa$-gen. over $\check{V}$ and $p \Vdash \check{p} \in \dot{G}$ for $p \in \mathbb{C}^\kappa$.)

In N we now define for $\alpha < \kappa$:

X_α = the smallest $X \prec H_\theta$ s.t.

$\alpha \cup \{p,$

Set: $C = \{\alpha \mid \alpha = X_\alpha \cap \kappa\}$. Then C is cub in κ and there is $\tau \in C$ which is regular. Set $X = X_\tau$ and let $\sigma : \bar{M} \xrightarrow{\sim} X$, where $\bar{M}$ is transitive. Then $\sigma : \bar{M} \prec M$, $\tau = \mathrm{crit}(\sigma)$, $\sigma(\tau) = \kappa$. Hence $\sigma \restriction L_\tau[A_0] = \mathrm{id}$, where $L_\tau[A_0] = H^N_\tau$. As before, $\sigma(\mathbb{C}^\tau) = \mathbb{C}^\kappa$. Since

(4) $p \Vdash^{\bar{M}}_{\mathbb{C}^\tau} \varphi(t_1, \dots, t_m) \longleftrightarrow p \Vdash^{M}_{\mathbb{C}^\kappa} \varphi(\sigma(t_1), \dots, \sigma(t_m))$,

σ extends to a $\tilde{\sigma} : \bar{M}[\bar{G}] \prec M[G] = M'$, (where $\bar{G} = G \cap \mathbb{C}^\tau$), defined by:

$$\tilde{\sigma}(t^{\bar{G}}) = \sigma(t)^G.$$ Set $\bar{M}' = \bar{M}[\bar{G}]$.

Then $\tilde{\sigma}(A \cap \tau) = A$ and $\tilde{\sigma} \restriction L_\tau[A] = \mathrm{id}$, where $L_\tau[A] = H^{N[\bar{G}]}_{\omega_1}$. It follows that $\tilde{\sigma}(\mathbb{P}^\tau) = \mathbb{P}$. Since:

(5) $q \Vdash^{\bar{M}'}_{\mathbb{P}^\tau} \varphi(t_1, \dots, t_m) \longleftrightarrow q \Vdash^{M'}_{\mathbb{P}} \varphi(\tilde{\sigma}(t_1), \dots, \tilde{\sigma}(t_m))$,

$\tilde{\sigma}$ extends to a $\sigma^* : \bar{M}'[\bar{H}] \prec M'[H]$ defined by $\sigma^*(t^{\bar{H}}) = \tilde{\sigma}(t)^H$ (where $\bar{H} = H \cap \mathbb{P}^\tau$).

Since (3) holds, where $p \in \bar{G}$, $q \in \bar{H}$, there is $\bar{\mathfrak{v}} \in M'[\bar{H}]$ which is a good model (in $M'[\bar{H}]$, hence in $N'[\bar{H}]$). Thus $\sigma^*(\bar{\mathfrak{v}}) = \mathfrak{v}$ is a good model in $N'[H]$.

It follows exactly as before that:

(6) $\in_{\mathfrak{v}}$ is an end extension of $\in_{\bar{\mathfrak{v}}}$;

$$A_{\mathfrak{v}} = A_{\bar{\mathfrak{v}}}\ ,\ B_{\mathfrak{v}} = B_{\bar{\mathfrak{v}}}\ .$$

As before, this implies:

(7) There is $D \in N'[H]$ which is a devilish set for $\mathfrak{v}$.

But $\bar{H}$ is $\mathbb{P}^{\tau}$-generic over $N' = N[\bar{G}][G']$, where $\bar{G} = G \cap \mathbb{C}^{\tau}$, $G' = G \cap \mathbb{C}^{\kappa}_{\tau}$. Since $\mathbb{C}^{\kappa}_{\tau}$, $\mathbb{P}^{\tau} \in N[\bar{G}]$, it follows that G' is $\mathbb{C}^{\kappa}_{\tau}$-generic over $N[\bar{G}][\bar{H}]$. We can then repeat the argument in the proof of Lemma 2 ((11)–(14)) to show:

(8) There is no devilish set in $N'[\bar{H}]$.

As before, however, there is a devilish set in $N'[H] = N'[\bar{H}][H]$,

where H is $\mathbb{P}_{\bar{H}}$ - generic over $N'[\bar{H}]$.
Hence there is $q \in H$ s.t.
$q \Vdash_{\mathbb{P}_{\bar{H}}}$ there is a devilish set.
Let $H_0 \times H_1$ be $\mathbb{P}_{\bar{H}} \times \mathbb{P}_{\bar{H}}$ - generic over $N'[\bar{H}]$ with $q \in H_i$ $(i = 0, 1)$.
Let $D_i \in N'[H_i]$ be devilish.
Since $\mathbb{P}_{\bar{H}} \times \mathbb{P}_{\bar{H}}$ is ω_1-distributive, we obtain a contradiction exactly as before, arguing in $N'[\bar{H}][H_0 \times H_1]$.

QED (Lemma 3)