This work was inspired by work of my former student, Thoralf Räsch. In the literature the "canonical counterexample" to $(\alpha, \alpha^+) \to (\omega_1, \omega_2)$ is the nonexistence of a special Aronszajn tree on ω_1. (At least I have never seen another example.) In his dissertation Räsch shows that, in fact, $(\alpha, \alpha^+) \to (\omega, \omega_1)$ can fail in the presence of such a tree. The language employed in his counter-example is the theory:

$$ZFC^- + A \text{ is the largest cardinal} +$$
$$+ 2^\alpha \leq A \text{ for all } \alpha < A,$$

where A is the designated predicate. Hence not only does CH fail in his model, but the language which says it holds has no gap 1 model. In his ground model he assumes GCH and the existence of an inaccessible. He then uses Mitchell's forcing, collapsing the inaccessible to become ω_2 while simultaneously making CH false. Unlike Mitchell, however, he does not require the inaccessible to be Mahlo. Thus, if e.g. he takes it to be the first inaccessible in L, there will still be a special Aronszajn tree in the extension. Räsch's construction carries over mutatis mutandis to any regular $\beta > \omega$.

In the present paper we consider an analogous problem: The "canonical counterexample" to $(\alpha, \alpha^{++}) \longrightarrow (\omega, \omega_2)$ is the non-existence of a Kurepa tree. We produce a model in which the gap 2 cardinal conjecture fails at ω, although $\Diamond^+_{\omega_1}$ still holds. (Hence there is a Kurepa tree.) The language used in our counter-example is the theory:

$ZFC^- + GCH + A^+$ is the largest cardinal $+ \square_{A^+}$

(This, of course, involves a predicate for the $\square_{A^+}$ sequence.) Hence in our model, not only does $\square_{\omega_1}$ fail, but the theory which says it holds has no gap 2 model at ω. In the ground model we assume GCH and the existence of a Mahlo cardinal (since $\square_{\omega_1}$ holds if ω_2 is not Mahlo in L). ~~In~~ an initial forcing we collapse the Mahlo cardinal to become ω_2, using the usual collapsing conditions. In that extension the above language $\mathcal{L}$ fails to have a model, but Kurepa's hypothesis also fails. We then perform a second extension which makes $\diamondsuit^+_{\omega_1}$ true, while still admitting no model of $\mathcal{L}$. The construction can be carried out mutatis mutandis for any regular $\beta > \omega$ in place of ω.

~~An~~ one respect our result is weaker than that of Räsch. Räsch shows that $(\alpha, \alpha^+) \to (\omega_1, \omega_2)$ can fail when ω_2 is the first inaccessible in L, which is provably the best possible result. Our construction requires that ω_2 be Mahlo in L. We can see no reason that $(\alpha, \alpha^{++}) \to (\omega, \omega_2)$ could not fail with ω_2 being the first inaccessible in L, but a counter-example would have to employ a language different from that we used.