§1 Insertions

Set: $\eta(+) = \begin{cases} \eta & \text{if } \eta \text{ is a limit ordinal} \\ \eta+1 & \text{if not} \end{cases}$

We regard $\emptyset$ as being the unique normal iteration of length $\emptyset$

Def Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration of M of length η. Let $I' = \langle\langle M'_i\rangle, \langle\nu'_i\rangle, \langle\pi'_{ij}\rangle, T'\rangle$ be a normal iteration of M of length η'.

Let $e : \eta(+) \to \eta'(+)$ be a normal function s.t. $e(0) = 0$.

e is an <u>insertion</u> of I into I' iff there is a sequence $\langle\sigma_i \mid i<\eta\rangle$ of <u>insertion maps</u> satisfying:

(a) $\sigma_0 = \mathrm{id}$, $\sigma_i : M_i \xrightarrow[\Sigma^*]{} M'_{e(i)}$

(b) $i \leq_T j \longleftrightarrow e(i) \leq_{T'} e(j)$. If $i \leq_T j$ then $\sigma_j \pi_{ij} = \pi'_{e(i),e(j)} \sigma_i$.

(c) $e(i+1) = j+1$ for a $j < \eta'$ s.t. $e(i) \leq_{T'} j$.

Define $\tilde{e} : \eta \to \eta'$ by: $\tilde{e}(i) = e(i+1) - 1$.

(Hence $\tilde{e} = \emptyset$ if $\eta = \emptyset$.) For $i < \eta$ set:

$\tilde{\pi}_i = \pi'_{e(i),\tilde{e}(i)}$, $\tilde{\sigma}_i = \tilde{\pi}_i \sigma_i$

(d) $\tilde{\pi}\sigma_i(\nu_i) = \nu'_{\tilde{e}(i)}$. More precisely, one of the following holds:

- $\nu_i \in M_i \wedge \sigma_i(\nu_i) \in \mathrm{dom}(\tilde{\pi}_i) \wedge \nu'_{\tilde{e}(i)} = \tilde{\sigma}_i(\nu_i)$
- $\nu_i \in M_i \wedge \mathrm{dom}(\tilde{\pi}_i) = M'_{e(i)} \| \sigma_i(\nu_i) \wedge \nu'_{\tilde{e}(i)} = On \cap M'_{\tilde{e}(i)}$
- $\nu_i \notin M_i \wedge \mathrm{dom}(\tilde{\pi}_i) = M'_{e(i)} \wedge \nu'_{\tilde{e}(i)} = On \cap On \cap M'_{\tilde{e}(i)}$

(e) $\sigma_i \upharpoonright \lambda_l = \tilde{\sigma}_l \upharpoonright \lambda_l$ for $l < i < \eta$

Def The identical insertion is $\mathrm{id} \restriction \eta(+)$ with $\sigma_i = \mathrm{id} \restriction M_i$.

Note The map σ_i is total on M_i, although $\tilde{\sigma}_i$ can be partial.

Note e is definable from $\tilde{e} : \eta \to \eta'$ by:
$$e(i) = \mathrm{lub}\{\tilde{e}(h) \mid h < i\} \text{ for } i \leq \eta(+).$$

Note The insertion maps $\langle \sigma_i \mid i < \eta \rangle$ are uniquely determined by e, but we have yet to prove this fact.

Note $\tilde{e}$ is order preserving, since:
$$h < i \longrightarrow h+1 \leq i \longrightarrow e(h+1) = \tilde{e}(h)+1 \leq e(i) \longrightarrow$$
$$\longrightarrow \tilde{e}(h) < e(i) \leq \tilde{e}(i).$$

Note We shall often write e_i for $e(i)$

Remark We use by and large the notation of [NFS] and [FSIM] in dealing with normal iteration. If
$$I = \langle \langle M_i \mid i < \mu \rangle, \langle \nu_i \mid i+1 < \mu \rangle, \langle \pi_{ij} \mid i \leq_T j < \mu \rangle, T \rangle$$
is an iteration, then $T(i+1)$ is the immediate predecessor of $i+1$ in T. We then have:
$$\pi_{\tau_i, i+1} : M_i^* \longrightarrow_F^* M_{i+1},$$
where $M_i^* = M_h \| \eta$ and η is maximal s.t. τ_i is a cardinal in $M_h \| \eta$. If $M_i^* \neq M_h$, we call $i+1$ a truncation point. We also sometimes write i^* for τ_i.

Lemma 1 The following hold.

(1) $h \leq_T i \leftrightarrow e_h \leq_{T'} e_i$

prf. By (b)

(2) $\tilde{\sigma}_i \upharpoonright \lambda_h = \tilde{\sigma}_h \upharpoonright \lambda_h$ for $h \leq i$, $i+1 < \gamma$.

proof

Trivial for $h = i$. Let $h < i$. Then $\mathrm{crit}(\tilde{\pi}_i) \geq \lambda'_l$ for $l < e_i$. Hence $\mathrm{crit}(\tilde{\pi}_i) \geq \lambda_{\tilde{e}_h} = \tilde{\sigma}_h(\lambda_h)$, and $\tilde{\sigma}_i \upharpoonright \lambda_h = \tilde{\pi}_i \sigma_i \upharpoonright \lambda_h = \tilde{\pi}_i \tilde{\sigma}_h \upharpoonright \lambda_h = \tilde{\sigma}_h \upharpoonright \lambda_h$.

QED (2)

(3) Let $\zeta = T(i+1)$. Then $\kappa'_{\tilde{e}_i} < \lambda'_{\tilde{e}_\zeta}$

proof

$\kappa'_{\tilde{e}_i} = \tilde{\sigma}_i(\kappa_i) \in \tilde{\sigma}_i{}''\lambda_\zeta = \tilde{\sigma}_\zeta{}''\lambda_\zeta \subset \tilde{\sigma}_\zeta(\lambda_\zeta) = \lambda'_{\tilde{e}_\zeta}$.

(4) Let $\zeta = T(i+1)$. Then $\sigma_\zeta(\kappa_i) \geq \lambda'_j$ for $j < e_\zeta$.

proof

$e_\zeta = \mathrm{lub}_{j<\zeta}\, \tilde{e}_j$. Hence it suffices to show:

$\sigma_\zeta(\kappa_i) \geq \lambda'_{\tilde{e}_j}$ for $j < \zeta$. But $\kappa_i \geq \lambda_j$ for $j < \zeta$. Hence $\sigma_\zeta(\kappa_i) \geq \sigma_\zeta(\lambda_j) = \tilde{\sigma}_j(\lambda_j) = \lambda_{\tilde{e}_j}$ QED (4)

Note Let $\zeta = T(i+1)$. Then $e_{i+1} = \tilde{e}_{i+1} + 1$, and (3), (4) tell us that $T'(e_{i+1})$ lies between e_ζ and $\tilde{e}_\zeta$. The full determination of $T'(e_{i+1})$ is as follows:

Let $\xi = T(i+1)$. Then:

$\kappa'_{\tilde{e}_i} = \tilde{\sigma}_i(\kappa_i) = \tilde{\sigma}_\xi(\kappa_i) = \tilde{\pi}_\xi \sigma_\xi(\kappa_i)$ where $\tilde{\pi}_\xi = \pi'_{e_\xi, \tilde{e}_\xi}$.

But then $\kappa'_{\tilde{e}_i} < \lambda'_{\tilde{e}_i} = \tilde{\sigma}_i(\lambda_i)$. Hence there is a least j s.t. $e_\xi \leq_{T'} j \leq_{T'} \tilde{e}_\xi$ and

$\pi'_{e_\xi, j}(\sigma_\xi(\kappa_i)) = \kappa'_{\tilde{e}_i} < \lambda'_j$. But then:

$\kappa'_{\tilde{e}_i} < \mathrm{crit}(\pi'_{j, \tilde{e}_\xi})$, since if $j < \tilde{e}_\xi$ and

$\kappa = \mathrm{crit}(\pi'_{j, \tilde{e}_\xi})$ it is easily seen that

$\pi'_{j, \tilde{e}_\xi}(\kappa) \geq \lambda'_j$, where $\pi'_{j, \tilde{e}_\xi}(\kappa'_{\tilde{e}_i}) = \kappa'_{\tilde{e}_i} < \lambda'_j$.

Then:

(5) Let ξ, i, j be as above. Then $j = T'(\tilde{e}_i + 1)$.

proof.

$\kappa'_{\tilde{e}_i} < \lambda'_j$ is given. We claim:

<u>Claim</u> $\kappa'_{\tilde{e}_i} \geq \lambda'_h$ for $h < j$.

At $j = e_\xi$, this is immediate by (4). Now let $j > e_\xi$. Then $j = \mathrm{lub}\, A$, where

$A = \{h \mid e_\xi \leq_{T'} h+1 \leq_{T'} j\}$.

Hence it suffices to show:

<u>Claim</u> $\kappa'_{\tilde{e}_i} \geq \lambda'_h$ for $h \in A$.

Suppose not. Let $h \in A$ be least s.t. $\kappa'_{\tilde{e}_i} < \lambda'_h$.

Let $\tau = T'(h+1)$. Then $e_\xi \leq_{T'} \tau$. Then

$\mathrm{rng}(\pi_{e_\xi, h+1}) \subset \mathrm{rng}(\pi_{\tau, h+1})$.

But $\mu'_{\tilde{e}_i} \in \mathrm{rng}(\pi'_{e_\xi, h+1})$. Hence $\mu'_{\tilde{e}_i} \notin [\mu_h, \lambda_h)$, since $\mathrm{rng}(\pi'_{\tau, h+1}) \cap [\mu_h, \lambda_h) = \emptyset$. Hence $\mu'_{\tilde{e}_i} < \mu_h < \lambda_\tau$ and $\pi'_{\tau, \tilde{e}_i}(\mu'_{\tilde{e}_i}) = \mu'_{\tilde{e}_i}$.

Hence $\pi'_{e_i, \tau}(\sigma_i(\mu_i)) = \mu'_{\tilde{e}_i}$. Contradiction!, since $\tau < j$. QED (5)

<u>Def</u> $i^* = \xi$, $e_i^* = j$ where $\xi = T(i+1)$ and j is defined as above. We set:

$$\pi_i^* = \pi'_{e_i, e_i^*}, \quad \sigma_i^* = \pi_i^* \sigma_i .$$

Then:

(6) $M'^{*}_{\tilde{e}_i} = M'_{e_i^*} \| \mu$ where μ is maximal s.t. $\tau_{\tilde{e}_i}$ is a cardinal in $M'_{e_i^*} \| \mu$.

(7) $\sigma_i^* \upharpoonright M_i^* : M_i^* \xrightarrow[\Sigma^*]{} M'^{*}_{\tilde{e}_i}$, where:

(8) If $M_i^* \in M_{i^*}$, then $\sigma_i(M_i^*) \in M'_{e_i}$ and $\pi_i^* \upharpoonright \sigma_i(M_i^*) : \sigma(M_i^*) \xrightarrow[\Sigma^*]{} M'^{*}_{\tilde{e}_i}$

(<u>Note</u> We <u>cannot</u> conclude that $\sigma_i^*(M_i^*) \in M'_{e_i^*}$.)

(9) Let $M_i^* = M_{i^*}$. Then $M'_{e_i^*} = M'^{*}_{\tilde{e}_i}$ and $\pi_i^* : M'_{e_\xi} \xrightarrow[\Sigma^*]{} M'^{*}_{\tilde{e}_i}$

By (8), (9) we have:

(10) Let $h \leq_T j$. Then $\pi_{h,j}$ is total on M_h iff π'_{e_h, e_j} is total on M'_{e_h}

proof.

Case 1 $h = j$ (trivial)

Case 2 $i \leq_T j = i+1$ is a successor ordinal.

Let $\xi = i^* = T(j)$. Then $h \leq_T \xi$ and $e_h \leq_{T'} e_\xi$.
The conclusion follows by the induction hypothesis and (8), (9).

Case 3 $i \leq_T j$ and j is a limit ordinal.
Pick l s.t. $h \leq_T l \leq_T j$, $\pi_{l,j}$ is total on M_l,
and $\pi'_{e_l e_j}$ is total on M'_{e_l}. The conclusion follows
by the induction hypothesis applied to l.

QED (10) *)

(11) $\tilde{\sigma}_i(x) = \sigma_i^*(x)$ for $x \in \mathbb{P}(\kappa_i) \cap M_i^*$.

proof
$\tilde{\sigma}_i(x) = \tilde{\sigma}_{i^*}(x) = \tilde{\pi}_i \sigma_{i^*}(x) = \pi_i^* \sigma_{i^*}(x) = \sigma_i^*(x)$
since $i^* = T(i+1) \leq i$ and $x \in J^{E^{M_i}}_{\tau_i}$ and
$\tau_i < \lambda_{i^*}$.

QED (11)

*) Note Taking $h = T(j)$, $j = i+1$ in (10),
we can conclude:
$i+1$ is a truncation point in I iff
$e_{i+1} = \tilde{e}_i + 1$ is a truncation point in I'.

<u>Note</u> We write $\pi : M \xrightarrow[F]{} M'$ to mean that M' is the ultrapower of M by the extender F with canonical embedding π. If $n \le \omega$, we write $\pi : M \xrightarrow[F]{(n)} M'$ to mean that $\mathrm{crit}(F) < \rho^n_M$ and M' is the $\Sigma_0^{(n)}$-ultrapower of M. $\pi : M \xrightarrow[F]{*} M$ mean that $\pi : M \xrightarrow[F]{(n)} M'$ where n is maximal s.t. $\mathrm{crit}(F) < \rho^n_M$ and F is close to M in Steel's sense. If this holds, then π is Σ^*-preserving. All of these notions are defined in [FSIM] as are the notions: $\langle \sigma, g \rangle : \langle M, F \rangle \longrightarrow \langle M', F' \rangle$ and $\langle \sigma, g \rangle : \langle M, F \rangle \xrightarrow{*} \langle M', F' \rangle$, which we shall shortly use.

(12) $\langle \sigma_i^* \restriction M_i^*, \tilde{\sigma}_i \restriction \lambda_i \rangle : \langle M_i^*, F \rangle \longrightarrow \langle M'^*_{\tilde{e}_i}, F' \rangle$ where $F = E^{M_i}_{\nu_i}$, $F' = E^{M'_{\tilde{e}_i}}_{\nu'_{\tilde{e}_i}}$.

proof.

$\alpha \in F(X) \longleftrightarrow \tilde{\sigma}_i(\alpha) \in \tilde{\sigma}_i(F(X)) = F'(\tilde{\sigma}_i(X)) = F'(\sigma_i^*(X))$ by (11).

QED (12)

(13) $\sigma_i \pi_{\xi^*, i+1} = \pi'_{e_i^*, \tilde{e}_i+1} \cdot \sigma_i^*$

proof.

Let $\xi = i^* = T(i+1)$. Then:

$\sigma_{i+1} \pi_{\xi, i+1} = \pi'_{e_\xi, \tilde{e}_i+1} \sigma_\xi = \pi'_{e_i^*, \tilde{e}_i+1} \pi_i^* \sigma_\xi =$

$= \pi_{e_i^* \tilde{e}_i+1} \sigma_i^*$. QED(13)

We are now in a position to prove that the sequence $\langle \sigma_i \mid i < \eta \rangle$ of insertion maps is uniquely determined by e. Let $\langle \sigma_i' \mid i < \eta \rangle$ be a second such sequence. By induction on i we prove: $\sigma_i = \sigma_i'$. If $i = 0$, then $\sigma_0 = \sigma_0' = \mathrm{id}$. Now let $\sigma_h = \sigma_h'$ for $h \le i$,

We prove: $\sigma_{i+1} = \sigma_{i+1}'$.

Let $n \le \omega$ be least s.t. $\kappa_i < \rho^n_{M_i^*}$.

Then $\pi_{i^*, i+1} : M_i^* \xrightarrow[F]{(n)} M_{i+1}$ and

$\pi'_{e_i^*, \tilde{e}_i+1} : M'^*_{\tilde{e}_i} \xrightarrow[F']{(n)} M'_{\tilde{e}_i+1}$

By (12), however, there is exactly one $\sigma : M_{i+1} \xrightarrow[\Sigma_0^{(n)}]{} M'_{\tilde{e}_i+1}$ s.t.

(14) $\sigma(\pi_{i^*, i+1}(f)(a)) = \pi'_{e_i^*, \tilde{e}_i+1}(\sigma_i^*(f))(\tilde{\sigma}_i(a))$

for $f \in \Gamma^{(n)}(M_i^*, \kappa_i)$.

By (13), σ_{i+1} satisfies this condition.

Hence σ_{i+1} is uniquely determined.

Finally, let $\mu < \eta$ be a limit ordinal and let $\sigma_i = \sigma_i'$ for $i < \mu$. Then σ_μ is the unique $\sigma : M_\mu \to M'_{e_\mu}$ s.t. $\sigma \pi_{i,\mu} = \pi'_{e_i, e_\mu} \sigma_i$ for $i < \mu$.

We have thus shown:

(15) $\langle \sigma_i \mid i < \eta \rangle$ is uniquely determined by $\langle I, I', e \rangle$.

<u>Note</u> The notions $\Gamma(M, \kappa)$, $\Gamma^{(n)}(M, \kappa)$ and $\Gamma^*(M, \kappa)$ are defined in [FSIM].

Lemma 2 Let I, I', e be as above, let $A \subset \tau_\eta$ be $\Sigma_1(M_\eta \| \nu_\eta)$ in p and let $A' \subset \tau'_{\tilde{e}_\eta}$ be $\Sigma_1(M'_{\tilde{e}_\eta} \| \nu'_{\tilde{e}_\eta})$ in $\tilde{\sigma}_\eta(p)$ by the same definition.
Then A is $\Sigma_1(M^*_\eta)$ in a q and A' is $\Sigma_1(M'^*_{\tilde{e}_\eta})$ in $\sigma^*_\eta(q)$ by the same definition.
proof
Suppose not. Let I be a counterexample of length $\eta+2$ with η chosen minimally.
Let $\xi = T(\eta+1)$.

(1) $\xi < \eta$
proof Suppose not, Hence $\eta = \xi$.
Then $M^*_\eta = M_\eta \| \mu$ where $\mu \geq \nu$. Let A be $\Sigma_1(M_\eta \| \nu_\eta)$ in p. Then A is $\Sigma_1(M^*_\eta)$ in a q. Let A' be $\Sigma_1(M'_{\tilde{e}_\eta} \| \nu'_{\tilde{e}_\eta})$ in $\tilde{\sigma}_\eta(p)$ by the same definition. Then A' is $\Sigma_1(M'_{\tilde{e}_\eta} \| \tilde{\sigma}_\eta(\mu))$ in $\tilde{\sigma}_\eta(q)$. But:

$$\pi'_{\xi^*, \tilde{e}_\eta} : M'^*_{\tilde{e}_\eta} \xrightarrow[\Sigma^*]{} M'_{\tilde{e}_\eta} \| \tilde{\sigma}_\eta(\mu)'$$

and $\mathrm{crit}(\pi'_{\xi^*, \tilde{e}_\eta}) > \tau'_{\tilde{e}_\eta}$. Hence A' is $\Sigma_1(M'^*_{\tilde{e}_\eta})$ in $\sigma^*_\eta(q)$, since $\tilde{\sigma}_\eta(q) = \pi_{\xi^*, \tilde{e}_\eta} \sigma^*(q)$.

QED(1)

(2) $\nu_\eta = On_{M_\eta}$ and $\rho^1_{M_\eta} \leq \tau_\eta$

proof.
Suppose not. Then as before $A \in J^{E^{M_\eta}}_{\lambda_\xi}$ and $J^{E^{M_\eta}}_{\lambda_\xi} = J^{E^{M^*_\eta}}_{\lambda_\xi} \subset M^*_\eta$. !

If A is $\Sigma_1(M_\eta \| \nu_\eta)$ in p, then obviously $A' = \tilde{\sigma}_\eta(A)$ is $\Sigma_1(M'_{\tilde{e}_\eta} \| \nu_{\tilde{e}_\eta})$ in $\tilde{\sigma}_\eta(p)$ by the same definition. We again have $\mathrm{crit}(\pi'_{e^*_\eta, \tilde{e}_\eta}) > \tau'_{\tilde{e}_\eta}$. Hence A is $\Sigma_1(M^*_\eta)$ in the parameter A and A' is $\Sigma_1(M^{**}_{\tilde{e}_\eta})$ in $A' = \tilde{\sigma}_\eta(A) = \sigma^*_\eta(A)$ by the same definition. QED (2)

(3) η is not a limit ordinal.

Suppose not. Let A, A', p, p' be as above. By (1) we have: $\xi < \eta$, where $\xi = T(\eta+1)$.

By (2), $M_\eta = M_\eta \| \nu_\eta$ is active and $\tilde{\sigma}_\eta(\nu_\eta) = \nu'_{\tilde{e}_\eta}$.

Hence $M'_{\tilde{e}_\eta} = M'_{\tilde{e}_\eta} \| \nu'_\eta$ and the map

$\tilde{\pi}_\eta = \pi'_{e_\eta, \tilde{e}_\eta}$ is total on M'_{e_η}.

Pick $l \leq_T \eta$ s.t.

- $\mathrm{crit}(\pi_{l,\eta}) > \lambda_\xi$
- $\pi_{l,\eta}$ is total on M_l
- $p \in \mathrm{rng}(\pi_{l,\eta})$.

Set: $\bar{p} = \pi_{l,\eta}^{-1}(p)$.

Then A is $\Sigma_1(M_l)$ in $\bar{p}$ by the same definition as it is $\Sigma_1(M_\eta)$ in p.

Define a potential iteration $\bar{I}$ of length $l+2$ extending $I | l+1$ by stipulating:

$\bar{\nu}_l = \pi_{l,\eta}^{-1}(\nu_\eta)$. Then $\bar{M}_l = M_l \| \bar{\nu}_l$. Since $\pi_{l,\eta}(\kappa_\eta) = \kappa_\eta$, it follows easily that:

$\bar{M}^*_l = M^*_\xi$. Define $\bar{e} : l+2 \to \eta'$ by:

$\bar{e} \upharpoonright l+1 = e | l+1$, $\bar{e}_{l+1} = \tilde{e}_\eta + 1$.

Then $\langle \sigma_i \mid i \le \ell \rangle$ is the insertion of $\bar{I}$ into I' given by $\langle \sigma, \vec{e} \rangle$. Then A is $\Sigma_1(\bar{M}_\ell)$ in $\bar{p}$ and A' is $\Sigma_1(\bar{M}'_{\tilde{e}_\ell})$ in $p' = \tilde{\sigma}_\ell(\bar{p})$ by the same definition. By the minimality of η we conclude that A is $\Sigma_1(\bar{M}^*_\ell)$ in q and A' is $\Sigma_1(M'^*_{\tilde{e}_\ell})$ in $q' = \tilde{\sigma}^*_\ell(q)$ by the same definition.

Clearly $\bar{\kappa}_\ell = \kappa_\eta$. Using this we get:

$\bar{M}^*_\ell = M^*_\eta$, $\bar{\pi}^*_\ell = \pi^*_\eta$, $\bar{\sigma}^*_\ell = \sigma^*_\eta$. Hence:

A is $\Sigma_1(M^*_\eta)$ in q and A' is $\Sigma_1(M'^*_{\tilde{e}_\eta})$ by the same definition.

Contradiction. QED (3)

Now let $\eta = j+1$, $h = T(\eta)$. (Recall $\xi = T(\eta+1)$)

Then $\pi_{h,\eta} \upharpoonright M^*_j : M^*_j \longrightarrow_{\Sigma_1} M_\eta = \langle J^E_{\nu_\eta}, E_{\nu_\eta} \rangle$.

Hence:

(4) $M^*_j = \langle J^{\bar{E}}_{\bar{\nu}}, \bar{E}_{\bar{\nu}} \rangle$ where $\bar{E}_{\bar{\nu}} \ne \emptyset$

(5) $\tau_\eta < \kappa_j$

proof

$\tau_\eta < \lambda_j$, since $\tau_\eta = \kappa_\eta^{+M_\eta}$ and $\kappa_\eta < \lambda_h \le \lambda_j$, where λ_j is inaccessible in M_η. But $\kappa_\eta, \tau_\eta \in \operatorname{rng}(\pi_{h,\eta})$ by (4), where $[\kappa_j, \lambda_j) \cap \operatorname{rng}(\pi_{h,\eta}) = \emptyset$. QED (5)

(6) $\rho^1_{M^*_j} \leq \tau_\eta$ (Hence $\pi_{h,\eta} : M^*_j \xrightarrow[E_{\nu_j}]{} M_\eta$ is a Σ_0 ultrapower)

prf.

Suppose not. Then $\pi_{h,\eta} {}'' \rho^1_{M^*_j} \subset \rho^1_{M_\eta}$. Hence

$\tau_\eta = \pi_{h,\eta}(\tau_\eta) < \rho^1_{M_\eta}$ by (5), contradicting (2)

QED(6)

(Note (6) says that $\bigwedge x^1\, x^1 \neq \tau_\eta$ in M^*_j, which is a $\Pi_1^{(1)}$ condition on τ_η.)

(7) $\rho^1_{M'^*_{\tilde{e}_j}} \leq \tau'_{\tilde{e}_\eta} = \sigma^*_j(\tau_\eta)$.

proof

Since $\sigma^* \restriction M^*_j : M^*_j \xrightarrow[\Sigma^*]{} M'^*_{k_j}$, we have:

$$\rho^1_{M'^*_j} \leq \sigma^*_j(\tau_\eta),$$

But, since $\tau_\eta < \kappa_j < \lambda_h$ we have:

$$\tau'_{\tilde{e}_\eta} = \tilde{\sigma}_\eta(\tau_\eta) = \tilde{\sigma}_h(\tau_h) = \pi'_{e^*_j \tilde{e}_h} \sigma^*_j(\tau_\eta) = \sigma^*_j(\tau_\eta)$$

since $\mathrm{crit}(\pi_{k^*_i, \tilde{k}_h}) > \sigma^*_j(\kappa_j) > \sigma^*_j(\tau_\eta)$. QED(7)

Recall that $\eta = j+1$ and $e_\eta = \tilde{e}_j + 1$.

Moreover $e^*_j = T'(\tilde{e}_j + 1)$, where $\kappa_{\tilde{e}_j} = \sigma^*_j(\kappa_j)$ is the critical point of $\pi'_{e^*_j, \tilde{e}_j+1}$. Hence by (7) and $\sigma^*_j(\kappa_j) > \sigma^*_j(\tau_\eta)$ we have:

(8) $\pi_{\tilde{e}^*_j, e_\eta} : M'^*_j \xrightarrow[E_{\nu'_j}]{} M'_\eta$ is a Σ_0-ultrapower.

We recall that A is $\Sigma_1(M_\eta)$ in a parameter p and A' is $\Sigma_1(M'_{\tilde{\eta}})$ in $p' = \tilde{\sigma}_\eta(p)$. Moreover, $\tilde{\pi}_\eta = \pi'_{e_\eta, \tilde{e}_\eta}$ is total on $M'_{e_\eta} = M'_{\tilde{e}+1}$, since $M'_{\tilde{e}_\eta} = M'_{\tilde{e}_\eta} \| \tilde{\sigma}_\eta(\nu_\eta)$. Crit $(\pi'_{e_\eta, \tilde{e}_\eta}) \geq \lambda'_{\tilde{e}_j}$, where $\lambda'_{\tilde{e}_j} > \kappa'_{\tilde{e}_j} > \tau'_{\tilde{e}_\eta} = \tilde{\sigma}^*_j(\tau_\eta)$; Hence

A' is $\Sigma_1(M'_{k_\eta})$ in $p'' = \sigma_\eta(p) = \tilde{\pi}_\eta^{-1}(p')$ by the same definition. Using this we prove:

(a) A is $\Sigma_1(M_j^*)$ in a q and A' is $\Sigma_1(M'^*_{\tilde{e}_j})$ in $\sigma_j^*(q)$ by the same definition.

proof

Let $p = \pi_{h,\eta}(f)(\alpha)$, where $f \in M_j^*$ and $\alpha < \lambda_j$. Then $p = \pi_{e_j^*, e_\eta}(f')(\alpha')$, where $f' = \sigma_j^*(f)$ and $\alpha' = \tilde{\sigma}_j(\alpha)$.

Let $F = E_{\nu_j}^{M_j}$, $F' = E^{M'_{\tilde{e}_j}}_{\nu_{\tilde{e}_j}}$. By the minimality of η we know that F_α is $\Sigma_1(M_j^*)$ in a parameter a and $F'_{\alpha'}$ is $\Sigma_1(M'^*_{\tilde{e}_j})$ in $a' = \sigma_j^*(a)$ by the same definition.

The map $\pi = \pi_{h,\eta} : M_j^* \to M_\eta$ is cofinal by (2). But then, letting:

$$A(\xi) \leftrightarrow \bigvee y\, B(\xi, y, p)$$

where B is Σ_0, we have:

$$A(\xi) \leftrightarrow \bigvee u \in M_j^* \; \bigvee y \in \pi(u)\; B(\xi, y, \pi(f)(\alpha))$$
$$\leftrightarrow \quad " \quad \{\gamma < \kappa_j \mid \bigvee y \in u\; B(\xi, y, f(\gamma))\} \in F_\alpha$$

Hence A is $\Sigma_1(M_j^*)$ in $q = \langle a, f\rangle$.

Similarly $\pi' = \pi_{e_j^*, e_\eta} : M'^*_{e_j^*} \to M'_{e_\eta}$ is cofinal by (8) and, letting:

$$A'(\xi) \leftrightarrow \bigvee y\, B'(\xi, y, p''),\; f' = \sigma_j^*(f)$$

we have:

$$A'(\xi) \leftrightarrow \bigvee u \in M'^*_{\tilde{h}_j} \; \{\gamma < \kappa'_{\tilde{h}_j} \mid \bigvee y \in u\, B(\xi, y, f'(\gamma))\} \in F'_{\alpha'}$$

by the same argument. Hence A' is $\Sigma_1(M'^*_{\tilde{e}_j})$ in $q' = \langle a', f'\rangle = \sigma_j^*(q)$ by the same definition. QED (9)

Now extend $I|h+1$ to a potential iteration $\bar{I}$ of length $h+2$ by setting: $\bar{\nu}_h = On \cap M_j^* = \pi_{h,\eta}^{-1}(\nu_\eta)$. Extend $I'|k^*+1$ to $\bar{I}'$ of length $h+2$ by setting: $\bar{\nu}'_{e_j^*} = On \cap M'^*_{\tilde{e}_j} = \pi'^{-1}_{\tilde{e}_j, e_\eta}(\nu'_k)$ (recall: $e_\eta = \tilde{e}_j + 1$). If we define $\bar{e}$ by: $\bar{e}_i = e_i$ for $i \le h$, $\bar{e}_{h+1} = e_j^* + 1$, it is easily seen that $\bar{e}$ is a potential insertion of I into I',

But $\bar{M}_h \| \bar{\nu}_h = M_j^*$ and $\bar{M}'_{\tilde{\bar{e}}_h} \| \bar{\nu}'_{\tilde{\bar{e}}_h} = M'^*_{\tilde{\bar{e}}_j}$, where $\tilde{\bar{e}}_h = \overset{*}{e}_j$. Since $\bar{\kappa}_h = \kappa_\eta$ and $\bar{\kappa}_{\tilde{\bar{e}}_h} = \sigma_j^*(\kappa_\eta) = \kappa'_{\tilde{e}_\eta}$, we have:

$\bar{M}_h^* = M_\eta^*$; $\bar{M}'^*_{\tilde{\bar{e}}_h} = M'^*_{e_\eta} = M'^*_{\tilde{e}_\eta}$. Since A is $\Sigma_1(\bar{M}_h^*)$ in q and A' is $\Sigma_1(M'_{\tilde{\bar{e}}_h})$ in $q' = \sigma_j^*(q) = \tilde{\bar{\sigma}}_h(q)$ by the same def, we can conclude by the minimality of j that A is $\Sigma_1(\bar{M}_\eta^*)$ in an r and A' is $\Sigma_1(M'^*_{\tilde{h}_\eta})$ by the same definition.

QED (Lemma 2)

Extending insertions

By Lemma 1 we have:

Lemma 3 Let e be a potential insertion of I into I', where $lh(I) = \eta+2$, $lh(I') = \eta'+2$.

Extend I, I'

$$\pi = \pi_{\eta^*, \eta+1} : M^*_\eta \xrightarrow[E_{\nu_\eta}]{*} M_{\eta+2}$$

$$\pi' = \pi'_{\eta'^*, \eta'+1} : M'^*_{\eta'} \xrightarrow[E'_{\nu'_{\eta'}}]{*} M'_{\eta'+2}$$

If we define:

$$\tilde{\sigma}_{\eta+1}(\pi(f)(\alpha)) = \pi'(\tilde{\sigma}^*_\eta(f))(\tilde{\sigma}_\eta(\alpha))$$

for $f \in \Gamma^*(\kappa_\eta, M^*_\eta)$, $\alpha < \lambda_\eta$,

Extend e by setting: $\hat{e}_{\eta+2} = \tilde{e}_\eta + 2 = e_{\eta+1}+1$.

Then $\hat{e}$ inserts $\hat{I}$ into $\hat{I}'$ with insertion maps $\langle \sigma_i \mid i \le \eta+1 \rangle$.

Note The $\hat{e}$ defined above is minimal at $\eta+1$ in the sense:

Def Let e insert I into I', $i < lub(lh(I))$. e is minimal at i iff $e_{i+1} = e_i + 1$ (i.e. $e_i = \tilde{e}_i$).

We obviously have:

<u>Lemma 4</u> Let $lh(I) = \eta + 1$ and let e insert I into I'. If e is minimal at η and $e_\eta \underset{T'}{\leq} \xi'$, then is an insertion of I into I', where $f(i) = e(i)$ for $i < \eta$ and $f(\eta) = \eta'$. (Hence $\tilde{e}_\eta = \eta'$.)

<u>Lemma 5</u> Let e be an insertion of I into I', where $lh(I) = \eta$, $lh(I') = \eta'$, $e_\eta = \eta'$. Let $\nu \in M_\eta$ s.t. $\nu > \nu_i$ for $i < \eta$. We can then extend I, I' to potential iterations $\hat{I}, \hat{I}'$ by setting $\hat{\nu}_\eta = \nu$, $\hat{\nu}'_{\eta'} = \sigma_\eta(\nu)$.

Finally:

<u>Lemma 6</u> Let $lh(I) = \eta$, $lh(I') = \eta'$ and let e be an insertion of I into I', where η is limit ordinal and $\eta' = \sup_{i<\eta} e_i$. Let b be a well founded cofinal branch in I. Let $b' = \{i < \eta' \mid \bigvee j \in b \; i \underset{T'}{\leq} j\}$, be a well founded branch in I'. If we extend I to $\hat{I}$ and I' to $\hat{I}'$ by setting $T''\{\eta\} = b$, $T'''\{\eta'\} = b'$, then e extends uniquely to an insertion of $\hat{I}$ into $\hat{I}'$ with $e(\eta) = \eta'$ and $\sigma_\eta \pi_{i\eta} = \pi'_{e_i e_\eta} \sigma_i$.

Composing insertions

Lemma 7 Let e insert I into I' and f insert I' into I''. Then fe inserts I into I''. Moreover, $\widetilde{fe} = \tilde{f}\tilde{e}$ and $\sigma_i^{fe} = \sigma_{e_i}^{f} \circ \sigma_i^{e}$.

proof.

We define: $\sigma_i^{fe} = \sigma_{e_i}^{f} \cdot \sigma_i^{e}$ and verify that (a)–(e) hold.

Trivially

(a) $\sigma_0^{fe} = id$, $\sigma_i^{fe} : M_i \xrightarrow[\Sigma]{} M_i''$

(b) ~~Ab~~ $i \leq_T i'$, then

hence $fe(i) \leq fe(i')$. But then

$\sigma_{i'}^{fe} \pi_{ii'} = \sigma_{e_{i'}}^{f} \sigma_{i'}^{e} \pi_{ii'} = \sigma_{e_{i'}}^{f} \pi_{e_i e_{i'}} \sigma_i^{e} =$

$= \pi''_{fe(i), fe(i')} \sigma_{e_i}^{f} \sigma_i^{e}$. QED

(c) $fe(i+1) = f(\tilde{e}_i + 1) = \tilde{f}\tilde{e}(i) + 1$

(~~Hence~~ $\widetilde{fe} = \tilde{f}\tilde{e}$.) We have:

$\tilde{\pi}_i^{fe} = \pi''_{fe(i), \tilde{f}\tilde{e}(i)} =$

$= \pi''_{f\tilde{e}(i), \tilde{f}\tilde{e}(i)} \cdot \pi''_{fe(i), f\tilde{e}(i)}$,

Note that $\pi''_{f\tilde{e}(i), \tilde{f}\tilde{e}(i)} = \tilde{\pi}^{f}_{\tilde{e}(i)}$.

and $\pi'_{e(i), \tilde{e}(i)} = \tilde{\pi}_i^{e}$.

Thus:

$$\tilde{\sigma}_i^{fe} = \tilde{\pi}_i^{fe}\, \sigma_i^{fe} = \tilde{\pi}^f_{\tilde{e}(i)} \cdot \pi''_{fe(i), f\tilde{e}(i)} \cdot \sigma_i^{fe}$$

$$= \tilde{\pi}^f_{\tilde{e}(i)} \cdot \pi''_{fe(i), f\tilde{e}(i)} \cdot \sigma^f_{e_i} \cdot \sigma^e_i =$$

$$= \quad " \quad , \sigma^f_{\tilde{e}(i)}\, \tilde{\pi}_i\, \sigma^e_i = \tilde{\sigma}^f_{\tilde{e}(i)} \cdot \tilde{\sigma}^e_i$$

Thus $\tilde{\sigma}_i^{fe} = \tilde{\sigma}^f_{\tilde{e}(i)} \cdot \tilde{\sigma}^e_i$.

This proves (d), since:

$$\tilde{\sigma}_i^{fe}(\nu_i) = \sigma^f_{\tilde{e}(i)}\, \tilde{\sigma}^e_i(\nu_i) = \sigma^f_{\tilde{e}(i)}(\nu'_{\tilde{e}_i}) = \nu''_{f\tilde{e}(i)}.$$

(e) Let $i > l$, $\xi < \lambda_l$. Then

$$\sigma_i^{fe}(\xi) = \sigma^f_{e_i}\, \sigma^e_i(\xi) = \sigma^f_{e_i}\, \tilde{\sigma}^e_l(\xi),$$

But $\tilde{\sigma}^e_l(\xi) < \tilde{\sigma}^e_l(\lambda_l) = \lambda'_{\tilde{e}(l)}$.

Hence:

$$\sigma_i^{fe}(\xi) = \sigma^f_{e_i}\, \tilde{\sigma}^e_l(\xi) = \tilde{\sigma}^f_{\tilde{e}(l)} \cdot \tilde{\sigma}^e_l(\xi) =$$

$$= \tilde{\sigma}^{fe}_l(\xi).$$ QED (Lemma 7)

Now let I^{ξ} be an iterate of M for $\xi \in \Gamma$, where $e^{\xi,\mu}$ inserts I^{ξ} into I^{μ} for $\xi \le \mu$ (We take $e^{\xi,\xi} = id$).

Let $e^{\xi\mu} = e^{\zeta\mu} e^{\xi\zeta}$ for $\xi \le \zeta \le \mu < \Gamma$

Let $\langle \sigma_i^{\xi\mu} \mid i < lh(I^{\xi}) \rangle$ be the sequence of insertion maps given by $e^{\xi,\mu}$, (with $\sigma_i^{\xi,\xi} = id$). Then for $\mu \le \xi \le \zeta$; $i < lh(I^{\mu})$

- $\tilde{e}^{\xi,\zeta} \tilde{e}^{\mu,\xi} = \tilde{e}^{\mu,\zeta}$
- $\sigma_i^{\mu\zeta} = \sigma^{\xi,\zeta}_{e_i^{\mu,\xi}} \cdot \sigma_i^{\mu,\xi}$
- $\tilde{\sigma}_i^{\mu\zeta} = \tilde{\sigma}^{\xi\zeta}_{\tilde{e}_i^{\mu,\xi}} \cdot \tilde{\sigma}_i^{\mu\xi}$

Suppose that Γ is a limit ordinal.

<u>Def</u> By a <u>good</u> <u>limit</u> of

$$\langle I^{\xi} \mid \xi < \Gamma \rangle, \quad \langle e^{\xi\mu} \mid \xi \le \mu < \Gamma \rangle$$

we mean $\langle I, \langle e^{\xi} \mid \xi < \Gamma \rangle \rangle$ s.t.

- e^{ξ} inserts I^{ξ} into I
- $e^{\mu} \cdot e^{\xi\mu} = e^{\xi}$ for $\xi \le \mu < \Gamma$
- If $i < lh(I)$, then $i = \tilde{e}^{\xi}(h)$, for some $\xi < \Gamma$, $h < lh(I^{\xi})$.

It is easily seen that $\langle I, \langle e^{\xi} \mid \xi < \Gamma \rangle \rangle$ is uniquely determined by $\langle I^{\xi} \rangle$, $\langle e^{\mu,\xi} \rangle$ if it exists.

When does a good limit exist?

Each $\tilde{e}^{\xi\mu}$ is $<$-preserving. Let $<_\xi = \in \cap \eta_\xi^2$ where $\eta_\xi = lh(I^\xi)$. Let $\langle \eta, <_\eta \rangle, \langle \tilde{e}^\xi \mid \xi < \Gamma \rangle$ be any direct limit of $\langle \langle \eta_\xi, <_\xi \rangle \mid \xi < \Gamma \rangle, \langle \tilde{e}^{\xi\mu} \mid \xi \leq \mu < \Gamma \rangle$.

If $<_\eta$ is ill founded, then obviously no good limit exists.

Assume that $<_\eta$ is well founded. We assume w.l.o.g. that η is an ordinal and $<_\eta = \in \cap \eta^2$.

Def For $\xi < \Gamma$, $i \leq \eta_\xi(+)$ set:

$$e^\xi(i) =: \mathrm{lub}\{\tilde{e}^\xi(h) \mid h < i\}$$

It follows easily that:

- $e^\xi : \eta^\xi(+1) \longrightarrow \eta(+)$ is a normal function
- $e^\xi e^{\zeta\xi} = e^\zeta$ for $\zeta \leq \xi < \Gamma$
- $e^\xi(0) = 0$; $e^\xi(i+1) = \tilde{e}^\xi + 1$.

(which defines $\tilde{e}^\xi$ from e^ξ).

<u>Lemma 8</u> Let $h < \eta_\xi$. For sufficient $\mu \geq \xi$ we have:
$e^\xi(h) = \tilde{e}^\mu e^{\xi,\mu}(h)$.

proof.

Since $e^\xi(h) < \eta$ there is μ s.t. $\xi \leq \mu < \Gamma$ and
$e^\xi(h) = \tilde{e}^\mu(j)$ where $j < \eta_\mu$. Let $k = e^{\xi,\mu}(h)$.
Then $e^\mu(k) = e^\xi(h)$.

<u>Claim</u> $k = j$

Suppose not.

<u>Case 1</u> $k > j$.

Then $j \leq \tilde{e}^{\xi,\mu}(l)$ for an $l < h$, since
$e^{\xi,\mu}(h) = \text{lub}\{\tilde{e}^{\xi,\mu}(l) \mid l < h\}$. Hence
$\tilde{e}^\mu(j) \leq \tilde{e}^\xi(l) < \sigma^\xi(h) = \tilde{e}^\mu(j)$ Contr!

<u>Case 2</u> $k < j$

Then $\tilde{e}^\xi(h) = \tilde{e}^\mu(k) < \tilde{e}^\mu(j) = e^\xi(h) \leq \tilde{e}^\xi(h)$

Contradiction! QED (Lemma 8).

Def Let $0 < \gamma < \eta$, $\xi < \Gamma$. Set:

$\eta_\xi^{(\gamma)} = \{ i < \eta_\xi \mid \tilde{e}_\xi(i) < \gamma \}$, $\tilde{e}^\xi_{(\gamma)} = \tilde{e}^\xi \mid \eta_\xi^{(\gamma)}$,

$e^\xi_{(\gamma)} = e^\xi \upharpoonright \mathrm{lub}(\eta_\xi^{(\gamma)})$, $I^\xi_{(\gamma)} = I^\xi \mid \eta_\xi^{(\gamma)}$.

For $\xi \leq \mu < \Gamma$ set:

$\tilde{e}^{\xi,\mu}_{(\gamma)} = \tilde{e}^{\xi,\mu} \upharpoonright \eta_\xi^{(\gamma)}$, $e^{\xi,\mu}_{(\gamma)} = e^{\xi,\mu} \upharpoonright \mathrm{lub}(\eta_\xi^{(\gamma)})$

Then $e^{\xi,\mu}_{(\gamma)}$ is an insertion of $I^\xi_{(\gamma)}$ into

$I^\mu_{(\gamma)}$ with $\tilde{e}^{\xi,\mu}_{(\gamma)}(i) = e^{\xi,\mu}_{(\gamma)}(i+1) - 1$ and

$e^{\xi,\mu}_{(\gamma)}(i) = \mathrm{lub}\{ \tilde{e}^{\xi,\mu}_{(\gamma)}(h) \mid h < i \}$ for $i \leq \mathrm{lub}(\eta_\xi^{(\gamma)})$

(Except that $\eta_\xi^{(\gamma)} = \emptyset$ is possible. In this case $e^{\xi,\mu}_{(\gamma)} = \tilde{e}^{\xi,\mu}_{(\gamma)} = \emptyset$.)

Note that by Lemma 8, for sufficient $\xi < \Gamma$ we have $\tilde{e}^\xi(0) = e^\xi(0) = 0$, hence $\eta_\xi^{(\gamma)} \neq \emptyset$.)

We now try inductively to show that there is a good limit $\langle I_{(\gamma)}, \langle e^\xi_{(\gamma)} \mid \xi < \Gamma \rangle\rangle$ of $\langle I^\xi_{(\gamma)} \mid \xi < \Gamma \rangle$, $\langle e^{\xi,\mu}_{(\gamma)} \mid \xi \leq \mu < \Gamma \rangle$.

($I_{(\gamma)}$ is then uniquely determined, if it exists, and $\mathrm{lh}(I_{(\gamma)}) = \gamma$. If $I_{(\gamma)}$ exists for all γ, ~~and~~ $I = \bigcup_{\gamma < \eta} I_{(\gamma)}$, then $\langle I, e^\xi \mid \xi < \Gamma \rangle$ is the good limit of $\langle I^\xi \mid \xi < \Gamma \rangle$, $\langle e^{\xi,\mu} \mid \xi \leq \mu < \Gamma \rangle$.

Without further assumptions it does not follow that $I^{(\gamma)}$ always exists. We shall try to determine necessary and sufficient condition for its existence.

Case 1 $\gamma = 1$

Then $I_{(\gamma)} = \langle M \rangle$ and $I^{\zeta}_{(\gamma)} = \langle M \rangle$ if $\eta^{\zeta}_{(\gamma)} > 0$ (hence $\eta^{\zeta}_{(0)} = 1$). We have

$\sigma^{\zeta}_{(\gamma) 0} = \mathrm{id} \upharpoonright M.$

We now try to show that if $I_{(\gamma)}$ exists and $\gamma < \eta$, then $I_{(\gamma+1)}$ exists.

Case 2 $\gamma = \beta + 1$.

Set $\nu_\beta = \tilde{\sigma}^{\zeta}_{\beta}(\nu_{\beta}^{M^{\zeta}_{\beta}})$ for $\zeta < \Gamma$ s.t. $\tilde{e}^{\zeta}(\bar{\beta}) = \beta$.

This extends $I_{(\gamma)}$ to a potential it $I^{+}_{(\gamma)}$ of length $\gamma + 1$. Similarly $\nu_{\bar\beta}^{M^{\zeta}_{\bar\beta}}$ extends $I^{\zeta}_{(\bar\gamma)}$ to a potential it $I^{\zeta +}_{(\bar\gamma)}$ of length $\bar\gamma + 1$, where $\bar\gamma = \bar\beta + 1$. $e^{\zeta} \upharpoonright \bar\gamma + 1$ is then a potential insertion of $I^{\zeta +}_{(\bar\gamma)}$ into $I^{+}_{(\gamma)}$.

By our extension lemmas, $e^{\zeta} | \bar\gamma + 1 = e^{\zeta}_{(\gamma+1)}$ inserts $I^{\zeta}_{(\bar\gamma+1)}$ into $I_{(\gamma+1)}$. This holds for all such ζ, $\bar\gamma = \bar\gamma_{\zeta}$. Moreover

$e^{\mu}_{(\gamma)} \cdot e^{\zeta,\mu} \upharpoonright \bar\gamma + 1 = e^{\zeta}_{(\gamma)}$ when $\zeta < \mu < \Gamma$.

Finally we note that $\gamma = \tilde{e}^{\zeta}(\bar\gamma)$,

QED (Case 2)

<u>Case 3</u> γ is a limit ordinal.
In this case the existence of $I(\gamma)$ is not given. However, we can formulate a simple criterion for its existence on the assumption that M is <u>uniquely</u> <u>iterable</u> (i.e. M satisfies UBH and is normally iterable.). In this case $I_{(\gamma)}$ has a unique cofinal well founded branch b. Hence the only thing $I_{(\gamma+1)}$ could be is I' obtained from $I_{(\gamma)}$ by setting $b = T``\{\gamma\}$.

<u>Claim</u> I' is a good limit iff whenever $\xi < \Gamma$, $\tilde{e}^{\xi}(\bar{\gamma}) = \gamma$ and $h \leq_{T^{\xi}} \bar{\gamma}$, then $e^{\xi}(h) \in b$. *)

proof.
The necessity of this condition is obvious. To see that it is sufficient, note that it enables us to define the insertion maps $\tilde{\sigma}^{\xi}_{\bar{\gamma}} : M^{\xi}_{\bar{\gamma}} \xrightarrow{\Sigma^*} M_b$ as follows:
If $\tilde{e}^{\xi}(\bar{\gamma}) \neq e^{\xi}(\gamma)$, then $e^{\xi}(\gamma) \in b$ and:
$\tilde{\sigma}^{\xi}_{\bar{\gamma}} = \pi_{e^{\xi}(\gamma),b} \cdot \sigma^{\xi}_{\bar{\gamma}}$.
If $\tilde{e}(\bar{\gamma}) = e^{\xi}(\gamma)$, then $\bar{\gamma}$ is a limit ordinal.
Let $\bar{b} = T^{\xi}``\{\bar{\gamma}\}$. Then $e^{\xi}``\bar{b}$ lies cofinally in b and we define $\tilde{\sigma}^{\xi}_{\bar{\gamma}}$ by:
$\tilde{\sigma}^{\xi}_{\bar{\gamma}} \pi^{\xi}_{i,\bar{\gamma}} = \pi'_{e^{\xi}(i),b} \cdot \sigma^{\xi}_{i}$ for $i \in \bar{b}$.

QED (Claim)

*) <u>Note</u> It suffices that this condition holds for cofinally many $\xi < \gamma$.

Unique Iterability

The *unique branch hypothesis* (UBH) for a premouse M says that any normal iteration of M of limit length has at most one cofinal well founded branch. Hence, for such an M, there is only one possible iteration strategy: Pick the unique cofinal well founded branch. We say that M is *uniquely iterable* if it is normally iterable and satisfies UBH. Unique iterability is the assumption used in the last proof and, unless otherwise specified, will continue to be throughout the rest of this paper. However, it can be replaced by a less restrictive assumption:

Def Let M be a premouse and Σ an iteration strategy for M. Σ is *insertion invariant* wrt. M iff whenever I, I' are normal iterations of M, $e: I \longrightarrow I'$ is an insertion of I into I', and I' is by Σ, then I is by Σ.

Def M is *invariantly* iterable iff it is successfully normally iterable by an insertion invariant strategy Σ.

If, instead of assuming unique iterability, we assume that all normal iterations considered are by an insertion invariant strategy Σ, then the argument in Case 3 above still goes through, taking b as the unique cofinal branch in $I_{(\gamma)}$ given by Σ.

<u>Note</u> We are deliberately being slightly careless with the notion of unique iterability, since in practice one often uses the relativized notion:

<u>Def</u> M is <u>uniquely</u> γ-<u>iterable</u> iff M is normally γ-iterable and satisfies UBH for iterations of length $< \gamma$.

The notion of <u>invariant</u> γ-iterability is defined similarly.

We shall generally state our theorems as though the assumption is unique ∞-iterability, since the relativization to appropriate γ is, in most cases, obvious. (Where it is not, we shall be more specific.)