§3 Mirrors

We here make use of the theory of pseudo projecta as developed in NFS §4. We assume all of NFS §9 up to and including the proof of Lemma 4. We assume, in particular, the def. of:

$M \models \varphi(x_1, \ldots, x_n) \bmod \rho$ where $\rho = \langle \rho_i \mid i < \omega \rangle$ is a sequence of pseudo projecta;

$\pi : M \xrightarrow[\Sigma^*]{} N \bmod \rho$; $\rho = \min \rho'$;

$\pi : M \xrightarrow[\Sigma^*]{} N \min \rho$.

We prove a generalization of the construction used to prove Lemma 5 in NFS §9.

<u>Def</u> Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration of M. By a <u>mirror</u> of I we mean a sequence $I' = \langle\langle M_i'\rangle, \langle\pi_{ij}'\rangle, \langle\sigma_i\rangle, \langle\rho^i\rangle\rangle$ of the same length s.t.

(a) $\sigma_i : M_i \xrightarrow[\Sigma^*]{} M_i'$ mod ρ^i, where M_i' is a premouse.

(b) π_{ij}' is defined for $i \leq_T j$ and is a partial map of M_i' to M_j'. Moreover $\sigma_j \pi_{ij} = \pi_{ij}' \sigma_i$.

Set : $\nu_i' = \sigma_i(\nu_i) =: \begin{cases} \sigma_i(\nu_i) \text{ if } \nu_i \in M_i \\ On \cap M_i' \text{ if not} \end{cases}$ for $i+1 < lh(I)$

<u>Note</u> Let $F = E^{M_i}_{\nu_i}$, $F' = E^{M_i'}_{\nu_i'}$. Since $\sigma_i : M_i \xrightarrow[\Sigma_0]{} M_i'$ and M_i' is a premouse, we have:

$F' \neq \emptyset$ and : $\xi \in F(X) \leftrightarrow \sigma_i(\xi) \in F'(\sigma_i(X))$

Set : $\kappa_i' = crit(F')$, $\lambda_i' = \lambda(F') =: F'(\kappa_i')$,

$\tau_i' = \tau(F') = \tau^{+M_i' \| \nu_i'}$.

(c) Let $h = T(i+1)$. Then $\sigma_h \upharpoonright \tau_i + 1 = \sigma_i \upharpoonright \tau_i + 1$.

Hence $\mathbb{P}(\kappa_i') \cap M_i' = \mathbb{P}(\kappa_i') \cap M_i'^*$ where :

Set : $M_i'^* = M_h' \| \mu$ where μ is maximal s.t. τ_i' is a cardinal in $M_h' \| \mu$. (Thus $M_i'^* = \sigma_i(M_i^*)$ if $M_i^* \in M_h$. If not, then $M_i'^* = M_h'$, since ρ_0^h is cardinally absolute in M_h'.)

(d) Let $h = T(i+1)$. Then $\pi_{h,i+1}' : M_i'^* \xrightarrow[\Sigma^*]{} M_{i+1}'$ s.t. $\kappa_i' = crit(\pi_{h,i+1}')$ and

$$\xi \in \pi_{h,i+1}'(X) \longleftrightarrow \xi \in F'(X)$$

for $\xi < \lambda_i'$, $X \in \mathbb{P}(\kappa_i') \cap M_i'^* = \mathbb{P}(\kappa_i') \cap M_i' \| \nu_i'$.

(e) The π'_{ij} commute ($\pi'_{ij}\pi'_{hi} = \pi''_{hj}$) and for limit μ: $M'_\mu = \bigcup_{i T \mu} \operatorname{rng}(\pi'_{i\mu})$.

Note It follows that π'_{ij} is a total function on M_i & is Σ^*-preserving, if no $h \in (i,j]_T$ is a truncation point. Moreover $\operatorname{crit}(\pi'_{ij}) = \kappa'_l$, where $i = T(l+1)$, $l \leq_T j$.

Note By (d) we have:

$$\langle \sigma_h \upharpoonright M^*_i, \sigma_i \upharpoonright \lambda_i \rangle : \langle M^*_i, F \rangle \longrightarrow \langle M^{*\prime}_i, F' \rangle$$

where $F = E^{M_i}_{\nu_i}$, $F' = E^{M'_i}_{\nu'_i}$.

Note It follows inductively that:

- ν'_h is a cardinal in M_i for $h < i$.
- $\sigma_i \upharpoonright \lambda_h = \sigma_h \upharpoonright \lambda_h$ for $h \leq i$,
- $\kappa'_i \geq \lambda'_h$ for $h < i$,

(Hence $\pi'_{ij} \upharpoonright \lambda'_h = \mathrm{id}$ for $h < i$)

(For all of this we only need $\sigma_i : M_i \longrightarrow_{\Sigma_0} M'_i$.)

(f) If π_{ij} is total on M_i, then
$$\pi'_{ij}{}''\rho^i_n \subset \rho^j_n \leq \pi'_{ij}(\rho^i_n) \quad \text{for } n<\omega.$$

(g) Let $h = T(i+1)$, where $i+1$ is a truncation point. Set: $\rho^{*i} = \min(\langle \rho^n_{M^*_i} \mid i<\omega\rangle)$. Then $\pi'_{h,i+1}{}''\rho^{*i}_n \subset \rho^{i+1}_n \leq \pi'_{h,i+1}(\rho^{*i}_n)$ for $n<\omega$.

(h) If $\mu < lh(I)$ is a limit, then for all $n<\omega$, we have:
$$\rho^{\mu}_n = \rho^i_n \quad \text{for sufficiently large } i<\mu.$$

This defines the concept of mirror.

Note: By (7) we have:

$$\langle \sigma_h \upharpoonright M^*_i, \sigma_i \upharpoonright \lambda_i\rangle : \langle M^*_i, F\rangle \longrightarrow \langle M'^*_i, F'\rangle$$

where $h = T(i+1)$ as in (7).

<u>Lemma 1</u> Let I be of limit length η. Let I' be a mirror of I. Let b be a cofinal branch in T which is well founded wrt. I' — i.e.

$$\langle M_i' \mid i \in b \rangle, \langle \pi_{ij}' \mid i \leq j \text{ in } b \rangle$$

has a well founded (hence transitive) limit. There are unique $\hat{I}, \hat{I}'$ of length $\eta+1$ s.t. $\hat{I}$ extends I, $\hat{I}'$ extends I', $\hat{T}''\{\eta\} = b$ and $\hat{I}'$ is a mirror of $\hat{I}$.

proof.

b is obviously a well founded branch in I. This gives us $\hat{I}$. But then we have

$$\hat{M}_\eta', \langle \hat{\pi}_{i\eta}' \mid i \leq_T \eta \rangle.$$

There is a unique $\sigma_\eta : \hat{M}_\eta \to \hat{M}_\eta'$ s.t. $\sigma_\eta \pi_{i\eta} = \pi_{i\eta}' \sigma_i$ for $i \leq_T \eta$. We must define ρ^μ s.t.

$$\sigma_\eta : \hat{M}_\eta \xrightarrow[\Sigma^*]{} \hat{M}_\eta' \text{ min } \rho^\mu.$$

We first note:

(1) ~~Let~~ $n < \omega$. Then ρ_n^{i} stabilizes at some $i \leq_T \mu$ (i.e. if $i \leq_T i' \leq_T \mu$ then $\rho_n^{i'} = \pi_{ii'}'(\rho_n^{i})$).

pf. of (1)

Suppose not. Pick i s.t. there is no truncation pt. j with $i \leq_T j \leq_T \eta$.

Then there are j_n s.t. $i = j_0$ and $j_n \lneq_T j_{n+1} \lneq_T \eta$ with:

$$\pi_{j_n j_{n+1}}(\rho_n^{j_n}) > \rho_n^{j_{n+1}},$$

Hence $\pi_{j_n \eta}(\rho_n^{j_n}) > \pi_{j_{n+1} \eta}(\rho_n^{j_{n+1}})$

for $n < \omega$. Contr!

If we then set:

$\rho_n^u =: \rho_n^i$ if ρ_n^h stabilizes at i

we easily get:

(2) $\sigma_\eta : \hat{M}_\eta \longrightarrow_{\Sigma^*} \hat{M}_\eta'$ min ρ^u.

QED (Lemma 1)

Lemma 2 Let I be an iteration of length $\eta+1$. Let I' be a mirror of I. Extend I to a potential it. of length $\eta+2$ by appointing a suitable ν_η. Set $\nu'_\eta = \sigma_\eta(\nu_\eta)$. This gives $M^*_\eta, M'^*_\eta, \varkappa_\eta, \varkappa'_\eta, \tau_\eta, \tau'_\eta, \lambda_\eta, \lambda'_\eta$, and $\rho^{*\eta}$. Let $\xi = T(\eta+1)$ in $\hat{I}$. Let

$$(*) \quad \pi' = \pi'_{\xi,\eta+1} : M'^*_\eta \xrightarrow[\Sigma^*]{} M'_{\eta+1}$$

s.t. $\zeta \in F'(X) \leftrightarrow \zeta \in \pi'(X)$ for $\zeta < \lambda'_\eta$,

$$F = E^{M_\eta}_{\nu_\eta}, \quad F' = E^{M'_\eta}_{\nu'_\eta}.$$

(This extends $\langle\langle M'_i\rangle, \langle \pi'_{ij}\rangle\rangle$ to length $\eta+2$.) Extend I to an it. of length $\eta+2$ by:

$$\pi_{\xi,\eta+1} : M^*_\xi \xrightarrow[F]{*} M_{\eta+1}.$$

Then there exist $\sigma_{\eta+1}, \rho^{n+1}$ s.t.

$$\sigma_{\eta+1} : M_{\eta+1} \xrightarrow[\Sigma^*]{} M'_{\eta+1} \ \min(\rho^{n+1})$$

and for all $n<\omega$:

$$\pi'_{\xi,\eta+1} \rho^{*,\eta}_{M} \subset \rho^{\eta+1}_{n} \leq \pi_{\xi,\eta+1}(\rho^{*,\eta}_{m}),$$

(Here $\rho^{*,\eta} = \rho^\xi$ if $M^*_\eta = M_\xi$; otherwise $\rho^{*,\eta} = \min(\langle \rho^n_{M^*_\eta} \mid n<\omega\rangle)$.)

(Hence the mirror pair $\langle I, I' \rangle$ extends to a mirror pair $\langle \hat{I}, \hat{I}' \rangle$ of length $\eta + 2$.

If we have a mirror pair $\langle I, I' \rangle$ of length $\eta+1$ and we extend both by appointing ν_η, $\nu'_\eta = \sigma_\eta(\nu_\eta)$, we call the resulting pair a <u>potential mirror pair</u> of length $\eta + 2$.

Lemma 2 follows by NFS §9 Lemma 4 from:

<u>Lemma 3</u> Let $\langle I, I' \rangle$ be a potential mirror pair of length $\eta + 2$. Let $\xi = T(\eta+1)$. Set:

$$\rho^* = \begin{cases} \rho^\xi & \text{if } M^*_\eta = M_\xi \\ \min(\langle \rho^i_{M^*_\xi} \mid i < \omega \rangle) & \text{if } M^*_\eta \in M_\xi \end{cases}$$

Then:

$$\langle \sigma_\xi \restriction M^*_\eta, \sigma_\eta \restriction \lambda_\eta \rangle : \langle M^*_\eta, F \rangle \xrightarrow{**} \langle M'^*_\eta \| \rho^*_0, F' \rangle$$

where $F = E^{M_\eta}_{\nu_\eta}$, $F' = E^{M'_\eta}_{\nu'_\eta}$.

We derive Lemma 3 from an even stronger lemma. We first define:

Def Let M be acceptable. Let $\kappa \in M$ be inaccessible in M s.t. $\mathbb{P}(\kappa) \cap M \in M$. $A \subset \mathbb{P}(\kappa) \cap M$ is strongly $\Sigma_1(M)$ in the parameter p iff there is $B \subset M$ s.t. B is $\Sigma_0(M)$ and:

- $x \in A \leftrightarrow \bigvee z\, B(z, x, p)$
- If $v \in M$ s.t. $v \subset \mathbb{P}(\kappa)$ and $\overline{\overline{v}}^M \leq \kappa$, then

$$\bigvee u \in M \bigwedge x \in v \bigvee z \in u\, (B(z, x, p) \vee B(z, \kappa \setminus x, p))$$

We prove:

Lemma 4 Let $\langle I, I' \rangle, \eta, \xi, p^*$ be as in Lemma 3. Let $A \subset \mathbb{P}(\kappa_\eta)$ be strongly $\Sigma_1(M_\eta \| \nu_\eta)$ in p. Let $A' \subset \mathbb{P}(\kappa'_\eta)$ be $\Sigma_1(M'_\eta \| \nu'_\eta)$ in $p' = \sigma_\eta(p)$, by the same def. Then there is $q \in M^*_\eta$ s.t.

- A is strongly $\Sigma_1(M^*_\eta)$ in q
- Let A'' be $\Sigma_1(M^*_\eta)$ in $q' = \sigma_\xi(q)$ by the same Σ_1 definition. Then $A'' \subset A'$.

Before proving Lemma 4 we show that it implies Lemma 3.

Lemma 5 Let $\langle I, I' \rangle, \eta, \xi, \rho^*$ etc. satisfy Lemma 4. Then:

(a) $\langle \sigma_\xi \upharpoonright M^*_\xi, \sigma_\xi \upharpoonright \lambda_\eta \rangle : \langle M^*_\xi, F \rangle \xrightarrow{**} \langle M'^*_\eta, F' \rangle$.

(b) $\langle \sigma_\xi \upharpoonright M^*_\eta, \sigma_\xi \upharpoonright \lambda_\eta \rangle : \langle M^*_\xi, F \rangle \xrightarrow{**} \langle M^*_\eta | \rho^*_0, F' \rangle$.

proof

We first prove (a). Let $\alpha < \lambda_\eta$; $\alpha' = \sigma_\xi(\alpha)$. Then F_α is strongly $\Sigma_1(M_\eta || \nu_\eta)$ in α, since:

$$X \in F_\alpha \longleftrightarrow \bigvee Y\, (Y = F(X) \wedge \alpha \in Y)$$

where for all $u \in M_\xi || \nu_\eta$ s.t. $u \subset \mathbb{P}(\kappa_\eta)$ and $\bar{\bar{u}} < \kappa$ in $M_\eta || \nu_\eta$ we have:

$$\bigvee u \in M_\eta || \nu_\eta \wedge X \in u \bigvee Y \in u\, ((Y = F(X) \wedge \alpha \in Y) \vee$$
$$\vee (Y = F(\kappa_\eta \setminus X) \wedge \alpha \in Y))$$

Note that $F'_{\alpha'}$ is $\Sigma_1(M'_\eta || \nu'_\eta)$ in α' by the same definition. By our assumption there is $q \in M^*_\eta$ s.t.

- $\bar{G} = F_\alpha$ is strongly $\Sigma_1(M^*_\eta)$ in q
- Let G be $\Sigma_1(M'^*_\eta)$ in $q' = \sigma_\xi(q)$ by the same definition. Then $G \subset F'_{\alpha}$.

Let $X \in \bar{G} \longleftrightarrow \bigvee z\, \bar{B}(z, X, q)$, where $\bar{B}$ is $\Sigma_0(M^*_\xi)$ and verifies that $\bar{G}$ is strongly $\Sigma_1(M^*_\xi)$ in q. Thus, if we define:

$$X \in \bar{H} \leftrightarrow (X : \kappa_\eta \to \mathbb{P}(\kappa_\eta)) \wedge \bigvee u \bigwedge i < \kappa \bigvee z \in u$$

$$(B(z, X, q) \vee B(z, \kappa_\eta \setminus X, q)),$$

in M_η^*, then $\bar{H} = ({}^{\kappa_\eta}\mathbb{B}(\kappa_\eta)) \cap M_\eta^*$. If H have the same def. in $M_\eta^{\prime *}$, then obviously:

$$X \in H \to (X \in G \vee \kappa_\eta \setminus X \in G),$$

This proves (a).

To prove (b) note that if we define G' over $M_\eta^{\prime *}|\rho_0^*$ in q' as G was defined over $M_\eta^{\prime *}$ in q, then obviously:

$$G' \subset G \subset F_d',$$

If we then define H' over $M_\eta^{\prime *}|\rho_0^*$ in q' as H was defined over $M_\eta^{\prime *}$, then

$$H' \subset H \subset {}^{\kappa_\eta'}\mathbb{P}(\kappa_\eta');$$

and:

$$X \in H' \to X \in G' \vee \kappa_\eta' \setminus X \in G'$$

as before. This proves (b).

QED (Lemma 5)

We now turn to the proof of Lemma 4.
Suppose not, Let η be the least counterexample.
We again have fixed ν_η and $\nu'_\eta = \sigma_\xi(\nu_\eta)$, which gives us $\kappa_\eta, \kappa'_\xi, \tau_\xi, \tau'_\xi, \lambda_\eta, \lambda'_\eta$, $\xi = T(\eta+1)$, M^*_ξ, M'^*_η and p^*.

(1) $\xi < \eta$

proof

Suppose not. ~~Let~~ $A \subset \mathbb{P}(\kappa_\xi)$ be strongly $\Sigma_1(M_\eta \| \nu_\eta)$ in p and let $A' \subset \mathbb{P}(\kappa'_\xi)$ be $\Sigma_1(M'_\eta \| \nu'_\eta)$ in $p' = \sigma_\xi(p)$ by the same definition. Clearly τ_η is a cardinal in $M_\eta \| \nu_\eta$, so $M^*_\xi = M_\eta \| \mu$ for a $\mu \geq \nu_\eta$. Similarly $M'^*_\eta = M'_\eta \| \mu'$ where:

$$\mu' = \begin{cases} \sigma_\xi(\mu) & \text{if } \mu \in M_\eta \\ On \cap M_\eta & \text{if not,} \end{cases}$$

Now suppose $\nu_\eta \in M^*_\eta$ (i.e. $\mu > \nu_\xi$). Then $A \in M^*_\xi$ and $A' \in M'^*_\eta$ where $\sigma_\xi(A) = A'$. Then A is trivially strongly $\Sigma_1(M^*_\xi)$ in the parameter A and A' is $\Sigma_1(M'^*_\eta)$ in $A' = \sigma_\xi(A)$ by the

same definition, where $A' \subset A'$.
Contradiction!
Now let $M^*_\eta = M_\eta \| \nu_\eta$. Then $M'^*_\eta = M'_\eta \| \nu'_\eta$ and A' is $\Sigma_1(M'^*_\eta)$ definable in $p' = \sigma_\xi(p)$ by the same definition. But A is strongly $\Sigma_1(M^*_\eta)$ in p, since $M^*_\eta = M_\eta \| \nu_\eta$.
Contradiction! QED (1)

(2) $\nu_\eta = On \cap M_\eta$

proof

Suppose not. Then $\lambda_\xi > \tau_\eta$ is inaccessible in M_η.
Hence $A \in J^{EM_\eta}_{\lambda_\xi} = J^{EM_\xi}_{\lambda_\xi} \subset M^*_\eta$.
Similarly $A' \in J^{EM'_\eta}_{\lambda'_\xi} = J^{EM'_\xi}_{\lambda'_\xi} \subset M'^*_\eta \,|\, \rho^*_0$.
Thus A is strongly $\Sigma_1(M^*_\xi)$ in the parameter A and A' is $\Sigma_1(M'^*_\eta)$ in $A' = \sigma_\xi(A)$ by the same definition.
Contradiction! QED (2)

(3) $\tau_\eta \geq \rho^1_{M_\eta}$

prf. Suppose not. Then $\tau_\eta < \rho^1_{M_\eta}$. Hence $A \in J^{E^{M_\eta}}_{\rho^1_{M_\eta}}$, since $A \subset J^{E^{M_\eta}}_{\tau_\eta}$. Hence $A \in J^{E^{M_\eta}}_{\lambda_\xi} = J^{E^{M_\xi}}_{\lambda_\xi} \subset M^*_\eta$. Hence A is strongly $\Sigma_1(M^*_\xi)$ in the parameter A.

Now let A'' be $\Sigma_1(M_\eta|\rho^?_0)$ in $p' = \sigma_\xi(p)$ by the same definition. Then $A'' \subset A'$. But since

$$\sigma_\xi : M_\eta \xrightarrow[\Sigma^*]{} M'_\eta \ \min(\rho ?),$$

we have: $A'' = \sigma_\xi(A)$. But λ'_ξ is inaccessible in M'_η; hence:

$$A'' \in J^{E^{M_\eta}}_{\lambda'_\xi} = J^{E^{M_\xi}}_{\lambda'_\xi} \subset M'^*_\eta .$$

Hence $A'' = \sigma_\xi(A)$ is $\Sigma_1(M'^*_\eta)$ in $A'' = \sigma_\xi(A)$ by the same definition. Contradiction!

QED(3)

(4) η is not a limit ordinal.

proof

Suppose not. Pick $\bar\eta \leq_T \eta$ s.t. $\bar\eta = \mu + 1$, $\pi_{\bar\eta\eta}$ is total on $M_{\bar\eta}$, $\kappa_\mu = \text{crit}(\pi_{\bar\eta,\eta}) > \lambda_\xi$ and $p \in \text{rng}(\pi_{\bar\eta\eta})$. Then $\pi'_{\bar\eta\eta}$ is total on $M'_{\bar\eta}$, $\kappa'_\mu = \text{crit}(\pi'_{\bar\eta,\eta}) > \lambda'_\xi$ and $p' \in \text{rng}(\pi'_{\bar\eta,\eta})$, where $p' = \sigma_\eta(p)$. Set $\bar p = \pi_{\bar\eta\eta}^{-1}(p)$, $\bar p' = \pi'^{-1}_{\bar\eta,\eta}(p')$. Then $\sigma_{\bar\eta}(\bar p) = p$. Then

$$M_{\bar\eta} = \langle J_{\bar\nu}^{E^{M_{\bar\eta}}}, \bar F \rangle, \; M'_{\bar\eta} = \langle J_{\bar\nu'}^{E^{M'_{\bar\eta}}}, \bar F' \rangle.$$

Extend the mirror $\langle I | \bar\eta+1, I' | \bar\eta+1 \rangle$ to a potential mirror $\langle \bar I, \bar I' \rangle$ of length $\bar\eta + 2$ by setting: $\bar\nu_{\bar\eta} = \bar\nu$, $\bar\nu'_{\bar\eta} = \bar\nu'$.

Then $\bar M^*_{\bar\eta} = M^*_\eta$, $\bar M'^*_{\bar\eta} = M'^*_\eta$,

$\xi = \bar T(\bar\eta+1) = T(\eta+1)$ and

$$\sigma_\xi \upharpoonright M^*_{\bar\eta} : \bar M^*_{\bar\eta} \longrightarrow_{\Sigma^*} \bar M'^*_{\bar\eta} \text{ min } \rho^*,$$

It is easily seen that A is $\Sigma_1(M_{\bar\eta})$ in $\bar p$ and A' is $\Sigma_1(M'_{\bar\eta})$ in $\bar p'$ by the same definition. By the minimality of η we conclude that there is $q \in M^*_\eta = \bar M^*_{\bar\eta}$ s.t. A is strongly $\Sigma_1(M^*_\eta)$ in q and A is $\Sigma_1(M'^*_\eta)$ in $q' = \sigma_\xi(q)$ by the same def. Contr! QED (4)

Now let $\eta = \mu+1$. Let $\xi = T(\mu+1)$. Then

$\pi_{\xi\eta} : M^*_\mu \xrightarrow[\Sigma^*]{} M_\eta$ and $\kappa_\mu = \mathrm{crit}(\pi_{\xi\eta})$.

Hence M^*_μ has the form $\bar{M} = \langle J^{\bar{E}}_{\bar{\nu}}, \bar{F}\rangle$ where $\bar{F} \neq \emptyset$

Set: $\bar{\kappa} = \mathrm{crit}(\bar{F})$, $\bar{\tau} = \tau(\bar{F}) =: \bar{\kappa}^{+\bar{M}}$, $\bar{\lambda} = \lambda(\bar{F}) =: \bar{F}(\bar{\kappa})$.

Similarly M'^*_μ has the form $\bar{M}' = \langle J^{\bar{E}'}_{\bar{\nu}'}, \bar{F}'\rangle$ and we define $\bar{\kappa}', \bar{\tau}', \bar{\lambda}'$ accordingly.

Set: $\pi = \pi_{\xi\eta}$, $\pi' = \pi'_{\xi\eta}$.

(5) $\kappa_\mu > \bar{\kappa}$,

since otherwise $\kappa_\eta = \pi(\bar{\kappa}) \geq \pi(\kappa_\mu) = \lambda_\mu \geq \lambda_\xi > \kappa_\eta$. Contr! QED(5)

But then $\kappa_\mu > \bar{\tau}$ and hence $\bar{\tau} = \tau_\eta$, $\bar{\kappa} = \kappa_\eta$

Similarly $\kappa'_\mu > \bar{\tau}'$ and $\bar{\tau}' = \tau'_\eta$, $\bar{\kappa}' = \kappa'_\eta$.

But then:

(6) $\kappa_\mu > \rho^1_{\bar{M}}$,

since otherwise $\rho^1_{M_\eta} \geq \pi(\kappa_\mu) = \lambda_\mu > \tau_\eta$

Contr! by (3). QED(6)

Hence, since $\pi : \bar{M} \xrightarrow[E_{\nu_\eta}]{*} M_\eta$, we have:

(7) $\pi : \bar{M} \xrightarrow[E_{\nu_\mu}]{} M_\eta$ is a Σ_0 ultraproduct and $\rho^1_{\bar{M}} = \rho^1_{M_\eta}$.

Recall that A is strongly $\Sigma_1(M_\eta)$ in p and A' is $\Sigma_1(M'_\eta)$ in $p' = \sigma_\eta(p)$ by the same definition. By (7) we know:

(8) $p = \pi(f)(\alpha)$ where $\alpha < \lambda_\mu$, $f \in \bar{M}$ and $f : \kappa_\mu \to \bar{M}$. Hence:

(9) $p' = \pi'(f')(\alpha')$ where $f' = \sigma_\xi(f)$, $\alpha' = \sigma_\mu(\alpha)$.

proof

$p' = \sigma_\eta(\pi(f)(\alpha)) = (\sigma_\eta \pi(f))(\sigma_\eta(\alpha)) = (\pi' \sigma_\xi(f))(\sigma_\mu(\alpha))$

QED(9)

Let A be strongly $\Sigma_1(M_\eta)$ in p as witnessed by $\bigvee z\, B(z, x, p)$, where B is $\Sigma_0(M_\eta)$. Set:

$$B_0(u, x, p) \longleftrightarrow: \bigvee z \in u\ B(z, x, p).$$

Then A is strongly $\Sigma_1(M_\eta)$ in p as witnessed by $\bigvee u B_0(u, x, p)$. Note that for all u, u':

(10) $(B_0(u, x, p) \wedge u \subset u') \longrightarrow B_0(u', x, p)$.

Let B_1 be $\Sigma_0(\bar{M})$ by the same definition as B_0 over M_η. Set $\tilde{F} =: E_{\nu_\mu}^{M_\mu}$, $\tilde{F}' =: E_{\nu'_\mu}^{M'_\mu}$.

By the cofinality of the map $\pi : \bar{M} \longrightarrow M_\eta$ and (10) we have:

(11) $Ax \longleftrightarrow \bigvee u \in \bar{M}\ B_0(\pi(u), x, p)$

$$\longleftrightarrow \bigvee u\ \{\gamma < \kappa_\mu \mid B_1(u, x, f(\gamma))\} \in \tilde{F}_\alpha.$$

But $\tilde{F}_\alpha$ is strongly $\Sigma_1(M_\mu \| \nu_\mu)$ in α. Hence:

(12) There is $q \in \bar{M}$ s.t.

(a) $G = \tilde{\bar{F}}_\alpha$ is strongly $\Sigma_1(\bar{M})$ in q

(b) Let G' be $\Sigma_1(\bar{M}')$ in $q' = \sigma_s(q)$ by the same definition. Then $G' \subset \tilde{\bar{F}}'_{\alpha'}$ where $\alpha' = \bar{\sigma}_u(\alpha)$.

Let $\bigvee z\, G_0(z, X, q)$ witness the fact that G is strongly $\Sigma_1(\bar{M})$. Then:

(13) $AX \longleftrightarrow \bigvee v\, (v$ is transitive $\wedge$

$$\wedge \bigvee_{\in v} u \bigvee_{\in v} z \bigvee_{\in v} Y \bigvee_{\in v} y\, (Y = \{\gamma \mid B_1(u, X, f(\gamma))\} \wedge$$

$$\wedge\, G_0(y, Y, q)).$$

$\longleftrightarrow \bigvee v\, B_2(v, X, r)$ where $r = \langle f, q \rangle$ and B_2 is $\Sigma_0(\bar{M})$ in r.

We now claim:

(14) A is strongly $\Sigma_1(\bar{M})$ in r as witnessed by $\bigvee v\, B_2(v, X, r)$.

proof.

It suffices to show:

<u>Claim</u> Let $w \subset \mathbb{P}(\bar{\kappa}) \cap \bar{M}$ s.t. $\bar{\bar{w}} \leq \bar{\kappa}$ in $\bar{M}$. (Hence $\bar{\bar{w}} \leq \bar{\kappa}$ in M_γ). There is $v \in \bar{M}$ s.t.

$\bigwedge X \in w\, (B_2(v, X, r) \vee B_2(v, \bar{\kappa} \setminus X, r))$.

(Hence $\bigwedge X \in w\, \bigvee z \in u\, (B_2(z, X, r) \vee B_2(z, \bar{\kappa} \setminus X, r))$ with $u = \{v\}$.)

For the sake of simplicity we assume w.l.o.g. that $X \in W \leftrightarrow \bar{\kappa} \setminus X \in W$. There is then $v \in M_\gamma$ s.t.

$$\bigwedge X \in W \bigvee z \in v (B_0(z, X, p) \vee B_0(z, \bar{\kappa} \setminus X, p)).$$

Hence

$$\bigwedge X \in W (B_1(v, X, p) \vee B_1(v, \bar{\kappa} \setminus X, p).$$

We may assume w.l.o.g. that $v = \pi(u)$ for a $u \in \bar{M}$. Set: $\theta(X) =: \{\gamma \mid B_2(u, X, f(\gamma)\}$ for $X \in W$. Then:

$$\bigwedge X \in W (\theta(X) \in \tilde{F}_\alpha \vee \theta(\bar{\kappa} \setminus X) \in \tilde{F}_\alpha),$$

Let $\beta \in \bar{M}$ s.t. $\{\kappa_\mu, f, v\} \subset J_\beta^{\bar{E}} = \bar{M}|\beta$.

Then $\theta(X) \in \bar{M}|\beta$ for $X \in W$. It follows easily that

$$Z = \{\theta(X) \mid X \in W\} \in \bar{M} \text{ and } \bar{\bar{Z}} \leq \bar{\kappa} < \kappa_\mu$$

in $\bar{M}$. But then there is $U \in \bar{M}$ s.t. $\bar{M}|\beta \subset U$, $Z \subset U$, U is transitive and:

$$\bigwedge Y \in Z \bigvee z \in U (G_0(z, Y, q) \vee G_0(z, \kappa_\mu \setminus Y, q)).$$

Hence:

$$\bigwedge X \in W (B_2(v, X, r) \vee B_2(v, \bar{\kappa} \setminus X, r))$$

QED (14)

Finally, we show:

(15) Let A'' be $\Sigma_1(\bar{M}')$ in $\Omega' = \sigma_\zeta(\Omega)$ by the same definition. Then $A'' \subset A'$.

proof.

We assume $V\nu\, B_2'(\nu, X, \Omega')$ where B_2' has the same $\Sigma_0(\bar{M}')$ definition as B_2 over $(\bar{M})$ and $X \in \mathbb{P}(\bar{\tau}') \cap \bar{M}'$. Then there are $u, z, Y, y \in \bar{M}'$ s.t. $Y = \{\gamma \mid B_1'(u, X, f'(\gamma))\} \cap G_0'(y, Y, q')$

where G_0', B_1' have the same Σ_0 definitions over $\bar{M}'$ as G_0, B_1 over $\bar{M}$.

But then $Y \in G'$, $Y = \{\gamma \mid B_1'(u, X, f'(\gamma))\}$, where $G' \subset \tilde{F}_{\alpha'}'$ $(\alpha' = \sigma_\mu(\alpha))$. But then $\alpha' \in \pi'(Y)$. Hence $B_0'(\pi'(u), X, \pi'(f)(\alpha))$ where $\pi'(f)(\alpha) = p'$, where B_0' has the same Σ_0 definition over M_η' as B_0 over M_η.

Hence $A'X$. QED (15)

Now extend $\langle I|\zeta+1, I'|\zeta+1\rangle$ to a potential mirror $\langle \hat{I}, \hat{I}'\rangle$ of length $\zeta+2$ by setting: $\nu_\zeta = \bar{\nu}$, $\nu_\zeta' = \bar{\nu}'$. Then $\bar{M} = M_\zeta \| \bar{\nu}$, $\bar{M}' = M_\zeta' \| \bar{\nu}'$. Since $\hat{\kappa}_\zeta = \kappa_\eta$ and $\hat{\tau}_\zeta = \tau_\eta$ we have: $\zeta = \hat{T}(\zeta+1)$ and $\hat{M}_\zeta^* = M_\eta^*$, $\hat{M}_\zeta'^* = M_\eta'^*$. By the minimality of η it follows that there is a parameter $s \in M^*_\eta$ s.t.

- A is strongly $\Sigma_1(M^*_\eta)$ in s
- ~~At~~ A''' has the same $\Sigma_1(M'^*_\eta)$ in $s' = \sigma_\xi(s)$,

then $A''' \subset A'' \subset A'$.

Contradiction!, since η was a counterexample.

This proves Lemma ~~4~~ and with it ~~Lemma~~ Lemma 3 and Lemma 2,