§6 Smooth Inserations

Def By a smooth inseration of length μ we mean $\underline{\mathcal{S}} = \langle \mathcal{S}_\xi \mid \xi < \mu \rangle$ s.t.

(a) $\mathcal{S}_\xi = \langle \langle I^i_\xi \rangle, \langle \nu^i_\xi \rangle, \langle e^{ij}_\xi \rangle, T_\xi \rangle$ is an inseration, where I^i_ξ is a normal iteration of M of length η^i_ξ.

(b) If $\xi + 1 < \mu$, then $\mathcal{S}_\xi$ has length $\eta_\xi + 1$ and $I^{\eta_\xi}_\xi = I^0_{\xi+1}$.

(c) If $\gamma < \mu$ is a limit ordinal, then there are at most finitely many $\xi < \gamma$ s.t. $\mathcal{S}_\xi$ has a truncation point on the main branch.

[**Def** We call $\xi + 1$ a truncation point in $\underline{\mathcal{S}}$ iff $\mathcal{S}_\xi$ has a truncation on the main branch.]

(d) There are partial inserations $e_{\xi\varsigma}$ $(0 \leq \xi \leq \varsigma < \mu)$ s.t.

(i) $e_{\xi,\varsigma}$ inserts $I^0_\xi \mid \gamma$ into I^0_ς.

(ii) If $\varsigma = \xi + 1$, then $e_{\xi,\varsigma} = (e^{0,\eta_\xi})^{\mathcal{S}_\xi}$

(iii) $e_{\xi,\varsigma} \cdot e_{\delta,\xi} = e_{\delta,\varsigma}$; $e_{\xi\xi} = \mathrm{id}$

(iv) Let $\gamma < \mu$ be a limit ordinal. Let $\xi_0 < \gamma$ s.t. there is no $\xi \in (\xi_0, \gamma)$ which is a truncation point. Then $\tilde{e}_{\xi\zeta}$ is a total insertion of I^0_ξ into I^0_ζ for $\xi_0 \leq \xi \leq \zeta < \gamma$. Moreover,

$$I^0_\gamma, \langle e_{\xi,\gamma} \mid \xi_0 \leq \xi < \gamma \rangle$$

is the good limit of

$$\langle I^0_\xi \mid \xi_0 \leq \xi < \gamma \rangle, \langle e_{\xi,\zeta} \mid \xi_0 \leq \xi \leq \zeta < \gamma \rangle.$$

(Note: Since $e_{\xi,\zeta}$ for $\xi \leq \xi_0 \leq \zeta$ can be defined by: $e_{\xi,\zeta} = e_{\xi_0,\zeta} \circ e_{\xi,\xi_0}$, we refer to $I^0_\gamma; \langle I^0_\xi \mid \xi < \gamma \rangle$ as the good limit of

$$\langle I^0_\xi \mid \xi < \gamma \rangle, \langle e_{\xi\zeta} \mid \xi \leq \zeta < \gamma \rangle.)$$

We leave it to the reader to prove by induction on μ:

<u>Lemma 1</u> Let $\mathbb{S}$ be a smooth iteration of limit length μ. If $\xi_0 < \mu$ s.t. (ξ_0, μ) has no truncation point, then $e_{\xi,\zeta}$ inserts I^0_ξ into I^0_ζ with $\tilde{e}_\xi(\eta_\xi) = \eta_\zeta$ for $\xi \leq \zeta < \gamma$.

Def M is uniquely smoothly iterable iff:

(1) If $\mathcal{I}$ is a smooth iteration of length $\mu+1$ and $\mathcal{I}_\mu$ has length $\eta_\mu+1$ and $E_\nu^{M'} \neq \emptyset$, where M' is the final model of $I_\mu^{\eta_\mu}$, and $\nu > \nu_\mu^i$ for $i < \eta_\mu$, then $\mathcal{I}_\mu$ extends to an iteration of length $\eta_\mu + 2$ with $\nu = \nu_{\eta_\mu}$.

(2) If $\mathcal{I}$ is an iteration of length $\mu+1$ and $\mathcal{I}_\mu$ is of limit length λ, then there is exactly one extension of $\mathcal{I}_\mu$ to an iteration of length $\lambda+1$

(3) If $\mathcal{I}$ is of limit length μ, then:

(a) There are at most finitely many truncation points below μ.

(b) $\langle I_\xi^0 \mid \xi < \mu\rangle$, $\langle \tilde{e}_{\xi,\zeta} \mid \xi \leq \zeta < \mu\rangle$ has a good limit $\langle I_\mu^0, \langle \tilde{e}_{\xi,\mu} \mid \xi < \mu\rangle\rangle$.

(Note (3)(a) is required to make sense of the notion of good limit in (3)(b), but does not, in itself, guarantee the existence of this limit, since the limiting process might yield an ill founded structure.)

Theorem 2 If M is uniquely normally iterable, then it is uniquely smoothly iterable.

proof

(1) holds, since $I^{\eta_\mu}_\mu$ is a normal iteration of M

(2) holds by § 2 Thm 4.

It remains to prove (3), which will take some effort, using Schlutzenberg's machinery.

In this case $\mathcal{S}$ is of limit length μ.

In order to simplify our notation, we set: $I_\xi =: I^0_\xi$ $(\xi < \mu)$. We note that $\mathcal{S}_\xi$ is uniquely recoverable from the pair $\langle I_\xi, I_{\xi+1}\rangle$. (To see this note that $\mathcal{S}_\xi \mid i+1$ is recoverable from I_ξ and $\langle \nu_h \mid h \leq i\rangle$.

But ν_h = that ν s.t. $E^{M(h)}_\nu \neq \emptyset$ and $\nu^M = \emptyset$, where $M_{(h)}$ is the top model of I_h and M is the top model of $I_{\xi+1}$.)

We also write η_i for η^0_i.

Theorem 2 then follows from:

Lemma 3 Let $\mathcal{I}$ be a smooth inceration of limit length. Then:

(a) If $i \leq j < \mu$, then I_j is an inflation of I_i with history $\langle a^{i,j}, \langle e_\alpha^{i,j} \mid \alpha \leq \eta_i \rangle\rangle$.

(b) If $(i, j]$ has no truncation point in $\mathcal{I}$, then $\eta_i = a^{i,j}_{\eta_j}$ and $\tilde{e}^{i,j}_{\eta_j} = \tilde{e}_{i,j}$.

(c) $\mathcal{I}$ has at most finitely many truncation points.

(d) $\langle I_i \mid i < \mu \rangle$, $\langle \tilde{e}_{i,j} \mid i \leq j < \mu \rangle$ has a good limit: I_μ, $\langle \tilde{e}_{i,\mu} \mid i < \mu \rangle$

(e) (a), (b) hold for μ in place of j.

proof

We prove Lemma 3 by induction on μ. Let it hold below μ.

Claim 1 (a), (b) hold

proof

By ind on $j < \mu$ we show that (a), (b) hold at j

Case 1 $j = 0$, (a), (b) are vacuously true.

<u>Case 2</u> $j' = h+1$.

(a), (b) hold for $i = h$ by §5 Theorem 5. Now let $i < h$. Then (a), (b) hold at i, h (with h in place of j') by the induction hypothesis. Since they also hold at h, j', the conclusion follows by §5 Thm 11.

<u>Case 3</u> $j' = \lambda$ is a limit ordinal.

Applying the induction hypothesis (of Lemma 3) to $S|\lambda$ shows that (a), (b) hold at λ. QED (Claim 1)

We now show that (c), (d), (e) hold at μ. At this point, however, we must be more precise about the degree of iterability assumed for M: We assume that M is uniquely normally $\Omega+1$-iterable, where Ω is regular and $\Omega > \mu, \eta_i$ for $i < \mu$. (It follows e.g. that is $\Theta > \omega$ is a limit cardinal with $cf(\Theta) > \mu$, and M is uniquely normally Θ-iterable, then M is uniquely smoothly Θ-iterable.

Trivially, the 1-step iteration of M:

$$\bar{I}_0 = \langle\langle M\rangle, \emptyset, \langle id\rangle, \emptyset\rangle \quad (\mathrm{lh}(\bar{I}_0)=1)$$

is an inflation of I_i for $i<\mu$. We attempt to construct a tower of successive iterations $\bar{I}_\xi$ of length $\xi+1$ s.t. I_ξ is an inflation of I_i for $i<\mu$. Our attempt will have only limited success. If we have constructed I_ξ for ξ below a limit ordinal λ, then we can indeed construct I_λ. In attempting to go from $\bar{I}_\xi$ to $\bar{I}_{\xi+1}$, however, we may encounter a "bad case" which blocks us from going further. Finally, we observe that we get a contradiction if $\bar{I}_{\Omega+1}$ exists. Hence the "bad case" must have occurred somewhere below Ω.

A close examination of this "bad case" then reveals that it is a very good case, in that it gives us (c), (d) and (e).

We attempt to successively construct:

$$\bar{I}_\xi = \langle\langle \bar{M}^\xi_\alpha\rangle, \langle \bar{\nu}^\xi_\alpha\rangle, \langle \bar{\pi}^\xi_{\alpha,\beta}\rangle, \bar{T}_\xi\rangle \text{ of length } \xi+1$$

such that:

(A) $\bar{I}_\xi$ is an inflation of i with history $\langle \bar{a}^{\xi,i}, \langle \bar{e}^{\xi,i}_\alpha \mid \alpha \le \xi\rangle$ for all $i<\mu$

(B) $\xi < \theta \to \bar{I}_\xi = \bar{I}_\theta | \xi$.

<u>Note</u> By (B) we can write: $\bar{M}_\alpha$, $\bar{\nu}_\alpha$, $\bar{e}^i$, $\bar{I}$, ; without reference to ξ. Similarly we write $\bar{a}^i_\alpha$, $\bar{e}^i_\alpha$ instead of $\bar{a}^{\xi,i}_\alpha$, $\bar{e}^{\xi,i}_\alpha$.

(C) Let $\alpha \le \xi$. Then $\alpha = \bigcup_{i<\mu} \tilde{\bar{e}}^{i}_\alpha{}''\bar{a}^i_\alpha$.

By (C) we have:

(1) $\alpha = \sup\{\bar{e}^i_\alpha(\bar{a}^i_\alpha) \mid i<\mu\}$
since $\bar{e}^i_\alpha(\bar{a}^i_\alpha) = \text{lub } \tilde{\bar{e}}^i_\alpha{}''\bar{a}^i_\alpha$.

Since $\tilde{\bar{e}}^i_\alpha(\bar{a}^i_\alpha) = \alpha$, (C) gives us:

$$\alpha+1 = \bigcup_{i<\mu} \text{rng}(\tilde{\bar{e}}^i_\alpha).$$

Since $\tilde{\bar{e}}^i_\alpha$ inserts $I_i | \bar{a}^i_\alpha+1$ into $\bar{I}|\alpha+1$, and $\tilde{\bar{e}}^i_\alpha \tilde{e}^{h,i}_{\bar{a}^i_\alpha} = \tilde{\bar{e}}^h_\alpha$, we have:

(2) Set: $e^{h,i}_{(\alpha)} = e^{h,i}_{a^i_\alpha}$. Then:

$I \mid \alpha+1, \langle \bar{e}^i_\alpha \mid i<\mu \rangle$

is the good limit of

$\langle (I_i \mid \bar{a}^i_\alpha + 1) \mid i<\mu \rangle \langle e^{h,i}_{(\alpha)} \mid h \leq i < \mu \rangle$.

Now set: $\tilde{\sigma}^i_{(\alpha)} =: \tilde{\sigma}^{\bar{e}^i_\alpha}_{\bar{a}^i_\alpha}$; $\tilde{\sigma}^{i,j}_{(\alpha)} = \tilde{\sigma}^{e^{i,j}_{(\alpha)}}_{\bar{a}^i_\alpha}$.

Then $\tilde{\sigma}^i_{(\alpha)} \tilde{\sigma}^{h,i}_{(\alpha)} = \tilde{\sigma}^h_{(\alpha)}$.

Set $\sigma^i_{(\alpha)} = \sigma^{e^{i,i}_{(\alpha)}}_{\bar{a}^i_\alpha}$. Then:

$$\sigma^i_{(\alpha)} : M^i_{\bar{a}^i_\alpha} \xrightarrow[\Sigma^*]{} \bar{M}_{\bar{\alpha}}, \text{ where } \bar{\alpha} = \bar{e}^i_{(\alpha)}(\bar{a}^i_\alpha),$$

since $\bar{e}^i_\alpha$ is an insertion. $\tilde{\sigma}^i_{(\alpha)}$, on the other hand, can be a partial function on $M^i_{\bar{e}^i_\alpha}$. However:

(3) $\tilde{\sigma}^i_{(\alpha)} : M^i_{\bar{a}^i_\alpha} \xrightarrow[\Sigma^*]{} \bar{M}_\alpha$ for sufficiently large $i<\mu$.

proof.

$\tilde{\sigma}^i_{(\alpha)} = \bar{\pi}_{\bar{e}^i(\bar{a}^i_\alpha),\alpha} \cdot \sigma^i_{(\alpha)}$. But then we can pick i big enough that there is no truncation in $(\bar{e}^i_\alpha(a^i_\alpha), \alpha]_T$. Hence $\bar{\pi}_{\bar{e}^i(\bar{a}^i_\alpha),\alpha}$ is Σ^*-preserving. QED (3)

We inductively construct I_ξ:

<u>Case 1</u> $\xi = 0$. $I_0 = \langle\langle M\rangle, \emptyset, \langle id\rangle, \emptyset\rangle$ is the unique 1-step iteration of M. (A)–(C) hold trivially.

<u>Case 2</u> $\xi = \theta + 1$ and $\bar{a}^i_\theta < \eta_i$ for arbitrarily large $i < \mu$. Let D be the set of such i. For sufficiently large $i \in D$ we have:

$$\tilde{\sigma}^i_{(\theta)} : M^i_{\bar{a}^i_\theta} \xrightarrow[\Sigma^*]{} \bar{M}_\theta$$

and

$$\tilde{\sigma}^{ij}_{(\theta)} : M^i_{\bar{a}^i_\theta} \xrightarrow[\Sigma^*]{} M^j_{\bar{a}^j_\theta} \text{ for } j:$$

For sufficiently large $i \in D$ we know:

$$\tilde{\sigma}^{ij}_{(\theta)}(\nu^i_{\bar{a}^i_\theta}) \geq \nu^j_{\bar{a}^j_\theta} \quad \text{for } j \in D \setminus i$$

Hence, for sufficiently large $i \in D$ we, in fact, have:

(1) $\tilde{\sigma}^{ij}_{(\theta)}(\nu^i_{\bar{a}^i_\theta}) = \nu^j_{\bar{a}^j_\theta}$ for $j \in D \setminus i$

(To see this, suppose not. Then there is a monotone sequence $\langle i_n \mid n < \omega\rangle$ s.t.

$$\tilde{\sigma}^{i_n i_{n+1}}_{(\theta)}(\nu^{i_n}_{\bar{a}^{i_n}_\theta}) > \nu^{i_{n+1}}_{\bar{a}^{i_{n+1}}_\theta}.$$

Set: $\gamma_n = \tilde{\sigma}^{i_n}_{(\theta)}(\nu^{i_n}_{\bar{a}^{i_n}_\theta})$. Then $\gamma_n > \gamma_{n+1}$ for $n < \omega$ and $\bar{M}_\theta$ is ill founded. Contradiction!)

Let $\bar{\nu} = \tilde{\sigma}^i(\nu^i_{\bar{\alpha}^i_\theta})$ for $i \in D$ s.t. (1) holds.

<u>Claim</u> $\bar{\nu} > \bar{\nu}_\delta$ in $\bar{I}_\theta$ for $\delta < \theta$.

proof.

Pick sufficiently large $i \in D$ s.t. $\delta \in \tilde{\bar{e}}^i_\theta{}''\bar{\alpha}^i_\theta$. Let $\tilde{\bar{e}}^i_\theta(\bar{\delta}) = \delta$. Then $\bar{\delta} < \nu^i_{\bar{\alpha}^i_\theta}$ and $\delta < \bar{\nu}$. QED (Claim).

We now apply our extension lemma §5 Lemma 9. Extend $\bar{I}_\theta$ to $\bar{I}_{\theta+1}$ by setting $\bar{\nu}_\theta = \bar{\nu}$. By §5 Lemma 9 we have: $\bar{I}_{\theta+1}$ is an inflation of I_i for $i \in D$ s.t. (1) holds.

But this set is cofinal in μ. Hence $\bar{I}_{\theta+1}$ is an inflation of every I_i. and (A) holds. (B) is trivial. (C) is also trivial since for sufficient $i \in D$ we have:

$$\tilde{\bar{e}}^i_{\theta+1}{}''\theta+1 = \operatorname{rng}(\tilde{\bar{e}}^i_\theta)$$

and $\theta + 1 = \bigcup_{i<\mu} \operatorname{rng}(\tilde{\bar{e}}^i_\theta)$.

QED (Case 2)

Case 3 $\xi = \theta + 1$ and Case 1 fails.
Then $\bar{a}^{i}_{\theta} = \eta_i$ for sufficiently large δ.
This is the "bad case" and $\bar{I}_{\theta+1}$ is undefined.

Case 4 $\xi = \lambda$ is a limit ordinal.
We assume that $\bar{I}_{\gamma}$ is defined for $\gamma < \lambda$ and set: $\tilde{I} = \bigcup_{\gamma < \lambda} \bar{I}_{\gamma}$. Then $\tilde{I}$ is an inflation ~~of~~ each I_i and satisfies (A) – (C). Let b be the unique well founded cofinal branch in $\tilde{I}$. Extend $\tilde{I}$ to $\bar{I}_{\lambda}$ of length $\lambda + 1$ by setting $\bar{T}``\{\lambda\} = b$.
By §5 Lemma 10 $\bar{I}_{\lambda}$ is an inflation of I_i for $i < \mu$ with history $\langle \bar{a}^{i}, \langle \bar{e}^{i}_{\alpha} \mid \alpha \leq \lambda \rangle \rangle$, where
$\bar{a}^{i}_{\lambda} = \sup_{\beta \in b} \bar{a}^{i}_{\beta}$; $\tilde{\bar{e}}^{i}_{\lambda} \upharpoonright \bar{a}^{i}_{\lambda} = \bigcup_{\beta \in b} \tilde{e}^{i}_{\beta} \upharpoonright a_{\beta}$,
$\tilde{e}^{i}_{\lambda}(\bar{a}^{i}_{\lambda}) = \lambda$. Clearly (A), (B) are satisfied. But so is (C) since

$$\bigcup_{i<\mu} \tilde{\bar{e}}^i_\lambda{}''\bar{a}^i_\lambda = \bigcup_{i<\mu}\bigcup_{\beta\in b} \tilde{\bar{e}}^i_\beta{}''\bar{a}^i_\beta .$$

$$\bigcup_{\beta\in b}\bigcup_{i<\mu} \tilde{\bar{e}}^i{}''\bar{a}^i_\beta = \bigcup b = \lambda .$$

QED (Case 4)

If the bad case did not occur, then $\bar{I}_\Omega$ exists, since M is uniquely $\Omega+1$-iterable. But this is contradictory, since:

<u>Lemma 4</u> If λ is a limit and $\bar{I}_\lambda$ exists, then $\mathrm{cof}(\lambda) \leq \mu$ or $\mathrm{cof}(\lambda) \leq \eta_i$ for some $i<\mu$.

proof

Suppose first that $\lambda > \bar{e}^i_\lambda(\bar{a}^i_\lambda)$ for all $i<\mu$, $\lambda = \sup_{i<\mu} \bar{e}^i_\lambda(\bar{a}^i_\lambda)$ by (1). Hence $\mathrm{cf}(\lambda)\leq\mu$. Otherwise $\lambda = \bar{e}^i_\lambda(\bar{a}^i_\lambda) = \sup \tilde{\bar{e}}^i_\lambda{}''\bar{a}^i_\lambda$. Hence $\bar{a}^i_\lambda$ is a limit ordinal. Letting $\langle \gamma_i \mid i<\varphi\rangle$ be a cofinal sequence in $\bar{a}^i_\lambda$, we have $\lambda = \sup_{i<\varphi} \tilde{\bar{e}}^i_\lambda(\gamma_i)$. Hence $\mathrm{cf}(\lambda)\leq\varphi\leq\eta_i$.

QED (Lemma 4)

Hence the "bad case" Case 3 must occur somewhere below Ω. Let it occur at $\zeta=\theta+1$. Then $\bar{I}_\theta$ is the final element of our tower.

For sufficiently large $i < \mu$ we have;

$\bar{a}^i_\theta = \eta_i$. Hence: $a^{ii'}_{\eta_i} = a^{ii'}_{\bar{a}^i_\theta} = a^i_\theta = \eta_i$ and:

$e^{ii'}_{\eta_i} \upharpoonright a^{ii'}_{\eta_i} = e_{ij} \upharpoonright \eta_i$ for $i \leq j$.

We first that (c) ~~of~~ ~~Lemma~~ 3 holds;

<u>Lemma 5</u> There are only finitely many truncation points $h+1 < \mu$ in $\mathfrak{S}$,

proof

Suppose not, Then there are cofinally many such $h+1$ ~~in~~ μ, Pick such a $j = h+1 > \delta$, where δ is chosen s.t. $(\bar{e}_\theta(\eta_i), \theta]_{\bar{T}}$ has no truncation point, and $\eta_l = \bar{a}^l_\theta$ for all $l \geq i$. (This is possible by (11)). By §5 Thm 5 there is a truncation point:

$$\alpha \in (e^{hi'}_{\eta_i}(a^{hi}_{\eta_i}), \eta_i]_{T_j}.$$

But we took h big enough that: $a^{hi'}_{\eta_j} = \eta_h$. Since $\bar{e}^{i'}_\theta e^{hi'}_{\eta_j} = \bar{e}^h_\theta$, we conclude by §1 Lemma 1 (4) that:

$$\bar{e}^j_\theta(\alpha) \in (\bar{e}^h_\theta(\eta_h), \theta]_{\bar{I}_\theta}$$

is a truncation point in $\bar{I}_\theta$, where $h \geq i$. Contradiction!

QED (Lemma 5)

Let $i_0 < \mu$ s.t. there is no truncation point in (i_0, μ). It follows easily that:

for $i_0 \leq i \leq j < \mu$ we have:

$$\bar{a}_\theta^{\,l} = \eta_l \ ; \ \tilde{e}_{ij} = \tilde{e}_{\eta_j}^{ij} \ , \ \text{and} \ \tilde{\bar{e}}_\theta^{\,j} \tilde{e}_{ij} = \tilde{\bar{e}}_\theta^{\,i}.$$

But then $\bar{I}_\theta$, $\langle \bar{e}_\theta^{\,i} \mid i_0 \leq i < \mu \rangle$ is the good limit of:

$$\langle I_i \mid i_0 \leq i < \mu \rangle, \langle \tilde{e}_{ij} \mid i_0 \leq i \leq j < \mu \rangle.$$

This proves (d) with:

$$I_\mu = \bar{I}_\theta \ ; \ \tilde{e}_{i\mu} = \tilde{\bar{e}}_\theta^{\,i} \text{ for } i_0 \leq i < \mu.$$

(Hence $\theta = \ln(I_\mu)$.)

(e) is then obvious by our construction.

QED

This proves Lemma 3 and, with it, Theorem 2.

By a straightforward modification of our proofs we get:

Let Σ be a successful insertion invariant strategy for M. Then M is uniquely smoothly Σ-insertable.