§1 Preliminaries

We find it most convenient to treat general iteration theory in terms of complete Boolean algebras. We write $\mathbb{A} \subseteq \mathbb{B}$ to mean that $\mathbb{A}, \mathbb{B}$ are Boolean algebras, $\mathbb{A}$ is complete, and $\mathbb{A}$ is completely contained in $\mathbb{B}$ (i.e. $\mathrm{id} \restriction \mathbb{A}$ is a complete homomorphism). We write "BA" as an abbreviation for "Boolean algebra". If $\mathbb{A} \subseteq \mathbb{B}$ and $b \in \mathbb{B}$ we define the function $h = h_{\mathbb{A}\mathbb{B}}$ by:

$$h(b) = \bigcap \{a \in \mathbb{A} \mid b \subset a\}.$$

Then $h(\bigcup_i b_i) = \bigcup_i h(b_i)$, $h(a \cap b) = a \cap h(b)$ if $a \in \mathbb{A}$, and $b = 0 \longleftrightarrow h(b) = 0$. (We denote the maximal and minimal elements of a BA by $1, 0$.)

For each complete BA we can form the canonical maximal $\mathbb{B}$-valued model $V^{\mathbb{B}}$. The elements of $V^{\mathbb{B}}$ are called names and there is a valuation function $\varphi \to [\![\varphi]\!]_{\mathbb{B}}$ attaching to each statement $\varphi = \varphi'(\vec{t})$ a truth value $[\![\varphi'(\vec{t})]\!] \in \mathbb{B}$. (Here φ is a ZFC formula and $t_1, \ldots, t_n$ are names.) If $A : V^{\mathbb{B}} \to \mathbb{B}$, we may also add a predicate $\mathring{A}$ to the language, setting:

$$[\![t \in \mathring{A}]\!] = \bigcup_{s \in V^{\mathbb{B}}} [\![s = t]\!] \wedge A(s).$$

We can enrich our language further by closing the Σ_0 formulae under infinite disjunction $\bigvee_{i \in I} \varphi_i$ and conjunction $\bigwedge_{i \in I} \varphi_i$, where I is any set. We shall generally take $V^{\mathbb{B}}$ as an <u>identity</u> <u>model</u> in the sense:

$$[\![s = t]\!] = 1 \to s = t. \quad ^{*)}$$

Thus e.g. $\{x \mid [\![x \in y]\!] = 1\}$ is a set, not a proper class. We also write:

$$a \Vdash \varphi \underset{Df}{\longleftrightarrow} (a \in \mathbb{B} \setminus \{0\} \wedge a \subset [\![\varphi]\!]).$$

$$\Vdash \varphi \longleftrightarrow_{Df} 1 \Vdash \varphi.$$

If $\mathbb{A} \subseteq \mathbb{B}$, there is a $\sigma : V^{\mathbb{A}} \to V^{\mathbb{B}}$ s.t.

$$[\![\varphi(\vec{t})]\!] = [\![\varphi(\sigma(\vec{t}))]\!] \qquad \mathbb{B},$$

where φ is a Σ_0 formula, $t_1, \ldots, t_n \in V^{\mathbb{B}}$, and $[\![s \in \sigma(t)]\!]_{\mathbb{B}} = \bigcap_{s' \in V^{\mathbb{A}}} ([\![s = s']\!]_{\mathbb{B}} \cap [\![s' \in t]\!]_{\mathbb{A}}$ for $s \in V^{\mathbb{B}}$, $t \in V^{\mathbb{A}}$.

*) If we have defined $V^{\mathbb{B}}$ without this property we can achieve it by a simple factorization.

Since it is unimportant what the actual elements of $V^{\mathbb{A}}, V^{\mathbb{B}}$ are, we can easily arrange that $\sigma = \mathrm{id}$, when dealing with specific $\mathbb{A}, \mathbb{B}$. We shall also follow this convention when dealing with a chain $\langle \mathbb{B}_i \mid i<\alpha \rangle$ s.t. $\mathbb{B}_i \subseteq \mathbb{B}_j$ for $i \leq j < \alpha$. The simplest BA is $2 = \{0,1\}$. V^{2} is isomorphic to V in an obvious sense. Let $x \mapsto \check{x}$ be the isomorphism of V onto V^{2}. Following the convention that the above map $\sigma : V^{2} \to V^{\mathbb{B}}$ is the identity we have: $[\![t \in \check{x}]\!] = \bigcup_{z \in x} [\![t = \check{z}]\!]$ in $V^{\mathbb{B}}$.

The predicate $\check{V}$ is, as usual, defined by $[\![t \in \check{V}]\!] = \bigcup_{x \in V} [\![t = \check{x}]\!]$.

If W is an inner model of ZFC$^-$ and $\mathbb{B} \in W$ is complete in the sense of W, we can of course form $W^{\mathbb{B}} = (V^{\mathbb{B}})_W$ by relativizing the definition of $V^{\mathbb{B}}$ to W. (However, we may not be able to define $W^{\mathbb{B}}$ internally as an identity model if W does not model full ZFC or have a definable well ordering.) If $\mathbb{B} \in W$ is complete in W we say that $G \subset \mathbb{B}$ is a $\mathbb{B}$-<u>generic set</u> <u>over</u> W iff G is an ultrafilter on $\mathbb{B}$ which is complete wrt. W in the sense:

$$\bigcup_{i\in X} b_i \in G \longleftrightarrow \forall i \in X\ b_i \in G$$

for $X \subset \mathbb{B}$, $X \in W$. (Equivalently, $G \subset \mathbb{B} \setminus \{0\}$ s.t. $G \cap \Delta \neq \emptyset$ whenever $\Delta \in W$ is dense in $\mathbb{B}\setminus\{0\}$.)

If $\overline{\overline{\mathcal{P}(\mathbb{B}) \cap W}} \leq \omega$ then there is a generic $G \ni b$ for each $b \in \mathbb{B}\setminus\{0\}$. If $\mathbb{B}$ is complete in V it is often useful in analysing $V^{\mathbb{B}}$ to work in a mythical larger universe in which:

(*) V is an inner model and $\overline{\overline{\mathcal{P}(\mathbb{B}) \cap V}} \leq \omega$.

This is harmless, since if $\mathbb{C}$ collapses $2^{\overline{\overline{\mathbb{B}}}}$ to ω, then $(*)(\check{V}, \check{\mathbb{B}})$ holds in $V^{\mathbb{C}}$.

We note that there is a unique term $\mathring{G} \in V^{\mathbb{B}}$ s.t. $\Vdash \mathring{G} \subset \check{\mathbb{B}}$ and $[\![\check{b} \in \mathring{G}]\!] = b$ for $b \in \mathbb{B}$. Thus, if G is generic over V, then $G = \mathring{G}^G$. We call $\mathring{G}$ the <u>canonical</u> $\mathbb{B}$-<u>generic</u> <u>name</u>.

Now let $\mathbb{A} \subseteq \mathbb{B}$. If G is $\mathbb{A}$-generic over V, then $G' = \{b \in \mathbb{B} \mid \forall a \in G\ a \subset b\}$ is a complete filter on $\mathbb{B}$ and we can form the factor algebra $\mathbb{B}/G'$ (which we normally denote by $\mathbb{B}/G$). If $\mathbb{B}$ is complete in V, then $\mathbb{B}/G$ is complete in $V[G]$. There is a canonical homomorphism $\sigma : \mathbb{B} \to \mathbb{B}/G$ s.t.

$$\sigma(b) \subset \sigma(c) \longleftrightarrow \neg b \cup c \in G.$$

When the context permits, we write b/G for $\sigma(b)$.

Now let $\mathbb{B}_0 \subseteq \mathbb{B}_1$ and let G_0 be $\mathbb{B}_0$ - generic over V, Let $\tilde{G}$ be $\mathbb{B}_1/G_0$ - generic over $V[G_0]$. Set $G_1 = G_0 * \tilde{G} =_{\mathrm{Df}} \{ b \in \mathbb{B}_1 \mid b/_{G_0} \in \tilde{G} \}$. Then G_1 is $\mathbb{B}_1$ - generic over V and $V[G_1] = V[G_0][\tilde{G}]$.

Conversely, if G_1 is $\mathbb{B}_1$ - generic over V and we set:

$$G_0 = \mathbb{B}_0 \cap G_1 \,,\; \tilde{G} = \{ b/_{G_0} \mid b \in G_1 \} ,$$

then G_0 is $\mathbb{B}_0$ - generic over V, $\tilde{G}$ is $\mathbb{B}_1/G_0$ - generic over $V[G_0]$ and $G_1 = G_0 * \tilde{G}$.

Omitting most proofs, we mention some basic facts:

<u>Fact 1</u> Let $\mathbb{A} \subseteq \mathbb{B}$ and let $h = h_{\mathbb{A}\mathbb{B}}$ be the function defined above. Then

$$h(b) = [\![\check{b}/\dot{G} \neq 0]\!]_{\mathbb{A}}$$

where $\dot{G}$ is the $\mathbb{A}$-generic name.

<u>Fact 2</u> Let $\mathbb{A} \subseteq \mathbb{B}_0 \subseteq \mathbb{B}_1$, G is $\mathbb{A}$-generic, and $\sigma_h : \mathbb{B}_h \to \mathbb{B}_h/G$ is the canonical homomorphism $(h = 0, 1)$, then there is a complete injective homomorphism $\delta : \mathbb{B}_0/G \to \mathbb{B}_1/G$ defined by

$$\delta(\sigma_0(b)) = \sigma_1(b)$$

<u>Note</u> Then $\delta : \mathbb{B}_0/G \xrightarrow{\sim} \mathbb{B}' \subseteq \mathbb{B}_1/G$. In such a context we shall often identify $\mathbb{B}_0/G$ with $\mathbb{B}'$ so as to have $\mathbb{B}_0/G \subseteq \mathbb{B}_1/G$.

<u>Fact 3</u> Let $\mathbb{A} \subseteq \mathbb{B}$ and $\Vdash_{\mathbb{A}} \mathring{b} \in \check{\mathbb{B}}/\dot{G}$, then there is a unique $b \in \mathbb{B}$ s.t.

$$\Vdash_{\mathbb{A}} \mathring{b} = \check{b}/\dot{G}.$$

proof

To see uniqueness, let $\Vdash \check{b}/G = \check{b}'/\dot{G}$. Then $\Vdash \check{b \setminus b'}/\dot{G} = 0$. Hence $h(b \setminus b') =$ $= [\![\check{b \setminus b'}/G \neq 0]\!] = 0$. Hence $b \setminus b' = 0$. Hence $b \subset b'$. Similarly $b' \subset b$.

To see existence, note that $\Delta =$
$= \{a \in \mathbb{A} \mid \forall b\ a \Vdash \mathring{b} = \check{b}/\mathring{G}\}$ is dense in $\mathbb{A} \setminus \{0\}$.
Let X be a maximal antichain in Δ.
Let $a \Vdash \mathring{b} = \check{b}_a/\mathring{G}$ for $a \in X$. Set
$b = \bigcup_{a \in X} a \cap b_a$. Then $\Vdash \mathring{b} = b/\mathring{G}$, since
if G is $\mathbb{A}$-generic there is $a \in X \cap G$
by genericity. Hence $\mathring{b}^G = b_a/G = b/G$.

QED (Fact 3)

<u>Fact 4</u> If $\mathbb{A} \subseteq \mathbb{B}$, then
$$a \Vdash_{\mathbb{A}} (\check{b}/\mathring{G} \subset \check{c}/\mathring{G}) \longleftrightarrow a \cap b \subset c.$$

<u>Fact 5</u> Let $\mathbb{A} \subset \mathbb{B} \subset \mathbb{C}$. Let G be $\mathbb{A}$-generic. Let $h = h_{\mathbb{B}\mathbb{C}}$ (in V) and $\tilde{h} = h_{\mathbb{B}/G,\, \mathbb{C}/G}$ (in $V[G]$). Then
$$h(c)/G = \tilde{h}(c/G) \text{ for all } c \in \mathbb{C}.$$

<u>Fact 6</u> If $\dot{\mathbb{B}} \in V^{\mathbb{A}}$ s.t.
$\Vdash_{\mathbb{A}} \dot{\mathbb{B}}$ is a complete BA, then
there is a complete $\mathbb{B} \supseteq \mathbb{A}$ s.t.
$$\Vdash_{\mathbb{A}} \check{\mathbb{B}}/\mathring{G} = \mathring{\mathbb{B}}.$$

proof (sketch)
Form the complete BA $\mathbb{A} * \mathring{\mathbb{B}}$
consisting of the points $\{b \mid \Vdash_{\mathbb{A}} b \in \mathring{\mathbb{B}}\}$
and with the operations:

$\neg b$ = that b' s.t. $\Vdash_{\mathbb{A}} b' = \neg b$

$\bigcup X$ = that b' s.t. $\Vdash_{\mathbb{A}} b' = \bigcup\{x^{\dot G} \mid x \in \check X\}$ in $\dot{\mathbb{B}}$

$\bigcap X$ = " " $\Vdash_{\mathbb{A}} b' = \bigcap\{$ " $\}$ ",

$\mathbb{A} * \dot{\mathbb{B}}$ is then a complete BA and there is a complete homomorphic injection $\sigma : \mathbb{A} \to \mathbb{A} * \dot{\mathbb{B}}$ defined by:

$\sigma(a)$ = that $\dot a$ s.t. $\Vdash (\check a \in \dot G \wedge \dot a = 1) \vee (\check a \notin \dot G \wedge \dot a = 0)$

Hence we can find $\mathbb{B}$, $\delta : \mathbb{A} * \dot{\mathbb{B}} \xleftrightarrow{\sim} \mathbb{B}$ s.t. $\delta\sigma = \mathrm{id}$. QED (Fact 6)

<u>Note</u> In the following we shall often deal with sequences $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ s.t. $\mathbb{B}_i \subseteq \mathbb{B}_j$ for $i \leq j < \alpha$. We then set: $h_i = h_{\mathbb{B}_i, \tilde{\mathbb{B}}}$, where $\tilde{\mathbb{B}} = \bigcup_{i<\alpha} \mathbb{B}_i$. We let $\dot G_i$ be the $\mathbb{B}_i$-generic name and write $\Vdash_i$ for $\Vdash_{\mathbb{B}_i}$ and $[\![\varphi]\!]_i$ for $[\![\varphi]\!]_{\mathbb{B}_i}$. We assume $V^{\mathbb{B}_i}$ is so defined that $\mathrm{id} \restriction V^{\mathbb{B}_i}$ is the canonical map $\sigma : V^{\mathbb{B}_i} \to V^{\mathbb{B}_j}$ mentioned above for $i \leq j < \alpha$. If G_h is $\mathbb{B}_h$-generic and $\delta_i : \mathbb{B}_i / G_h \to \tilde{\mathbb{B}} / G_h$ is the injective homomorphism of Fact 2, we shall, whenever the context permits, identify $\mathbb{B}_i / G_h$ with $\mathrm{rng}(\delta_i)$, so that $\delta_i = \mathrm{id}$.

Iterations

<u>Def</u> $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ is an <u>iteration</u> iff

- $\mathbb{B}_i \subseteq \mathbb{B}_j$ for $i \leq j < \alpha$ and $\mathbb{B}_0 = \{0, 1\}$
- $\mathbb{B}_\lambda$ is generated by $\bigcup_{i<\lambda} \mathbb{B}_i$ for limit $\lambda < \alpha$.

We adopt the notation introduced above in dealing with iterations.

By a $\mathbb{B}$-<u>sequence</u> we mean a $b = \langle b_i \mid i \in u \rangle$ s.t. $u \subset \{i \mid i+1 < \alpha\}$ and $b_i \in \mathbb{B}_{i+1}$ for $i \in u$.

We like to pretend that b_i is defined for all $i < \alpha$ and therefore set:

$$b_i = 1 \quad \text{if } i < \alpha \setminus \mathrm{dom}(b).$$

The <u>support</u> of b is defined by:

$$\mathrm{supp}(b) = \{i \mid b_i \neq 1\}.$$

<u>Def</u> If $\mathrm{supp}(b)$ is bounded in α, set:

$$b^* = \bigcap_{i<\alpha} b_i \quad (\text{in } \mathbb{B}_j \text{ where } \mathrm{supp}(b) \subset j)$$

<u>Def</u> b is a <u>good sequence</u> for $\mathbb{B}$ ($GS(b, \mathbb{B})$) iff

- b is a $\mathbb{B}$-sequence
- $h_i((b \upharpoonright j)^*) = (b \upharpoonright i)^*$ for $i \leq j < \alpha$
- $h_i(b_i) = 1$ for $i+1 < \alpha$.

<u>Note</u> Then $(b \upharpoonright i)^* \neq 0$ for all $i < \alpha$, since $h_0((b \upharpoonright i)^*) = \emptyset^* = 1$.

<u>Note</u> The condition $h_i(b_i)=1$ could be weakened to: $h_i(b_i) \supset (b\restriction i)^*$. However, we lose nothing by adopting the stronger requirement, since if b satisfies the weaker condition & we set $\tilde{b}_i = b_i \cup \neg h_i(b_i)$, then $\tilde{b}$ is a good sequence and $(\tilde{b}\restriction i)^* = (b\restriction i)^*$ for all $i<\alpha$.

<u>Def</u> Let $\mathbb{B} = \langle \mathbb{B}_i \mid i<\alpha \rangle$ be an iteration and let G_i be $\mathbb{B}_i$-generic. Set:

$$\mathbb{B}/G_i = \langle \mathbb{B}_{i+j}/G_i \mid j < \alpha - i \rangle.$$

Without proof we mention:

<u>Lemma 1.1</u> $\mathbb{B}/G_i$ is an iteration in $V[G_i]$

<u>Def</u> Let $\mathbb{B}$, G_i be as above and let $GS(b, \mathbb{B})$. Set:

$$b/G_i = \langle b_{i+j}/G_i \mid j+1 < \alpha - i \rangle.$$

<u>Lemma 1.2</u> $GS(b/G_i, \mathbb{B}/G_i)$ in $V[G_i]$

<u>Lemma 1.3</u> Let $i<\alpha$. Then

$$GS(b, \mathbb{B}) \longleftrightarrow (GS(b\restriction i, \mathbb{B}) \wedge$$
$$\wedge\ (b\restriction i)^* \Vdash_i GS(\check{b}/\dot{G}_i, \check{\mathbb{B}}/\dot{G}_i))$$

(where $\dot{G}_i$ is the $\mathbb{B}_i$-generic name)

Def b has countable support in $\mathbb{B}$ ($CS(b, \mathbb{B})$) iff $GS(b, \mathbb{B}) \wedge \overline{\overline{supp(b)}} \leq \omega$.

Def $\mathbb{B}$ is a countable support iteration iff for all limit $\lambda < \alpha$ we have:

(a) If $CS(b, \mathbb{B}\restriction\lambda)$, then $b^* = 0$ in $\mathbb{B}_\lambda$

(b) $\{b^* \mid CS(b, \mathbb{B}\restriction\lambda)\}$ is dense in $\mathbb{B}_\lambda \setminus \{0\}$.

Then:

Lemma 2.1 Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be a countable support iteration. Let $\lambda < \alpha$, $Lim(\lambda)$. If $CS(b, \mathbb{B}\restriction\lambda)$, then $h_i(b^*) = (b\restriction i)^*$ for $i < \lambda$.

However, the analogue of Lemma 1.3 can fail for the property $CS(b, \mathbb{B})$. This happens if cofinalities of ordinals are changed by the iteration. For this reason we shall employ revised countable support iterations. However, we use Donder's simplified notion in place of Shelah's definition.

Def Let $GS(b, \mathbb{B})$, $\lambda \leq \alpha$, $Lim(\lambda)$.

$U_\lambda(b) = U_\lambda(b, \mathbb{B}) =_{Df} \bigcup_{i<\lambda} \{a \in \mathbb{B}_i \setminus \{0\} \mid a \subset (b\restriction i)^*\}$

$S_\lambda(b) = S_\lambda(b, \mathbb{B}) =_{Df}$ the set of $a \in U_\lambda(b)$ s.t. either $\forall i < \lambda\ a \Vdash_i cf(\check{\lambda}) = \omega$, or else $a \subset (b\restriction i)^*$ for all $i < \lambda$.

Def b has revised countable support in $\mathbb{B}$ $(R(b, \mathbb{B}))$ iff for all limit $\lambda \leq \alpha$

$S_\lambda(b)$ is dense in $U_\lambda(b)$

Def $\mathbb{B}$ is a revised countable support iteration (RCS iteration) iff for all limit $\lambda < \alpha$,

(a) If $R(b, \mathbb{B}\restriction\lambda)$, then $b^* \neq 0$ in $\mathbb{B}_\lambda$

(b) $\{b^* \mid R(b, \mathbb{B}\restriction\lambda)\}$ is dense in $\mathbb{B}_\lambda \setminus \{0\}$

Lemma 2.2

(a) $CS(b, \mathbb{B}) \rightarrow R(b, \mathbb{B})$

(b) $R(b, \mathbb{B}) \longleftrightarrow R(b, \mathbb{B}\restriction i+1)$

if $\mathrm{supp}(b) \subset i < \alpha$

(c) $R(b, \mathbb{B}) \rightarrow R(b\restriction i, \mathbb{B}) \rightarrow R(b\restriction i, \mathbb{B}\restriction i)$

for $i < \alpha$,

We get the analogue of Lemma 1.3:

Lemma 2.3 Let $i < \alpha$. Then

$R(b, \mathbb{B}) \longleftrightarrow (R(b\restriction i, \mathbb{B}) \wedge$

$\wedge\, (b\restriction i)^* \Vdash_i R(\check{b}/\dot{G}_i, \check{\mathbb{B}}/\dot{G}_i))$,

where $\dot{G}_i$ is the $\mathbb{B}_i$-generic name.

Finally:

Lemma 2.4 Let $\mathbb{B}$ be an RCS iteration. Let $j < \alpha$. Then

(a) $R(b, \mathbb{B}\restriction j) \longleftrightarrow R(b, \mathbb{B})$

(b) $R(b, \mathbb{B}\restriction j) \rightarrow h_i(b^*) = (b\restriction i)^*$ for $i < j$

(c) $\{b^* \mid R(b, \mathbb{B}\restriction j)\}$ is dense in $\mathbb{B}_j$

<u>Lemma 2.3</u> Let $\mathbb{B}$ be an RCS iteration.
Let $\lambda < \alpha$ be a limit point,
$R(b, \mathbb{B}\restriction\lambda)$, $R(b', \mathbb{B}\restriction\lambda)$, then
$$b^* \subset b^{*\prime} \longleftrightarrow \bigwedge i < \lambda\ (b\restriction i)^* \subset (b'\restriction i)^*.$$

<u>Note</u> Let $\lambda < \alpha$ be a limit. Let $R(b, \mathbb{B}\restriction\lambda)$,
Clearly there is $c \subset b$ s.t. either
$(c\restriction i)^* \Vdash_i \mathrm{cf}(\check{\lambda}) = \omega$ for an $i < \lambda$
or else $\mathrm{supp}(c) \subset i$ for an $i < \lambda$
(hence $(c\restriction i)^* = c^*$).
Hence the set of such c^* is dense in $\mathbb{B}_\lambda$.
Donders original ~~de~~finition of "RCS - iteration" was that for each limit $\lambda < \alpha$ the set of such c^* (with $GS(c, \mathbb{B}\restriction\lambda)$) be dense in $\mathbb{B}_\lambda \setminus \{0\}$. This is easily seen to be equivalent to the present definition.