# §3 Examples

## §3.1 The transfer lemma for embeddings of ZFC⁻ models

We recapitulate and expand upon some facts developed in [J] §5. *)

**Def** Let $M = \langle |M|, \in, \ldots \rangle$ be a transitive ZFC⁻ model. Let $\pi : M \prec M'$, where $M'$ is transitive. $\pi$ is cofinal in $M'$ iff $M' = \bigcup_{u \in M} \pi(u)$.

In the following, we suppose $M, N, \ldots$ to be transitive ZFC⁻ models unless otherwise stated.

**Fact 1** Let $\pi : M \prec M'$ and set $\tilde{M} = M' | \bigcup_{u \in M} \pi(u)$. Then $\tilde{M} \prec M'$ and $\pi : M \prec \tilde{M}$ cofinally.

(The proof uses: Let $x_1, \ldots, x_n \in \tilde{M}$, $x_i \in \pi(u_i)$. Then $\tilde{M} \models \varphi(x_1, \ldots, x_n) \leftrightarrow \langle x_1, \ldots, x_n \rangle \in X$, where $X = \{\langle \vec{z} \rangle \in u_1 \times \ldots \times u_n \mid M \models \varphi(\vec{z})\}$.)

Hence:

**Fact 2** Let $\tau > \omega$ be regular in $M$, where $\pi : M \prec M'$. Set $\bar{H} = H_\tau^M$, $\tilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$, $\bar{\pi} = \pi \upharpoonright \bar{H}$. Then $\bar{\pi} : \bar{H} \prec \tilde{H}$ cofinally.

**Def** Let $\pi : M \prec M'$. Let $\tau$ be regular in $M$. $\pi$ is $\tau$-cofinal iff
$$M' = \bigcup \{\pi(u) \mid u \in M \wedge \bar{\bar{u}} < \tau \text{ in } M\}$$
(Hence $\tau$-cofinality implies cofinality.)

*) In ZFC⁻ "the axiom of choice reads": Every set is enumerable by an ordinal.

**Def** Let $\tau > \omega$ be regular in $M$, $\bar{H} = H_\tau^M$. Let $\bar{\pi} : \bar{H} \prec H$ cofinally. By a *liftup* of $\langle M, \bar{\pi} \rangle$ we mean a pair $\langle M', \pi \rangle$ s.t. $M'$ is transitive, $\pi \upharpoonright \bar{H} = \bar{\pi}$, and $\pi : M \prec M'$ $\tau$-cofinally.

(We also say: "$\langle M', \pi \rangle$ is a liftup of $M$ by $\bar{\pi}$".)

**Fact 3** Let $\langle M, \bar{\pi} \rangle$ be as above. There is at most one liftup $\langle M', \pi \rangle$.

proof:

Clearly, every element of $M'$ has the form $\pi(f)(x)$, where $f \in M$, $f : u \to M$ for a $u \in \bar{H}$, and $x \in \bar{\pi}(u)$. But

$M' \models \varphi(\pi(f_1)(x_1), \ldots, \pi(f_n)(x_n)) \longleftrightarrow$

$\longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\pi}(X)$, where

$X = \{\langle z_1, \ldots, z_n \rangle \mid \varphi(f_1(z_1), \ldots, f_n(z_n))\}$

(hence $X \in \bar{H}$).

This means that if $\langle M'', \pi' \rangle$ is a second liftup, we can define $\sigma : M' \stackrel{\sim}{\longleftrightarrow} M''$ by $\sigma(\pi(f)(x)) = \pi'(f)(x)$. Hence $\sigma = \mathrm{id}$, $M' = M''$. But $\pi(z) = \pi(\mathrm{const}_z(0)) =$ $\sigma(\pi(\mathrm{const}_z(0))) = \pi'(\mathrm{const}_z(0)) = \pi'(z)$ for $z \in M$; where $\mathrm{const}_z = \{\langle z, 0 \rangle\}$. QED (Fact 3)

<u>Note</u> By this analysis it follows easily that, if $\langle M', \pi\rangle$ is the liftup of $M$ by $\pi : \bar{H} \prec H$, where $\bar{H} = H_{\tau}^{M}$, and $\tau' = On \cap H$, then $\pi(\tau) = \tau'$ and $H = H_{\tau'}^{M'}$. $\bar{H}$ need not be an element of $M$, but if it is, it follows that $\pi(\bar{H}) = H$.

The proof of Fact 3 suggests a general method of constructing the liftup:

<u>Def</u> Let $M$ be a transitive $ZFC^-$ model with predicates $A_1, \ldots, A_n$. Let $\bar{H} = H_{\bar{\tau}}^{M}$, where $\bar{\tau}$ is regular in $M$, and let $\pi : \bar{H} \prec H$ cofinally.

$$\mathbb{D} = \mathbb{D}_{M,\pi} = \langle D, E, I, \tilde{A}_1, \ldots, \tilde{A}_n \rangle$$

is defined by:

$$D = \{\langle x, f\rangle \mid f \in M \wedge f : u \to M \text{ for a } u \in \bar{H} \wedge x \in \pi(u)\}$$

$$\langle x, f\rangle E \langle y, g\rangle \longleftrightarrow \langle x, y\rangle \in \pi(\{\langle z, w\rangle \mid f(z) \in g(w)\})$$

$$\langle x, f\rangle I \langle y, g\rangle \longleftrightarrow \langle x, y\rangle \in \pi(\{\langle z, w\rangle \mid f(z) = g(w)\})$$

$$\tilde{A}_i(\langle x, f\rangle) \longleftrightarrow x \in \pi\{z \mid A_i f(z)\}.$$

We then get <u>Łoś Theorem</u> in the form

$$\mathbb{D} \models \varphi(\langle x_1, f_1\rangle, \ldots, \langle x_n, f_n\rangle) \longleftrightarrow$$

$$\longleftrightarrow \langle x_1, \ldots, x_n\rangle \in \pi(\{\langle z_1, \ldots, z_n\rangle \mid M \models \varphi(f_1(z_1), \ldots, f_n(z_n))\})$$

The proof is by induction on $\varphi$ and is just like the proof of Łoś theorem for ultrapowers. Then $\mathbb{D} \models ZFC^-$ and $\mathbb{D}$ is an equality model with equality relation $I$.

This gives:

<u>Fact 4</u> The liftup of $\langle \bar{M}, \bar{\pi} \rangle$ exists iff iff $E$ is well founded.

prf. (sketch)

$(\rightarrow)$ Let $\langle M', \pi \rangle$ is the liftup, then $a E b \leftrightarrow k(a) \in k(b)$ for $a, b \in \mathbb{D}$, where $k$ is defined by $k(\langle x, f \rangle) = \pi(f)(x)$.

$(\leftarrow)$ Factor $\mathbb{D}$ by $I$ to get $\mathbb{D}^* = \mathbb{D}/I$. Let $[u]$ be the equivalence class of $u$ for $u \in \mathbb{D}$. Then $\mathbb{D}^*$ satisfies extensionality and has a well founded $\in$-relation. Hence there is $\sigma^* : \mathbb{D}^* \overset{\sim}{\longleftrightarrow} M'$, where $M'$ is transitive by Mostowski's isomorphism theorem. Set: $\sigma(u) = \sigma^*([u])$ for $u \in \mathbb{D}$. We can define $\pi : M \prec M'$ by $\pi(x) = \sigma(\langle 0, \mathrm{const}_x \rangle)$, where $\mathrm{const}_x = \{\langle x, 0 \rangle\}$ = the constant function $x$ on $\{0\}$. Set: $\tilde{\mathbb{D}} = \{\langle x, f \rangle \in \mathbb{D} \mid f \in \bar{H}\}$; $H' = \{\sigma(u) \mid u \in \tilde{\mathbb{D}}\}$.

$H'$ is easily seen to be transitive.

But $H = \{\bar{\pi}(f)(x) \mid \langle x, f \rangle \in \tilde{\mathbb{D}}\}$

Moreover:

$$\overline{\pi}(f)(x) \in \overline{\pi}(g)(y) \longleftrightarrow \langle x,y\rangle \in \overline{\pi}(\{\langle z,w\rangle \mid f(x)=g(y)\})$$
$$\longleftrightarrow \sigma(\langle x,f\rangle) \in \sigma(\langle y,g\rangle)$$

for $\langle x,f\rangle \in \tilde{\mathbb{D}}$. Hence there is an isomorphism $i' : H \overset{\sim}{\longleftrightarrow} H'$ defined by $i'(\overline{\pi}(f)(x)) = \sigma(\langle x,f\rangle)$. Hence $i = \mathrm{id}$, $H = H'$ and $\sigma(\langle x,f\rangle) = \overline{\pi}(f)(x)$ for $\langle x,f\rangle \in \tilde{\mathbb{D}}$. In particular,

$$\pi(z) = \sigma(\langle 0, \mathrm{const}_z\rangle) = \overline{\pi}(\mathrm{const}_z)(0) =$$
$$= \mathrm{const}_{\overline{\pi}(z)}(0) = \overline{\pi}(z) \quad \text{for } z \in \overline{H}$$

Hence $\pi \upharpoonright \overline{H} = \overline{\pi}$. But then for $\langle x,f\rangle, \langle y,g\rangle \in \mathbb{D}$, we have:

$$\sigma(\langle x,f\rangle) \in \sigma(\langle y,g\rangle) \longleftrightarrow \langle x,y\rangle \in \overline{\pi}(\{\langle z,w\rangle \mid f(z) \in g(w)\})$$
$$\longleftrightarrow \pi(f)(x) \in \pi(g)(y).$$

Hence there is $i : M' \overset{\sim}{\longleftrightarrow} M'' \subset M$ defined by $i(\sigma(\langle x,f\rangle)) = \pi(f)(x)$. $M''$ is easily seen to be transitive, however, so $i = \mathrm{id}$ and each $z \in M'$ has the form $\pi(f)(x)$, where $\langle x,f\rangle \in \mathbb{D}$. It follows easily that $\langle M', \pi\rangle$ is the liftup of $\langle M, \overline{\pi}\rangle$. QED (Fact 4)

This gives us the <u>interpolation lemma</u>:

__Fact 5__ Let $\pi' : \bar{M} \prec M'$. Let $\bar{\tau} \in \bar{M}$ be regular in $\bar{M}$ and set $\bar{H} = H^{\bar{M}}_{\bar{\tau}}$. Let $\bar{\pi} : \bar{H} \prec H$ cofinally. Then:

(a) The liftup $\langle M, \pi \rangle$ of $\langle \bar{M}, \bar{\pi} \rangle$ exists

(b) There is a unique $\sigma : M \prec M'$ s.t. $\sigma\pi = \pi'$ and $\sigma \upharpoonright H = \mathrm{id}$.

prf.

To prove (a) we note that $E$ is well founded, since $\langle x, f \rangle E \langle y, g \rangle \longleftrightarrow \pi'(f)(x) \in \pi'(g)(y)$. But for $\langle x_1, f_1 \rangle, \ldots, \langle x_n, f_n \rangle \in \mathbb{D}$ we have:

$$M \models \varphi(\pi(f_1)(x_1), \ldots, \pi(f_n)(x_n)) \longleftrightarrow$$

$$\longleftrightarrow M' \models \varphi(\pi'(f_1)(x_1), \ldots, \pi'(f_n)(x_n))$$

$$\longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\pi}(\{\vec{z} \mid \bar{M} \models \varphi(f_1(z_1), \ldots, f_n(z_n))\}).$$

Hence there is $\sigma : M \prec M'$ defined by $\sigma(\pi(f)(x)) = \pi'(f)(x)$ for $\langle x, f \rangle \in \mathbb{D}$. But this $\sigma$ is characterized by the above conditions. QED (Fact 5)

The structure $\mathbb{D}^*$ will be of interest to us, however, even if it is ill founded. An embedding $\tilde{\pi} : M \prec \mathbb{D}^*$ is definable by $\tilde{\pi}(x) = [\langle 0, \mathrm{const}_x \rangle]$. This embedding is cofinal in the sense that for every $z \in \mathbb{D}^*$ there is $u \in M$ s.t. $\mathbb{D}^* \models z \in \tilde{\pi}(u)$.

In dealing with ill founded models of set theory it is useful to work with solid structures in the following sense:

Def Let $\mathfrak{A} = \langle A, \in_{\mathfrak{A}}, \ldots \rangle$ model the extensionality axiom. $\mathfrak{A}$ is solid iff the well founded core $wfc(\mathfrak{A})$ is transitive and $\in_{\mathfrak{A}} \cap wfc(\mathfrak{A})^2 = \in \cap wfc(\mathfrak{A})^2$.

($wfc(\mathfrak{A})$ is the set of $x \in \mathfrak{A}$ s.t. $\in_{\mathfrak{A}} \cap X^2$ is well founded, where $X$ is the closure of $\{x\}$ under $\in_{\mathfrak{A}}$).

Clearly, every model is isomorphic to a solid model.

We note the following facts about solid models of $ZFC^-$:

Fact 6 Let $\mathfrak{A}$ be a solid model of $ZFC^-$.
Let $H = wfc(\mathfrak{A})$. Then

(a) $\omega \subset H$; $\alpha \in H \to \alpha + 1 \in H$

(b) If $\alpha \in H$, $x \in \mathfrak{A}$, and $\mathfrak{A} \models rn(x) \leq \alpha$,
then $x \in H$

(c) $H$ is admissible

prf.
(a), (b) are trivial. We prove (c).

(*Note* We take the replacement axiom of ZFC⁻ as reading:

$$\bigwedge x \bigvee y\, \varphi(x,y,\vec{z}) \to \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi(x,y,\vec{z})$$

for arbitrary formulae $\varphi$. The theory KP ("Kripke-Platek set theory") is obtained by restricting the fmla $\varphi$ in this schema – and in the separation schema – to $\Sigma_0$ formulae. A transitive structure is called *admissible* iff it satisfies KP.)

By (b), H is easily seen to satisfy $\Sigma_0$-separation, as well as the trivial existence axioms: "$\emptyset$ is a set", "$\{x,y\}$ is a set", "$\cup x$ is a set". We prove $\Sigma_0$ replacement. Let $u \in H$. Let $H \models \bigwedge x \bigvee y\, \varphi(x,y,\vec{z})$. Let $R(x,y)$ mean: "$\varphi(x,y,\vec{z})$ and $y$ is of minimal rank." Then there is $v \in \mathfrak{M}$ s.t. $\mathfrak{M} \models \bigwedge x \in u \bigvee y \in v\, R(x,y)$. But if we take $v$ as being of minimal rank in $\mathfrak{M}$, it must have rank $\in H$. Hence $v \in H$. QED (Fact 6)

*Note* It follows that if $u \in H$ is transitive and $\delta = On \cap H$, then $L_\delta(u)$ is admissible.

We now extend some of our definitions to solid models of $ZFC^-$.

<u>Def</u> Let $\mathfrak{A}$ be a solid model of $ZFC^-$; Let $\tau \in wfc(\mathfrak{A})$ be regular in $\mathfrak{A}$ and let $\bar{H} = H_\tau^{\mathfrak{A}}$ (hence $\bar{H} \subset wfc(\mathfrak{A})$). Let $\bar{\pi} : \bar{H} \prec H$ cofinally, where $H$ is transitive. $\langle \mathfrak{A}', \pi \rangle$ is a <u>liftup</u> of $\langle \mathfrak{A}, \bar{\pi} \rangle$ iff $\mathfrak{A}'$ is solid, $\pi \upharpoonright \bar{H} = \bar{\pi}$, and $\pi : \mathfrak{A} \prec \mathfrak{A}'$ is $\tau$-cofinal (i.e. for each $x \in \mathfrak{A}'$ there is $u \in \mathfrak{A}$ s.t. $\bar{\bar{u}} < \tau$ in $\mathfrak{A}$ and $\mathfrak{A}' \models x \in \pi(u)$).

A virtual repetition of the proof of Fact 3 gives:

<u>Fact 7</u> Let $\langle \mathfrak{A}, \bar{\pi} \rangle$ be as above. Up to isomorphism there is at most one liftup $\langle \mathfrak{A}', \pi \rangle$.

<u>Note</u> As before, $\pi(\tau) = \sup\{\bar{\pi}(\nu) \mid \nu < \tau\}$ in $\mathfrak{A}'$; hence $\pi(\tau) \in wfc(\mathfrak{A}')$ and $H = H_{\pi(\tau)}^{\mathfrak{A}'} \subset wfc(\mathfrak{A}')$. If $\bar{H} \in wfc(\mathfrak{A})$, then $H = \pi(\bar{H}) \in wfc(\mathfrak{A}')$. But we can then form $\mathbb{D}$ as before [taking $\langle x, f \rangle \in \mathbb{D}$ iff ($\mathfrak{A} \models f : u \to V$) for a $u \in \bar{H}$ and $x \in \bar{\pi}(u)$) ].

Repeating the proof of Fact 4 we get:

Fact 8 Let $\langle \mathfrak{A}, \bar{\pi} \rangle$ be as above. Then the liftup exists.

(Note The liftup $\langle \mathfrak{A}', \bar{\pi} \rangle$ is unique only up to isomorphism. But then $\mathrm{wfc}(\mathfrak{A}')$ is unique, by solidity.)

We now weaken our earlier definition of fullness to:

Def Let $N$ be a transitive ZFC$^-$ model s.t. $N = \langle L_\tau[A], \in, A, \ldots \rangle$. $N$ is almost full iff there is a solid model $\mathfrak{A}$ of ZFC$^-$ s.t. $N \in \mathrm{wfc}(\mathfrak{A})$, $N$ is regular in $\mathfrak{A}$, and $\mathfrak{A} \models V = L(N)$.

Then by the above we have:

Fact 9 Let $N = \langle L_\tau[A], \in, A, \ldots \rangle$ be almost full. Let $\pi : N \prec N'$ cofinally. Then $N'$ is almost full. (Moreover, if $\mathfrak{A}$ verifies the almost fullness of $N$ and $\langle \mathfrak{A}', \pi' \rangle$ is the liftup of $\langle \mathfrak{A}, \pi \rangle$, then $\mathfrak{A}'$ verifies the almost fulness of $N'$.)

By Fact 6:

<u>Fact 10</u> Let $N$ be almost full. There is $\delta$ s.t. $L_\delta(N)$ is admissible and $N$ is regular in $L_\delta(N)$.

<u>Def</u> $\delta_N$ = the least $\delta$ s.t. $L_\delta(N)$ is admissible.

A major tool will be the following <u>transfer lemma</u>:

<u>Fact 11</u> Let $\bar{N}$ be almost full. Let $\pi : \bar{N} \prec N$ cofinally. Let $x_1, \ldots, x_n \in \bar{N}$ and let $\varphi$ be a $\Pi_1$ formula. Then

$$L_{\delta_{\bar{N}}}(\bar{N}) \models \varphi(\bar{N}, \vec{x}) \rightarrow L_{\delta_N}(N) \models \varphi(N, \pi(\vec{x})).$$

proof.
Let $\bar{\Omega}$ witness the almost fulness of $\bar{N}$ and let $\pi' : \bar{\Omega} \prec \Omega$ be the liftup of $\langle \bar{\Omega}, \pi \rangle$. Obviously:

(1) $a \in \mathrm{wfc}(\bar{\Omega}) \rightarrow \pi'(a) \in \mathrm{wfc}(\Omega)$ by Fact 6

(2) $L_{\delta_{\bar{N}}}(\bar{N}) \subset \mathrm{wfc}(\bar{\Omega})$, $L_\delta(N) \subset \mathrm{wfc}(\Omega)$

Suppose not. Then there is a least $\alpha < \delta_N$ s.t. $L_\alpha(N) \models \neg\varphi(N, \pi(\vec{x}))$. Since $L_{\delta_N}(N)$ is an initial segment of $\Omega$, we have:

(3) $\Omega \models \alpha$ is least s.t. $L_\alpha(N) \models \neg\varphi(N, \pi(\vec{x}))$

(4) $\Omega \models \bigwedge \gamma \leq \alpha\ L_\gamma(N)$ is not admissible.

But then $\alpha = \pi'(\bar{\alpha})$, where in $\bar{\mathfrak{M}}$:
(5) $\bar{\alpha}$ is least s.t. $L_{\bar{\alpha}}(\bar{N}) \models \neg\varphi(\bar{N}, \bar{x}^0)$
(6) $\bigwedge \nu \leq \bar{\alpha}$ $L_\nu(\bar{N})$ is not admissible.
But $\bar{\alpha} \in \mathrm{wfc}(\bar{\mathfrak{M}})$ by (1). Hence
$\bar{\alpha} < \beta = \mathrm{On} \cap \mathrm{wfc}(\bar{\mathfrak{M}})$ and $L_\beta(\bar{N})$ is admissible. Thus (5), (6) hold in $L_\beta(\bar{N})$, since $L_\beta(\bar{N})$ is an initial segment of $\bar{\mathfrak{M}}$. Hence (5), (6) hold outright and $\alpha < \delta_{\bar{N}}$. Contr!

QED (Fact 11)

<u>Note</u> Fact 11 is actually a special case of a more general theorem:
If $\bar{N} = \langle L_\tau[A], \in, A, \ldots \rangle$ is a $ZFC^-$ model,
$\pi : \bar{N} \prec N$ cofinally, and $N$ is regular in $L_{\delta_N}(N)$, then the conclusion of Fact 11 holds (even if $\bar{N}$ is not regular in $L_{\delta_{\bar{N}}}(\bar{N})$).
We shall not need this, however, and do not prove it here, since our proof involves a modest application of fine structure theory.

## §3.2 Barwise Theory

In addition to the transfer lemma we shall make use of Barwise' theory of infinitary language on admissible structures. In the following let M be an admissible structure satisfying choice in the form: Every set is enumerable by an ordinal. In admissibility theory the basic three notions of recursion theory are redefined as follows:

M-recursive $= \underline{\Delta}_1(M)$

M-recursively enumerable $= \underline{\Sigma}_1(M)$

M-finite = element of M.

Barwise then developed an extension of first order logic involving formulae which are infinitely long but still M-finite. Thus a Barwise language on M is like predicate logic except that, whenever $\langle \varphi_i \mid i \in x \rangle \in M$ is a sequence of formulae, then $\bigwedge\limits_{i \in x} \varphi_i$ and $\bigvee\limits_{i \in x} \varphi_i$ are formulae. (Infinite blocks of quantifiers are not allowed, however.) The set of variables is M-infinite (i.e. we could have a variable $v_\xi$ for each $\xi \in On \cap M$). A language is then specified by fixing its predicates, constants, and function symbols.

The syntax is developed internally in such a way that the basic syntactical notions (e.g. "formula", "term", "sentence") are $\Delta_1(M)$. A <u>mathematical theory</u> $\mathcal{L} = \langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ then consists of a language $\mathcal{L}_0$ and a set $\mathcal{L}_1$ of axioms (all of which are sentences). $\mathcal{L}_1$ should be $\Sigma_1(M)$, if we wish to make use of the admissibility of M. We augment the usual predicate logical rules of inference by two infinitary rules:

$$\frac{\psi \to \varphi_i \quad (i \in x)}{\psi \to \bigwedge_{i \in x} \varphi_i} \qquad \frac{\varphi_i \to \psi \quad (i \in x)}{\bigvee_{i \in x} \varphi_i \to \psi}$$

for $\langle \varphi_i \mid i \in x \rangle \in M$.

A <u>proof</u> is then a (possibly infinite) sequence of formulae, each of which is an axiom or follows from the previous formulae by a rule of inference. If the axiom set $\mathcal{L}_1$ is $\Sigma_1(M)$, it turns out that every provable formula has a proof p which is M-finite (i.e. $p \in M$). From this we get the

M-finiteness lemma: If $\varphi$ is provable in $\mathcal{L}$, then it is provable from an M-finite $u \subset \mathcal{L}_\varphi$.

A model $\mathfrak{A}$ of the language $\mathcal{L}_0$ is described by fixing its domain of individuals $|\mathfrak{A}|$ and the interpretation $s^{\mathfrak{A}}$ of each predicate symbol, constant, or function symbols, just as in finitary predicate logic. We can then straightforwardly define truth ($\mathfrak{A} \models \varphi$) for $\mathcal{L}_0$-sentences $\varphi$ and satisfaction ($\mathfrak{A} \models \varphi[a_1, \dots, a_n]$) for $\mathcal{L}_0$ + formulae containing only finitely many free variables. We say that $\mathfrak{A}$ models the theory $\mathcal{L} = \langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ iff all axioms in $\mathcal{L}_1$ are true in $\mathfrak{A}$. The notion of proof is correct in the sense that, if $\mathfrak{A}$ models $\mathcal{L}$, then sentence provable in $\mathcal{L}$ is true in $\mathfrak{A}$.

The final stone in this mosaic is the completeness theorem for countable M: If M is is countable, then $\mathcal{L}$ is consistent iff $\mathcal{L}$ has a model.

This means that for any admissible $M$, we can make the completeness theorem true in a generic extension of $V$ simply by collapsing $M$ to $\omega$. In many cases we can then use this to prove properties of $V$.

We note that if $\mathcal{L}_1$ is $\Sigma_1(M)$ in parameters $\vec{p}$, then the statement "$\mathcal{L}$ is consistent" is uniformly $\Pi_1(M)$ in $\vec{p}$, since it says that $M$ contains no proof of a contradiction. (But by the foregoing, "$\mathcal{L}$ is consistent" is equivalent to: $\Vdash_{\mathbb{P}}$ "$\mathcal{L}$ has a model", where $\mathbb{P}$ is any set of conditions which collapses $M$ to $\omega$.) It is in this context that we will apply the transfer lemma: Let $N$ be almost full and let $\pi: N \prec N'$ cofinally. Let $\mathcal{L}$ be a theory on $L_{\delta_N}(N)$ s.t. the set of axioms $\mathcal{L}_1$ is $\Sigma_1(N)$ in parameters $N$ and $\vec{p} \in N$. Let $\mathcal{L}'$ have the same definition in $N'$, $\pi(\vec{p})$ over $L_{\delta_{N'}}(N')$. Then:

$\mathcal{L}$ is consistent $\longrightarrow$ $\mathcal{L}'$ is consistent.

In this paper we shall deal only with languages $\mathcal{L}$ on $M$ which contain a binary "$\in$-predicate" $\dot{\in}$ and a designated constant $\underline{x}$ for each $x \in M$. (We suppose $\dot{\in}$ and $\langle \underline{x} \mid x \in M \rangle$ to have a uniform $\Delta_1(M)$ definition over any admissible $M$.) We also suppose that the set of axioms $\mathcal{L}_0$ contains a __base theory__ consisting of:

- $ZFC^-$ (including the schemata of separation and replacement for all finite formulae of $\mathcal{L}_0$)
- the "defining" axioms for the constants $\underline{x}$ $(x \in M)$: $\bigwedge v \, (v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$.

We note that if $\mathfrak{A}$ is a solid model of $\mathcal{L}$, we then have $\underline{x}^{\mathfrak{A}} = x \in \mathrm{wfc}(\mathfrak{A})$ for all $x \in M$.

## § 3.4 The forcing $\mathbb{P}_A$.

Let $A \subset \omega_2$ be a stationary set of points of cofinality $\omega$. We define:

Def $\mathbb{P}_A$ = the set of $p : \alpha+1 \to A$ s.t. $\alpha < \omega_1$ and $p$ is a normal function.

$p \leq q$ in $\mathbb{P}_A \longleftrightarrow p \supset q$.

Hence if $G$ is $\mathbb{P}_A$-generic, $f = \bigcup G$ is a cofinal normal function $f : \omega_1 \to A$. It is easily established that $f$ adds no reals.

Lemma 1 $\mathbb{P}_A$ is subcomplete.

proof.

Let $\mathbb{P}_A \in H_\theta$. Let $\tau > \theta$ be regular. Let $N = \langle L_\tau[A], A, \ldots \rangle$ where $H_\tau \subset N$. Let $\sigma : \bar{N} \prec N$ s.t. $\sigma(\bar{\theta}, \bar{\mathbb{P}}) = \theta, \mathbb{P}_A$ and $\bar{N}$ is countable and full.

Claim $\sigma$ witnesses the subcompleteness of $\mathbb{P}_A$.

Let $\bar{\lambda}_0 = \mathrm{On} \cap \bar{N}$, $\bar{\lambda}_i = \sigma^{-1}(\lambda_i)$ $(i = 1, \ldots, m)$ where $\mathbb{P}_A \in H_{\lambda_i}$ + $\lambda_i \in (\omega_1, \theta)$ is regular $(i = 1, \ldots, m)$.

Let $\tilde{\lambda}_i = \sup \sigma'' \bar{\lambda}_i$ $(i = 0, n, m)$

Let $\sigma(\bar{s}) = s$

Claim 1 There is $\sigma_0 : \bar{N} \prec N$ with

(a) $\sup \sigma_0{}'' \omega_2^{\bar{N}} \in A$

(b) $\sigma_0(\bar{s}, \bar{\lambda}_i, \bar{\mathbb{P}}) = s, \lambda_i, \mathbb{P}$

(c) $\sup \sigma_0{}'' \bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, n, m)$

prf.

For $\alpha < \omega_2$ set $X_\alpha$ = the smallest $X \prec N$ s.t. $\alpha \cup \mathrm{rng}(\sigma) \subset X$. Set:

$C = \{\alpha < \omega_2 \mid \alpha = \omega_2 \cap X\}$. Then $C$ is club in $\omega_2$.

For $\alpha \in C$ set $\pi_\alpha : N_\alpha \xleftrightarrow{\sim} X_\alpha$. Then

(1) $\alpha = \mathrm{crit}(\pi_\alpha)$, $\pi_\alpha(\alpha) = \omega_2$

(2) Set $\sigma_\alpha = \pi_\alpha^{-1}\sigma$. Then

$\langle N_\alpha, \sigma_\alpha \rangle$ = the liftup of $\langle \bar{N}, \sigma_\alpha \restriction H_{\omega_3}^{\bar{N}} \rangle$

prf.

Form $\langle N', \sigma' \rangle$ = the liftup of $\langle \bar{N}, \sigma_\alpha \restriction H_{\omega_3}^{\bar{N}} \rangle$.

Then there is $\pi' : N' \prec N_\alpha$ s.t. $\pi'\sigma' = \sigma_\alpha$ and $\pi' \restriction H_{\omega_3}^{N'} = \mathrm{id}$. But then $\pi' \restriction \alpha = \mathrm{id}$, since $\alpha < \omega_3^{N_\alpha}$. Hence $\alpha \cup \mathrm{rng}(\sigma) \subset \mathrm{rng}(\pi')$. Hence $\mathrm{rng}(\pi') = \mathrm{rng}(\pi_\alpha)$, $\pi' = \pi_\alpha$. QED (2)

Now let $\langle N', \sigma' \rangle$ = the liftup of $\langle \bar{N}, \sigma \restriction H_{\omega_2}^{\bar{N}} \rangle$. Since

$\pi_\alpha \upharpoonright H_{\omega_2}^{N_\alpha} = \mathrm{id}$ and $\pi_\alpha \sigma_\alpha = \sigma$, we have $\sigma_\alpha \upharpoonright H_{\omega_2}^{\bar{N}} = \sigma \upharpoonright H_{\omega_2}^{\bar{N}}$. Hence:

(3) $\langle N', \sigma' \rangle =$ the liftup of $\langle \bar{N}, \sigma_\alpha \upharpoonright H_{\omega_2}^{\bar{N}} \rangle$.

Hence there is $\pi' : N' \prec N_\alpha$ s.t. $\pi' \sigma' = \sigma_\alpha$ and $\pi' \upharpoonright H_{\omega_2}^{N'} = \mathrm{id}$.

(Note It is in fact easily seen that if $\alpha_0 = \min C$, then $\alpha_0 = \omega_2^{N'}$, $N' = N_{\alpha_0}$, and $\pi' = \pi_\alpha^{-1} \pi_{\alpha_0}$.)

Clearly $\pi' : N' \prec N_\alpha$ cofinally, since $\sigma' : \bar{N} \prec N'$ cofinally and $\sigma_\alpha : \bar{N} \prec N_\alpha$ cofinally.

Since $\sigma' : \bar{N} \prec N$ is the liftup of $\langle \bar{N}, \sigma \upharpoonright H_{\omega_2}^{\bar{N}} \rangle$, we have:

(4) $\sigma'(\tau) = \sup \sigma'``\tau$ whenever $\tau \geq \omega_2^{\bar{N}}$ is regular in $\bar{N}$.

Similarly:

(5) $\sigma_\alpha(\tau) = \sup \sigma_\alpha``\tau$ whenever $\tau \geq \omega_3^{\bar{N}}$ is regular in $\bar{N}$.

Now let $\delta' = \delta_{N'}$. Let $\mathcal{L}'$ be the language on $L_{\delta'}(N')$ containing the base theory and with a new constant $\dot{\sigma}$ and the axioms:

- $\sigma' : \underline{\bar{N}} \prec \underline{N}'$ cofinally
- $\sigma'(\bar{s}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \underline{\sigma'(\bar{s})}, \underline{\sigma'(\bar{\mathbb{P}})}, \underline{\sigma'(\bar{\lambda}_i)}$ $(i = 1, \ldots, m)$
- $\sup \sigma' {}''\tau = \sigma'(\tau)$ whenever $\tau$ is regular in $\underline{\bar{N}}$.

$\mathcal{L}'$ is consistent, since it is modeled by $\langle H_{\omega_1}, \sigma' \rangle$. Moreover the theory $\mathcal{L}'$ is $\Sigma_1(L_{\delta'}(N'))$ in the parameters

$N', \bar{N}, \bar{s}, \bar{\lambda}_i, \sigma'(\bar{s}), \sigma'(\bar{\lambda}_i)$ $(i = 1, \ldots, m)$,

Now let $\delta_\alpha = \delta_{N_\alpha}$ and let $\mathcal{L}^\alpha$ be $\Sigma_1(L_{\delta_\alpha}(N_\alpha))$ by the same definition in the parameters:

$N_\alpha, \bar{N}, \bar{s}, \bar{\lambda}_i, \sigma_\alpha(\bar{s}), \sigma'(\bar{\lambda}_i)$ $(i = 1, \ldots, m)$

Then $\mathcal{L}^\alpha$ is consistent by the transfer lemma, since $\pi' : N' \prec N_\alpha$ is cofinal and $\pi'\sigma' = \sigma_\alpha$ and $\pi' \upharpoonright H_{\omega_1} = \mathrm{id}$.

By this we get:

(6) Let $\mathrm{cf}(\alpha) = \omega$. Then in $V$ there is a map $\sigma_1$ s.t.

- $\sigma_1 : \bar{N} \prec N'$ cofinally
- $\sigma_1(\bar{s}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \sigma_\alpha(\bar{s}), \sigma_\alpha(\bar{\mathbb{P}}), \sigma_\alpha(\bar{\lambda}_i)$ $(i = 1, \ldots, m)$
- $\sup \sigma_1 {}''\tau = \sigma_1(\tau)$ whenever $\tau \geq \omega_2^{\bar{N}}$ is regular in $\bar{N}$.

<u>Note</u> If $\alpha > \sup \sigma'' \omega_2^{\bar N}$, then we cannot have $\sigma_1 = \sigma_\alpha$, since $\sigma_\alpha \restriction \omega_2^{\bar N} = \sigma \restriction \omega_2^{\bar N}$.

proof of (6)

Let $Y \prec H_{\omega_2}$ be countable s.t. $N_\alpha, \sigma_\alpha \in Y$. $\alpha = \sigma_\alpha(\omega_2^{\bar N})$ is $\omega$-cofinal and $\sigma_\alpha(\tau)$ is $\omega$-cofinal whenever $\tau > \omega_2^{\bar N}$ is regular in $\bar N$ by (5). Hence $Y \cap \sigma_\alpha(\tau)$ is cofinal in $\sigma_\alpha(\tau)$ whenever $\tau \geq \omega_2^{\bar N}$ is regular in $\bar N$.
Let $k : \bar H \overset{\sim}{\longleftrightarrow} Y$, $k(\bar N_\alpha) = N_\alpha$, $k(\bar\sigma_\alpha) = \sigma_\alpha$, $k(\bar\delta_\alpha) = \delta_\alpha$, $k(\bar{\mathcal{L}}^\alpha) = \mathcal{L}^\alpha$. Then $k \restriction \bar N_\alpha : \bar N_\alpha \prec N_\alpha$ cofinally (since $On \cap N_\alpha$ has cofinality $\omega$) and $k''\bar\sigma_\alpha(\tau)$ is cofinal in $\sigma_\alpha(\tau)$ whenever $\tau \geq \omega_2^{\bar N}$ is regular in $\bar N$. $\bar{\mathcal{L}}^\alpha$ is consistent and therefore, by countability, has a solid model $\mathfrak{A}$.
Let $\bar\sigma_1 = \dot\sigma^{\mathfrak{A}}$. Then $\bar\sigma_1 \in wfc(\mathfrak{A})$ and!

- $\bar{\sigma}_1 : \bar{N} \prec \bar{N}_\alpha$ cofinally
- $\bar{\sigma}_1(\bar{\kappa}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \bar{\sigma}_\alpha(\bar{\kappa}), \bar{\sigma}_\alpha(\bar{\mathbb{P}}), \bar{\sigma}_\alpha(\bar{\lambda}_i)$.
- $\sup \bar{\sigma}_1 `` \tau = \bar{\sigma}_1(\tau)$ whenever $\tau \geq \omega_2^{\bar{N}}$ is regular in $\bar{N}$.

But then $\sigma_1 = k\bar{\sigma}_1$ has the desired properties,

QED (6).

Now let $\alpha \in A \cap C$. Then $cf(\alpha) = \omega$, Let $\sigma_1$ be as in (6) and set $\sigma_0 = \pi_\alpha \sigma_1$. Then $\alpha = \sup \sigma_0 `` \omega_2^{\bar{N}} \in A$, since $\pi_\alpha \upharpoonright \alpha = id$.
$\sigma_0(\bar{\kappa}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \pi\sigma_\alpha(\bar{\kappa}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \kappa, \mathbb{P}_A, \lambda_i$.
But $\sup \sigma_1 `` \bar{\lambda}_i = \sigma_1(\bar{\lambda}_i) = \sigma_\alpha(\bar{\lambda}_i) = \sup \sigma_\alpha `` \bar{\lambda}_i$.
Hence $\sup \sigma_0 `` \bar{\lambda}_i = \sup \pi `` \sigma_1(\bar{\lambda}_i) =$
$= \sup \pi `` \sigma_\alpha(\bar{\lambda}_i) = \sup \sigma `` \bar{\lambda}_i = \tilde{\lambda}_i$

QED (Claim 1)

Now let $\sigma_0$ be as in Claim 1 and let $\alpha = \sup \sigma_0 `` \omega_2^{\bar{N}} \in A$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{N}$. Set $\bar{g} = \bigcup \bar{G}$. Then $\bar{g} : \omega_1^{\bar{N}} \to \dot{\bar{A}}$ is normal and cofinal, where $\sigma(\bar{A}) = A$. But $\sigma_0(\bar{A}) = A$, since $A = \bigcup_{p \in \mathbb{P}} dom(p)$. Set $g = \sigma_0 \circ \bar{g}$

Then $g : \omega_1^{\bar N} \to A$ is normal with $\sup g''\omega_1^{\bar N} = \alpha \in A$. Set $p = g \cup \{\langle \alpha, \omega_1^{\bar N}\rangle\}$. Then $p \in \mathbb{P}_A$ and $\sigma_0(p)$

$$p \leq \sigma_0(q) \longleftrightarrow q \subset \bar q \longleftrightarrow q \in \bar G$$

for $q \in \bar{\mathbb{P}}$. Hence, if $G \ni p$ is $\mathbb{P}_A$-generic, then $\bar G = \sigma_0^{-1}{}''G$.

QED (Lemma 1)

<u>Note</u> $\mathbb{P}_A$ is provably not semiproper. Hence we have shown that not every subcomplete forcing is semiproper.

<u>Note</u> This example is rather special in the sense that $\sigma_0$ lies in $V$. That will not hold if - as in the next example - a regular cardinal becomes $\omega$-cofinal.

## § 3.5 Prikry forcing

Let $U$ be a normal measure on $\kappa$. Let $\mathbb{P} = \mathbb{P}_U$ be the set of Prikry conditions for adding a cofinal $\omega$-sequence to $\kappa$.

**Lemma 2** $\mathbb{P}$ is subcomplete.

We remember that $\mathbb{P}$ consists of all pairs $\langle s, X \rangle$ s.t. $X \in U$ and $s : n \to \kappa$ is monotone for some $n$.

$$\langle s', X' \rangle \leq \langle s, X \rangle \underset{\text{df}}{\longleftrightarrow} (s' \supset s \wedge X' \subset X \wedge$$
$$\wedge\ \mathrm{rng}(s') \setminus \mathrm{rng}(s) \subset X)$$

If $G$ is $\mathbb{P}$-generic, then
$$S = \bigcup \{ s \mid \bigvee X\ \langle s, X \rangle \in G \}$$
is called a Prikry sequence. $G$ is then recoverable from $S$ by:
$$G = \{ \langle s, X \rangle \mid s \subset S \wedge \mathrm{rng}(S) \setminus \mathrm{rng}(s) \subset X \}.$$
It can be shown that $S : \omega \to \kappa$ is a Prikry sequence iff $\mathrm{rng}(S)$ is almost contained in every $X \in U$,

Now let $\mathbb{P} \in H_\theta$. Let $\tau > \theta$ be regular and $N = \langle L_\tau[A], A, \in \rangle$ s.t. $H_\theta \subset N$. Let $\sigma : \bar{N} \prec N$ be countable and full s.t. $\sigma(\bar{\mathbb{P}}) = \mathbb{P}$.

<u>Claim</u> $\sigma$ witnesses the subcompleteness of $\mathbb{P}$.

Let $\lambda_i \in \mathrm{rng}(\sigma)$ s.t. $\mathbb{P} \in H_{\lambda_i}$ and $\lambda_i \in (\omega_1, \theta)$ is regular ($i = 1, \dots, m$).

Set: $\bar{\lambda}_i = \sigma^{-1}(\lambda_i)$.

Let $\sigma(\bar{s}) = s$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{N}$. We must show:

<u>Claim</u> There is $p \in \mathbb{P}$ which forces that whenever $G \ni p$ is $\mathbb{P}$-generic, then there is $\sigma_0 \in V[G]$ s.t.

(a) $\sigma_0 : \bar{N} \prec N$ cofinally

(b) $\sigma_0(\bar{s}, \bar{\mathbb{P}}, \bar{\lambda}_i) = s, \mathbb{P}, \lambda_i$ $(i = 1, \dots, m)$

(c) $\sup \sigma_0 {}'' \bar{\lambda}_i = \tilde{\lambda}_i =_{\mathrm{Df}} \sup \sigma {}'' \bar{\lambda}_i$

for $i = 1, \dots, m$.

(d) $\bar{G} = \sigma_0^{-1} {}'' G$.

Let $\langle N', \sigma' \rangle =$ the liftup of $\langle \bar{N}, \sigma \restriction H_{\bar{\kappa}}^{\bar{N}} \rangle$ where $\sigma(\bar{\kappa}) = \kappa$. Then $\sigma' : \bar{N} \prec N'$ cofinally and $\sup \sigma'{}''\tau = \sigma'(\tau)$ for all $\tau \geq \bar{\kappa}$ s.t. $\tau$ is regular in $\bar{N}$.

Let $\bar{g} : \omega \to \bar{\kappa}$ be the Prikry sequence engendered by $\bar{G}$. Set $g' = \sigma' \circ \bar{g}$. Then $g' : \omega \to \kappa' = \sigma'(\bar{\kappa})$ cofinally.

(1) $g'$ is a Prikry sequence for $N'$ (wrt. $U' = \sigma'(\bar{U})$)

prf.

We must show that $\mathrm{rng}(g')$ is almost contained in $X$ for every $X \in U'$. But $X = \sigma'(f)(\xi)$, where $f \in \bar{N}$, $f : \alpha \to \bar{U}$ for an $\alpha < \kappa$, and $\xi < \sigma(\alpha) = \sigma'(\alpha)$. Hence $\bar{Y} = \bigcap f''\alpha \in \bar{U}$ and $Y = \sigma'(\bar{Y}) = \bigcap \sigma'(f)''\sigma(\alpha) \in U'$. Hence $\bar{g}$ is almost contained in $\bar{Y}$ and $g'$ is almost contained in $Y \subset X$.

QED(1)

Now let $\langle N'', \sigma'' \rangle$ be the liftup of $\langle \bar{N}, \sigma \restriction [illegible] \rangle$, where $\mu = \bar{\kappa}^{++\bar{N}}$.

Then $\sigma''(\bar{\kappa}+\bar{N}) = \kappa^+$, $\sigma''(H^{\bar{N}}_{\bar{\kappa}+\bar{N}}) = H_{\kappa^+}$ and $\sigma''(\tau) = \sup \sigma''``\tau$ whenever $\tau \geq \bar{\kappa}^{+\bar{N}}$ is regular in $\bar{N}$.

Let $\delta' = \delta_{N'}$, $\delta'' = \delta_{N''}$. Let $\mathcal{L}'$ be the infinitary language on $L_{\delta'}(N')$ comprising the base theory, a new constant $\mathring{\sigma}$ and the further axioms:

- $\mathring{\sigma} : \underline{\bar{N}} \prec \underline{N'}$ cofinally
- $\mathring{\sigma}(\underline{\bar{\Lambda}}, \underline{\bar{\mathbb{P}}}, \underline{\bar{\lambda}_i}, \underline{\bar{\kappa}}) = \underline{\sigma'(\bar{\Lambda})}, \underline{\sigma'(\bar{\mathbb{P}})}, \underline{\sigma'(\bar{\lambda}_i)}, \underline{\sigma'(\bar{\kappa})}$
- $\sup \mathring{\sigma}``\tau = \mathring{\sigma}(\tau)$ whenever $\tau \geq \bar{\kappa}$ is regular in $\bar{N}$
- $\mathring{\sigma} \circ \underset{\sigma}{\bar{G}}$ is Prikry generic over $N'$

Then $\mathcal{L}'$ is consistent, since $\langle H_{\kappa^+}, \sigma' \rangle$ models $\mathcal{L}'$.

But $\langle N', \sigma' \rangle$ is the liftup of $\langle \bar{N}, \sigma'' \upharpoonright H^{\bar{N}}_{\bar{\kappa}} \rangle$ since $\sigma'' \upharpoonright H^{\bar{N}}_{\bar{\kappa}} = \sigma \upharpoonright H^{\bar{N}}_{\bar{\kappa}}$.

Hence there is unique $\pi : N' \prec N''$ cofinally, s.t. $\pi \upharpoonright H^{N'}_{\sigma'(\bar{\kappa})} = \mathrm{id}$ and $\pi\sigma' = \sigma''$. But then $\mathcal{L}''$

is consistent, where $\mathcal{L}''$ has the same definition over $L_{\delta''}(N'')$ in the parameters $\bar{\tau}, \bar{\mathbb{P}}, \bar{\lambda}_i, \bar{g}, \bar{u}, \sigma''(\bar{\tau}), \sigma''(\bar{\mathbb{P}}), \sigma''(\bar{\lambda}_i), \sigma''(\bar{u})$. (Note that $\sigma'(\bar{u}) = U \cap H^{N'}_{\sigma'(\bar{\tau}+\bar{N})}$ and $\sigma''(\bar{u}) = U$, since $\bar{u} = \{x \mid V_\lambda \langle \lambda, x\rangle \in \bar{\mathbb{P}}\}$.)

Now generically collapse $\delta''$ to $\omega$. In the resulting model $V[\tilde{G}]$ let $\mathfrak{A}$ be a solid model of $\mathcal{L}''$. Set $\sigma_1 = \mathring{\sigma}^{\mathfrak{A}}$, $g = \sigma_1 \circ \bar{g}$. Then

(2) $g$ is Prikry generic over $V$ since $\sigma''(\bar{u}) = U$,

Since $g$ is Prikry generic over $N''$ and $N''$ is regular in $L_{\delta''}(N'')$, $g$ is also Prikry generic over $L_{\delta''}(N'')$; hence

(3) $L_{\delta''}(N''[g])$ is admissible.

Let $\mathcal{L}^*$ be the language on $L_{\delta''}(N''[g])$ with the base, a constant $\mathring{\sigma}$, the axioms of $\mathcal{L}''$, and the axiom: $\underline{g} = \mathring{\sigma}''\underline{\bar{g}}$.

Then $\mathcal{L}^*$ is consistent, since $\langle H^{V[G]}_{\kappa^{++}}, \sigma'' \rangle$ is a model. But $\mathcal{L}^* \in V[g]$. We now virtually repeat the proof of (6) in §3.4 to get:

(4) In $V[g]$ there is $\sigma^*$ s.t.

- $\sigma^* : \bar{N} \prec N''$ cofinally
- $\sigma^*(\bar{\kappa}, \bar{\mathbb{P}}, \bar{x}_i) = \sigma''(\bar{\kappa}), \sigma''(\bar{\mathbb{P}}), \sigma''(\bar{x}_i)$ $(i = 1, \dots, m)$
- $\sigma^* \circ \bar{g} = g$
- $\sup \sigma^{*\,\prime\prime}\tau = \sigma^*(\tau)$ whenever $\tau \geq \bar{\kappa}^{++\bar{N}}$ is regular in $\bar{N}$.

proof (sketch)

We work in $V[g]$. As before let $Y \prec H_{\kappa^{++}}$ be countable s.t. $\bar{N}, N'', \sigma'' \in Y$. As before $Y \cap \sigma''(\tau)$ is cofinal in $\sigma''(\tau)$ whenever $\tau = \bar{\kappa}$ or $\tau \geq \bar{\kappa}^{++\bar{N}}$ is regular in $\bar{N}$. Let $k : \bar{H} \overset{\sim}{\longleftrightarrow} Y$, $k(\bar{N}'') = N''$, $k(\bar{\sigma}'') = \sigma''$, $k(\bar{\delta}'') = \delta''$, $k(\bar{\mathcal{L}}^*) = \mathcal{L}^*$.

Then $k \upharpoonright \bar{N}_\alpha : \bar{N}_\alpha \prec N_\alpha$ cofinally. Moreover $k''\bar{\sigma}''(\tau)$ is cofinal in $\sigma''(\tau)$ whenever $\tau = \bar{\kappa}$ or $\tau \geq \bar{\kappa}^{++\bar{N}}$ is regular in $\bar{N}$. $\bar{\mathcal{L}}^*$ is consistent + hence has a solid model $\mathfrak{A}$. Let $\sigma^* = \dot{\sigma}^{\mathfrak{A}}$. Then

$\bar{\sigma}^* \in wfc(\mathfrak{A})$ and:

- $\bar{\sigma}^* : \bar{N} \prec \bar{N}''$ cofinally
- $\bar{\sigma}^*(\bar{\kappa}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \bar{\sigma}''(\bar{\kappa}), \bar{\sigma}''(\bar{\mathbb{P}}), \bar{\sigma}''(\bar{\lambda}_i)$
- $\vec{g}^* = \bar{\sigma}^* \circ \bar{g}$ , where $k(\bar{g}^*) = g$
- $\sup \bar{\sigma}^*{}'' \tau = \bar{\sigma}^*(\tau)$ if $\tau = \bar{\kappa}$ or $\tau \geq \bar{\kappa}^{++\bar{N}}$ is regular in $\bar{N}$,

But then $\sigma^* = k\bar{\sigma}^*$ has the desired properties.

QED(4)

Since $\langle N'', \sigma'' \rangle$ = the liftup of $\langle \bar{N}, \sigma \upharpoonright H^{\bar{N}}_{\bar{\kappa}^{++}} \rangle$, there is $\pi_0 : N'' \prec N$ s.t. $\pi_0 \upharpoonright H^{\bar{N}}_{\bar{\kappa}^{++}} = id$ and $\pi_0 \sigma'' = \sigma$. Set: $\sigma_0 = \pi_0 \sigma^*$.

It follows easily that:

- $\sigma_0 : \bar{N} \prec N$ cofinally
- $\sigma_0(\bar{\kappa}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \kappa, \mathbb{P}, \lambda_i$
- $\sup \sigma_0{}'' \bar{\lambda}_i = \tilde{\lambda}_i$
- $g = \sigma_0 \circ \bar{g} = \sigma^* \circ \bar{g}$

But $g, G$ are interdefinable in $V[g] = V[G]$, where $G$ is a $\mathbb{P}$-generic set. Similarly for $\bar{g}, \bar{G}$ in $\bar{N}[\bar{G}]$. Hence

- $\bar{G} = \sigma_0^{-1}{}'' G$.

Since $G$ is $\mathbb{P}$-generic, there must be a $p \in G$ which forces all of this.

QED(Lemma 2)

This proof can easily be modified to show that $\mathbb{P}$ is subproper above $\mu$ for each $\mu < \kappa$, in the sense of the definition at the end of § 2: Letting $\sigma \upharpoonright H^{\bar{N}}_{\bar{\kappa}} : H^{\bar{N}}_{\bar{\kappa}} \prec \tilde{H}$ cofinally, we have $\tilde{H} = H_{\tilde{\kappa}}$, where $\tilde{\kappa} = \sup \sigma''\bar{\kappa} = \sigma'(\kappa)$, where $\langle N', \sigma' \rangle$ is defined as above. But then $\sigma \upharpoonright H^{\bar{N}}_{\bar{\mu}} \in \tilde{H}$ for $\bar{\mu} < \bar{\kappa}$. We can thus add to $\mathcal{L}'$ the axiom: $\dot{\sigma} \upharpoonright H^{\bar{N}}_{\bar{\mu}} = \underline{\sigma \upharpoonright H^{\bar{N}}_{\bar{\mu}}}$. Carrying this axiom with us through the rest of the proof, we arrive at $\sigma_0$ s.t. $\sigma_0 \upharpoonright H^{\bar{N}}_{\bar{\mu}} = \sigma \upharpoonright H^{\bar{N}}_{\bar{\mu}}$. QED

Our further examples will all be revisable $\mathcal{L}$-forcings in the sense of [J] §3. This will be true even of Namba forcing, since we have shown in [J] § 6 that Namba forcing is equivalent to such an $\mathcal{L}$-forcing.

From now on we assume a knowledge of [J] §3.

## § 3.6 Namba Forcing

Lemma 3 The forcing $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ of [J] §5. Example 1 (p. 7) is subcomplete.

In this forcing we start with a regular $\beta \geq \omega_2$ s.t. $2^\omega = \omega_1$ and $2^\beta = \beta$. $\mathbb{P}$ then collapses each regular $\gamma \in (\omega_1, \beta]$ to $\omega_1$, making it $\omega$-cofinal without collapsing reals. In [J] §6 we show that if $\beta = \omega_2$, then $\mathbb{B}A(\mathbb{P}) = \mathbb{B}.A(\mathbb{N})$, where $\mathbb{N}$ is Namba forcing. Hence:

Corollary 3.1 If $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$, then Namba forcing is subcomplete.

. . . .

We assume that the reader has a good understanding of [J] §3. We shall also make use of Corollary 2.8 in [J] §4, which says that if $G$ is $\mathbb{P}$-generic over $V$, then $G$ is definable from $\langle M^G, \pi^G, B^G \rangle$ by:

$$p \in G \leftrightarrow (M^p = M^G \upharpoonright (|p|+1) \wedge \pi^p = \pi^G \upharpoonright (|p|+1)^2 \wedge$$
$$\wedge\, b^p = (\pi^G_{|p|, \omega_1})^{-1} \text{''} B^G \wedge$$
$$\wedge \bigwedge \langle \bar{a}, a \rangle \in F^p\ \pi^G_{|p|, \omega_1} : \langle M^p_{|p|}, \bar{a} \rangle \prec \langle M, a \rangle).$$

We set: $M = L^A_\beta = \langle L_\beta[A], A \rangle$, where $L_\beta[A] = H_\beta$. Set: $N = \langle H_{\beta^+}, M, <, \ldots \rangle$, where $<$ well orders N. $\mathcal{L}$ is then the language on N which, in addition to the base theory has constants $\mathring{M}, \mathring{\pi}, \mathring{B}$, the "basic axioms" of [J] §3:

- $\dot{M} = \langle \dot{M}_i \mid i \leq \underline{\omega_1} \rangle$, $\dot{\pi} = \langle \dot{\pi}_{ij} \mid i \leq j \leq \underline{\omega_1} \rangle$
- $\dot{\pi}$ is a continuous commutative sequence of elementary embeddings $\dot{\pi}_{ij} : \dot{M}_i \prec \dot{M}_j$
- $\dot{M}_{\underline{\omega_1}} = \underline{M}$ ; $\dot{M}_i$ is countable and transitive for $i < \underline{\omega_1}$
- $\dot{\pi}_{ij} \upharpoonright \dot{\alpha}_i = id$, $\dot{\pi}_{ij}(\dot{\alpha}_i) = \dot{\alpha}_j$ where $\dot{\alpha}_i =_{\mathrm{df}} \omega_1^{\dot{M}_i}$
- $\dot{\beta}_i < \dot{\alpha}_{i+1}$ for $i < \underline{\omega_1}$, where $\dot{\beta}_i =_{\mathrm{df}} On \cap \dot{M}_i$
- $B \subset \bar{M}$
- $H_{\omega_1} = \underline{H_{\omega_1}}$,

and the further axioms:

- $\dot{\pi}_{i\underline{\omega_1}}$ is $\dot{\tau}_i$-cofinal in $\underline{M}$ $(i < \underline{\omega_1})$, where $\dot{\tau}_i =_{\mathrm{df}} \omega_2^{\dot{M}_i}$
- $\mathrm{rng}(\dot{\pi}_{i+1,\underline{\omega_1}})$ = the smallest $X \prec \underline{M}$ s.t. $\mathrm{rng}(\dot{\pi}_{i,\underline{\omega_1}}) \cup \{\dot{\alpha}_i\} \subset \dot{X}$

  for $i < \dot{\omega}_1$
- $\mathring{B} = \emptyset$

(Hence $\dot{B}$ will play no role and we shall ignore it.) In [J] §5 we show that $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is revisable and, therefore, adds no reals. If $G$ is $\mathbb{P}$-generic, then $\langle \underline{M}, \pi^G_{i\omega_1} \rangle$ is the liftup of

$\langle M_i^G, \pi_{i\omega_1}^G \upharpoonright H_{\tau_i^G}^{M_i} \rangle$ ($M^G$, $\pi^G$ and $\tau_i^G = \omega_1^{M_i^G}$ being defined in the obvious way.) If $\sigma(\bar{\gamma}) = \gamma$ where $\gamma \in (\omega_1, \beta]$ is regular, it follows that $\gamma \cap \mathrm{rng}(\pi_{i,\omega_1}^G)$ is cofinal in $\gamma$. Hence each regular $\gamma \in (\omega_1, \beta]$ becomes $\omega$-cofinal.

Now let $\theta$ be a cardinal s.t. $2^{2^\beta} < \theta$. Let $\lambda > \theta$ be regular and let $Q = \langle L_\lambda[A], A, \ldots \rangle$ s.t. $H_\theta \subset Q$. (We write "Q" instead of "N" to avoid confusion with the N just defined. We certainly have $N \in Q$ and hence $\mathbb{P} \in Q$.) Let $\sigma : \bar{Q} \prec Q$ s.t. $\bar{Q}$ is countable and full and $\sigma(\bar{\mathcal{L}}, \bar{\mathbb{P}}, \bar{\theta}) = \mathcal{L}, \mathbb{P}, \theta$.

<u>Claim</u> $\sigma$ witnesses the subcompleteness of $\mathbb{P}$.

proof. ...

Let $\sigma(\bar{s}) = s$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{Q}$. Let $\sigma(\bar{\lambda}_i) = \lambda_i$ $(i = 1, \ldots, m)$, where $\lambda_i < \theta$ is regular s.t. $\mathbb{P} \in H_{\lambda_i}$.

Since $\sigma(\bar{\mathcal{L}}) = \mathcal{L}$ we also have $\sigma(\bar{N}) = M$, $\sigma(\bar{N}) = \bar{N}$, where $\bar{\mathcal{L}}$ is a language on $\bar{N}$. Since $\bar{G}$ is $\bar{\mathbb{P}} = \mathbb{P}_{\bar{\mathcal{L}}}$ - generic over $\bar{Q}$, we get $M^{\bar{G}}, \pi^{\bar{G}}$ with $M^{\bar{G}}_{\alpha} = \bar{M}$ where $\alpha = \omega_1^{\bar{Q}}$. If $p = \langle M^p, \pi^p, \bar{F}^p \rangle \in \bar{\mathbb{P}}$, then $\sigma(p) = \langle M^p, \pi^p, F^p \rangle$, where $F^p = \{\langle \sigma(a), \bar{a} \rangle \mid \langle a, \bar{a} \rangle \in \bar{F}^p\}$.

We now prove:

<u>Claim 1</u> Let $\sigma : \bar{Q} \prec Q'$ cofinally. Let $\lambda'_i = \sup \sigma'' \bar{\lambda}_i$ $(i = 0, \dots, m)$ where $\bar{\lambda}_0 = On \cap \bar{Q}$. (Hence $Q' = \langle L_{\lambda'_0}[A], A, \dots \rangle$, where $Q = \langle L_\lambda[A], A, \dots \rangle$.) (Hence $Q' \prec Q$.) Let $\delta' = \delta_{Q'}$. Let $\mathcal{L}'$ be the language on $L_{\delta'}[Q']$ with the base theory (of § 3.2), a constant $\dot{\sigma}$ and axioms:

- $\dot{\sigma} : \underline{\bar{Q}} \prec \underline{Q'}$
- $\dot{\sigma}(\underline{\bar{\mathbb{P}}}, \underline{\bar{\lambda}}, \underline{\bar{\lambda}_i}) = \underline{\mathbb{P}}, \underline{\lambda}, \underline{\lambda_i}$ $(i = 1, \dots, m)$ (hence $\dot{\sigma}(\underline{\bar{M}}, \underline{\bar{N}}) = \underline{M}, \underline{N}$)
- $\langle \underline{M}, \dot{\sigma} \restriction \underline{\bar{M}} \rangle =$ the liftup of $\langle \underline{\bar{M}}, \dot{\sigma} \restriction \underline{H^{\bar{m}}_{\omega_2}} \rangle$
- $\sup \dot{\sigma}'' \bar{\lambda}_i = \lambda'_i$ $(i = 0, \dots, m)$,
- For $\nu < \omega_1$ let $X_\nu =$ the smallest $X \prec \underline{M}$ s.t. $\nu \cup \dot{\sigma}'' \underline{\bar{M}} \subset X$. Set $\dot{C} = \{\nu < \underline{\omega_1} \mid \nu = \underline{\omega_1} \cap X_\nu\}$. Then $otp(\dot{C}) = \underline{\omega_1}$.

<u>Then</u> $\mathcal{L}'$ is consistent.

proof of Claim 1

Let $\langle Q_0, \sigma_0\rangle$ be the liftup of $\langle \bar{Q}, \sigma \upharpoonright H^{\bar{Q}}_{\omega_2}\rangle$ (where, of course, $H^{\bar{Q}}_{\omega_2} = H^{\bar{M}}_{\omega_2}$). Let $\delta_0 = \delta_{Q_0}$. Let $\mathcal{L}_0$ be the language on $L_{\delta_0}(Q_0)$ with the base theory, a constant $\dot{\sigma}$, and the axioms: $\cdot\, H_{\omega_1} = \underline{H_{\omega_1}}$.

$\cdot\ \dot{\sigma} : \underline{\bar{Q}} \prec \underline{Q_0}$ is the liftup of $\langle \underline{\bar{Q}}, \dot{\sigma} \upharpoonright \underline{H^{\bar{Q}}_{\omega_2}}\rangle$

$\cdot\ \dot{\sigma}(\underline{\bar{\mathbb{P}}}, \underline{\bar{\tau}}, \underline{\bar{\lambda}_i}) = \underline{\sigma_0(\bar{\mathbb{P}})}, \underline{\sigma_0(\bar{\tau})}, \underline{\sigma_0(\bar{\lambda}_i)}$

$(i = 1, \ldots, m)$.

Then $\mathcal{L}_0$ is consistent, since $\langle H_{\omega_2}, \sigma_0\rangle$ models $\mathcal{L}_0$.

Now let $\langle Q_1, \sigma_1\rangle$ be the liftup of $\langle \bar{Q}, \bar{\sigma} \upharpoonright H^{\bar{Q}}_{\bar{\tau}}\rangle$, where $\bar{N} = H^{\bar{Q}}_{\bar{\tau}}$, $N = H^{Q}_{\tau}$, $\tau = \beta^+$. There is a unique $k : Q_0 \prec Q_1$ s.t. $k\sigma_0 = \sigma_1$ and $k \upharpoonright H^{\bar{Q}}_{\omega_1} = id$.

Clearly k takes $Q_0$ cofinally to $Q_1$ and k takes $\sigma_0(\bar{\lambda})$ cofinally to $\sigma_1(\bar{\lambda})$ whenever $\bar{\lambda} \geq \bar{\tau}$ is regular in $\bar{Q}$. Let $\mathcal{L}_1$ be the language on $L_{\delta_1}(Q_1)$ $(\delta_1 = \delta_{Q_1})$ defined in . . . . . ,

$\bar{\mathbb{P}}, \bar{\tau}, \bar{\lambda}_i, \sigma_1(\bar{\mathbb{P}}), \sigma_1(\bar{\tau}), \sigma_1(\bar{\lambda}_i)$ $(i = 1, \ldots, m)$

as $\mathcal{L}_0$ was in $Q_0, \bar{Q}, \ldots, \sigma_0(\bar{\tau}), \ldots, \sigma_0(\bar{\lambda}_i)$.

Then $\mathcal{L}_1$ is consistent by the transfer lemma. Generically collapse $\delta_1$ to $\omega$. This gives a solid model $\mathfrak{A}$ of $\mathcal{L}_1$. Let $\sigma^* = \dot{\sigma}^{\mathfrak{A}}$, Then $\langle Q_1, \sigma^* \rangle$ is the liftup of $\langle \bar{Q}, \sigma^* \restriction H_{\omega_2}^{\bar{Q}} \rangle$.

If $C^* = \dot{C}^{\mathfrak{A}}$ is defined from $\sigma^*$ as above, then $\mathrm{otp}(C^*) = \omega_1$, since $\omega_1$ is absolute in $\mathfrak{A}$.

Now let $k' : Q_1 \prec Q'$ be the unique map s.t. $k'\sigma_1 = \sigma$ and $k' \restriction H_{\tau_1}^{Q_1} = \mathrm{id}$, where $\tau_1 = \sigma_1(\bar{\tau})$. Then $k' : Q_1 \prec Q'$ cofinally and $k'$ takes $\sigma_1(\bar{\lambda}_i)$ cofinally to $\lambda'_i$ $(i = 1, \ldots, m)$, since $\sigma_1{}''\bar{\lambda}_i$ is cofinal in $\sigma_1(\bar{\lambda}_i)$. Set $\tilde{\sigma} = k'\sigma^*$. If $\tilde{C}$ is defined from $\tilde{\sigma}$ as $C^*$ was defined from $\sigma^*$, then $\tilde{C} = C^*$, since $k' \restriction H_{\omega_1} = \mathrm{id}$. Hence:

- $\tilde{\sigma} : \bar{Q} \prec Q'$ cofinally
- $\tilde{\sigma}(\bar{\mathbb{P}}, \bar{s}, \bar{\lambda}_i) = \mathbb{P}, s, \lambda_i$
- $\sup \tilde{\sigma}{}''\bar{\lambda}_i = \lambda'_i$ $(i = 0, \ldots, m)$
- $\mathrm{otp}\, \tilde{C} = \omega_1$.

The verifications are straightforward.

But then $\mathcal{L}'$ is consistent since it is modeled by $\langle H_\mu^{V[g]}, \tilde{\sigma} \rangle$, where $g$ generically collapses $\delta_1$ to $\omega$ and $\mu > \lambda$ is regular in $V[g]$.

QED (Claim 1)

We are now ready to prove the main claim that $\sigma$ witnesses the subcompleteness of $\mathbb{P}$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{Q}$. Let $p \in \mathbb{P}$ s.t. $\bar{Q}, \bar{G} \in H^{M^p}_{\omega_1}$ and $p$ conforms to:

$$N^* = \langle H_\mu, M, N, \mathbb{P}, \mathcal{L}', Q, \bar{Q}, \sigma, s, \lambda_i (i = \ldots m)\rangle$$

where $\mu > \bar{\bar{Q}}$ is regular. Set:

$$\tilde{N}^* = \tilde{N}^*(p, N^*) = \langle \tilde{H}, \tilde{M}, \tilde{N}, \tilde{\mathbb{P}}, \tilde{\mathcal{L}}, \tilde{Q}, \bar{Q}, \tilde{\sigma}, \tilde{s}, \tilde{\lambda}_i (i = \ldots m)\rangle$$

Then $|p| = \omega_1^{\tilde{N}^*}$ and:

(1) $\tilde{\mathcal{L}}$ is consistent, since $\mathcal{L}'$ is consistent.

Let $\mathfrak{A}$ be a solid model of $\tilde{\mathcal{L}}$ and let $\sigma^* = \dot{\sigma}^{\mathfrak{A}}$. Then

(2) $\sigma^* \in \mathrm{wfc}(\mathfrak{A})$ and

- $\sigma^* : \bar{Q} \prec \tilde{Q}'$ cofinally, where $\tilde{\sigma} : \bar{Q} \prec \tilde{Q}'$ cofinally. (Then $\tilde{Q}' \prec \tilde{Q}$)
- $\sigma^*(\bar{\mathbb{P}}, \bar{s}, \bar{\lambda}_i) = \tilde{\mathbb{P}}, \tilde{s}, \tilde{\lambda}_i$, where
- $\langle \tilde{M}, \sigma^* \upharpoonright \bar{M}\rangle =$ the liftup of $\langle \bar{M}, \sigma^* \upharpoonright H^{\bar{M}}_{\omega_2}\rangle$
- $\sup \sigma^{*}{}'' \bar{\lambda}_i = \tilde{\lambda}_i'$, where $\tilde{\lambda}_i' = \sup \tilde{\sigma}'' \bar{\lambda}_i$.

We now define a new condition $q \in \mathbb{P}$ by: $|q| = |p|$, $F^q = F^p$, $\pi^q \upharpoonright (\alpha+1)^2 = \pi^{\bar{G}}$, $M^q \upharpoonright (\alpha+1) = M^{\bar{G}}$, where $\alpha = \omega_1^{\bar{Q}}$ and $\bar{G}$ is the abovementioned $\bar{\mathbb{P}}$-generic set.

For $\alpha \le \nu \le |p|$ let $X_\nu$ = the smallest $X \prec M^p_{|p|}$ s.t. $\nu \cup \sigma^{*}{}''\bar{M} \subset X$. Let $\langle \nu_i \mid \alpha \le i \le |p| \rangle$ be the monotone enumeration of $C =$
$= \{\nu \mid \nu = |p| \cap X_\nu\}$. (Thus $\nu_\alpha = \alpha$, $\nu_{|p|} = |p|$.)

For $\alpha \le i \le |p|$ set: $\pi^q_{i,|p|} : M^q_i \overset{\sim}{\longleftrightarrow} X_{\nu_i}$.

(Thus $M^q_\alpha = \bar{M}$, $M^q_{|p|} = M^p_{|p|}$, $\pi^q_{\alpha,|p|} = \sigma^* \restriction \bar{M}$)

For $i \le j \le |p|$ we then set:

$$\pi^q_{ij} = (\pi^q_{j,|p|})^{-1} \circ \pi^q_{i,|p|}.$$

This defines $q$. Then

(3) $q \in \mathbb{P}$

prf.

We must show that $\mathcal{L}(q)$ is consistent. This is straightforward, since if $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$ we can turn it into a solid model of $\mathcal{L}(q)$ simply by replacing $\dot{M}^{\mathfrak{A}} \restriction (|p|+1)$ with $M^q$ and $\dot{\pi}^{\mathfrak{A}} \restriction (|p|+1)^2$ with $\pi^q$. QED(3)

$q$ is our "master condition". Let $G$ be $\mathbb{P}$-generic with $q \in G$. Then $\pi^G_{|q|\omega_1}$ extends uniquely to a $\pi^* : \bar{N}^* \prec N^*$ s.t. $F^q = F^p \subset \pi^*$. Then:

Set: $\sigma_0 = \pi^* \circ \sigma^*$.

(4) • $\sigma_0 : \bar{Q} \prec Q'$ cofinally

• $\sigma_0(\bar{\mathbb{P}}, \bar{s}, \bar{\lambda}_i) = \pi^*(\tilde{\mathbb{P}}, \tilde{s}, \tilde{\lambda}_i) = \mathbb{P}, s, \lambda_i$
$(i = 1, \dots, m)$

• $\langle M, \sigma_0 \upharpoonright \bar{M} \rangle =$ the liftup of $\langle \bar{M}, \sigma_0 \upharpoonright H^{\bar{M}}_{\omega_2} \rangle$.

Since $\tilde{\lambda}_i' = \sup \sigma'' \bar{\lambda}_i$ is $\omega$-cofinal in $\bar{N}^*$, it is taken cofinally to $\lambda_i'$ by $\pi^*$. Hence:

(5) $\sup \sigma_0'' \bar{\lambda}_i = \lambda_i' \quad (i = 0, \dots, m)$

It remains only to show:

<u>Claim</u> $\bar{G} = \sigma_0^{-1}{}'' G$.

prf.
Let $s \in \bar{G}$. <u>Claim</u> $\sigma_0(s) \in G$.
We first note:
(6) $M^s = M^{\sigma_0(s)}$, $\pi^s = \pi^{\sigma_0(s)}$, since $\sigma_0 \upharpoonright H^{\bar{M}}_{\omega_1} = \mathrm{id}$

(7) $M^s = M^{\bar{G}} \upharpoonright (|s|+1) = M^{\mathscr{G}} \upharpoonright (|s|+1) = M^G \upharpoonright (|s|+1)$.
Similarly: $\pi^s = \pi^G \upharpoonright (|s|+1)^2$
Now let $\langle a, \bar{a} \rangle \in F^s$. Then $\pi^G_{|s|\alpha} = \pi^{\mathscr{G}}_{|s|\alpha} = \pi^{\bar{G}}_{|s|,\alpha}$,
hence:
• $\pi^G_{|s|,\alpha} : \langle M^s_{|s|}, \bar{a} \rangle \prec \langle \bar{M}, a \rangle$
But if $\tilde{a} = \sigma^*(a)$, then
• $\pi^G_{\alpha,|p|} : \langle \bar{M}, a \rangle \prec \langle M^p_{|p|}, \tilde{a} \rangle$, since $\pi^G_{\alpha,|p|} = \pi^{\mathscr{G}}_{\alpha,|p|} = \sigma^* \upharpoonright \bar{M}$
If $a' = \pi^*(\tilde{a})$, then
• $\pi^G_{|p|,\omega_1} : \langle M^p_{|p|}, \tilde{a} \rangle \prec \langle M, a' \rangle$, since
$\pi^G_{|p|,\omega_1} = \pi^* \upharpoonright M^p_{|p|}$.

Putting this together:

(8.) $\pi^{G}_{|\bar{\nu}|,\omega_1} : \langle M^{\bar{\nu}}_{|\bar{\nu}|}, \bar{a} \rangle \prec \langle M, a' \rangle$, where

$\langle a', \bar{a} \rangle = \sigma_0(\langle a, \bar{a} \rangle) \in \sigma_0{}''F^{\nu} = \sigma_0(F^{\nu}) = F^{\sigma_0(\nu)}$.

By (6), (7), (8) and [J] §4 Cor 2.8 we conclude: $\sigma_0(\nu) \in G$.

QED

## § 3.7 Adding a mouse which iterates to a measure

*Lemma 4* The forcing of [J] § 5 Example 2 is subcomplete.

We shall call this forcing $\mathbb{P}^*$ to distinguish it from the forcing $\mathbb{P}$ of Example 1.

Let $\kappa$ be a measurable cardinal and $U$ a normal ultrafilter on $\kappa$. Let $\beta > \kappa$ be regular s.t. $2^\beta = \beta$. Set $M = \langle L_\beta[A], A, U \rangle$ where $L_\beta[A] = H_\beta$, $N = \langle H_{\beta^+}, M, <, \ldots \rangle$. In [J] § 5 we showed that if $G$ is $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ - generic (where $\mathcal{L}$ is as before with this $M, N$), then for every $\alpha < \omega_1$ there is $\bar{M} = \langle L_{\bar{\beta}}[\bar{A}], \bar{A}, \bar{U} \rangle$ which iterates to $M$ in exactly $\alpha$ many steps. Our intention now is to devise $\mathbb{P}^*$ s.t. if $G$ is $\mathbb{P}^*$-generic, then there is $\bar{M} \in V[G]$ which iterates to $M$ in $\omega_1$ many steps. We define a language $\mathcal{L}^*$ on $N$ and set $\mathbb{P}^* = \mathbb{P}_{\mathcal{L}^*}$. $\mathcal{L}^*$ is the language which, in addition to the basic axioms of [J] § 3, contains the axioms:

- $\dot{B} = \emptyset$
- $\underline{\omega_2} = \sup \dot{\pi}_{i\underline{\omega_1}}{}'' \omega_2^{\dot{M}_i}$ for $i \leq \underline{\omega_1}$
- Let $\langle M', \sigma \rangle$ be the liftup of $\langle \dot{M}_i, \dot{\pi}_{ij} \upharpoonright H_{\omega_2}^{\dot{M}_i} \rangle$. Then $M'$ iterates to $\dot{M}_j$ in $j-i$ steps. Moreover, if $k: M' \to \dot{M}_j$ is the iteration map, then $\dot{\pi}_{ij} = k\sigma$.

In this context we can make good use of the following lemma, which is implicit in the proof of [J] §5 Lemma 3:

<u>Lemma 4.1</u> Let $\langle \bar{M}_i \mid i \leq \alpha \rangle$ be an iteration with iteration maps $\bar{k}_{ij}$, where $\bar{M}_i = \langle L_{\bar{\beta}_i}[\bar{A}_i], \bar{A}_i, \bar{U}_i \rangle$, $\bar{U}_i$ being a normal ultrafilter on $\bar{\kappa}_i$ in $\bar{M}_i$. Let $\bar{H} = H_{\omega_2}^{\bar{M}_i}$ $(i \leq \alpha)$ and let $\sigma: \bar{H} \prec H$ cofinally. Let $\langle M_i, \sigma_i \rangle$ be the (transitive) liftup of $\langle \bar{M}_i, \sigma \rangle$. Then $\langle M_i \mid i \leq \alpha \rangle$ is an iteration with iteration maps $k_{ij}$, where $\sigma_j \bar{k}_{ij} = k_{ij} \sigma_i$.

Now let $\mathfrak{A}$ be a solid model of $\mathcal{L}'$ and set: $M_i = \dot{M}_i^{\mathfrak{A}}$, $\pi_{ij} = \dot{\pi}_{ij}^{\mathfrak{A}}$. Set $\sigma_{ij} = \pi_{ij} \upharpoonright H_i$, where $H_i = H_{\omega_1}^{M_i}$. Let $\langle M_i^j, \sigma_{ij}^i \rangle$ = the liftup of $\langle M_i, \sigma_{ij} \rangle$. For $h \leq i$ let $\langle M', \sigma_{ij}^h \rangle$ be the liftup of $\langle M_i^h, \sigma_{ij} \rangle$. Then $\langle M', \sigma_{ij}^h \cdot \sigma_{hi}^h \rangle$ is the liftup of $\langle M_h, \sigma_{hj} \rangle$. Hence $M' = M_h^j$. But then $\langle M_i^j \mid i \leq j \rangle$

is an iteration with maps $k_{hi}^{j}$ and we have:

(1) $k_{hi}^{l}\, \sigma_{jl}^{h} = \sigma_{jl}^{i}\, k_{hi}^{j}$ for $h \le i \le j \le l$.

$$\begin{tikzcd}
M_h^j \arrow[r, "k_{hi}^j"'] & M_i^j \arrow[r, "k_{ij}^j"'] & M_j = M_j^j \\
M_h^i \arrow[u, "\sigma_{ij}^h"] \arrow[r, "k_{hi}^i"'] & M_i \arrow[u, "\sigma_{ij}^i"] \arrow[ur, "\pi_{ij}"'] & \\
M_h \arrow[u, "\sigma_{hi}^h"] \arrow[ur, "\pi_{hi}"'] & &
\end{tikzcd}$$

This holds in particular if $\mathfrak{M}$ is defined by $\pi_{ij}^{\mathfrak{M}} = \pi_{ij}^{G}$, $\dot{M}_i^{\mathfrak{M}} = M_i^{G}$, where $G$ is $\mathbb{P}^*$-generic. $\langle M_i^{\omega_1} \mid i \le \omega_1 \rangle$ is then an iterat of $M_0^{\omega}$ to $M = M_{\omega_1}$ in $\omega_1$ many steps.

Now let $\theta$ be a cardinal s.t. $2^{2^{\beta}} < \theta$. Let $\lambda > \theta$ be regular and define $Q = \langle L_\lambda, [A], A, m \rangle$. Let $\sigma : \bar{Q} \prec Q$ as before, where $H_\theta \subset Q$. Let s.t. $\bar{Q}$ is countable and full. Let $\sigma(\bar{\mathcal{L}}^*, \bar{\mathbb{P}}^*, \bar{\theta}) = \mathcal{L}^*, \mathbb{P}^*, \theta$.

<u>Claim</u> $\sigma$ witnesses the subcompleteness of $\mathbb{P}^*$.

Set: $\bar{\lambda}_0 = On \cap \bar{Q}$, $\lambda_0 = On \cap Q = \lambda$. Set $\lambda'_i = \sup \sigma'' \bar{\lambda}_i$ $(i = 0, \ldots, m)$.

Claim 1 Let $\sigma : \bar{Q} \prec Q'$ cofinally. (Hence $Q' = Q | \lambda'_0$.) Let $\delta' = \delta_{Q'}$. Let $\mathcal{L}''$ be the language on $L_{\delta'}(Q')$ with the basic theory (cf §3.2), a constant $\dot{\sigma}$, and the further axioms:

- $\dot{\sigma} : \underline{\bar{Q}} \prec \underline{Q'}$
- $\dot{\sigma}(\underline{\bar{P}}, \underline{\bar{z}}, \underline{\bar{\lambda}_i}) = \underline{P}, \underline{z}, \underline{\lambda_i}$ $(i = 1, \ldots, m)$

  (hence $\dot{\sigma}(\underline{\bar{M}}, \underline{\bar{N}}) = \underline{M}, \underline{N}$)
- Let $\langle M', \sigma' \rangle$ be the liftup of $\langle \bar{M}, \dot{\sigma} \upharpoonright \underline{H^{\bar{M}}_{\omega_2}} \rangle$

  Then $M'$ iterates to $\underline{M}$ in exactly $\underline{\omega_1}$ many steps. Moreover, if $k$ is the iteration map, then $\dot{\sigma} \upharpoonright \underline{\bar{M}} = k\sigma'$.
- $\sup \dot{\sigma}'' \bar{\lambda}_i = \lambda'_i$ $(i = 0, \ldots, m)$
- For $\nu < \underline{\omega_1}$ set $X_\nu =$ the smallest $X \prec M'$ s.t. $\nu \cup \sigma'' \underline{\bar{M}} \subset X$. Set $\dot{C} = \{\nu < \underline{\omega_1} \mid \nu = \underline{\omega_1} \cap X_\nu\}$

  Then $\mathrm{otp}(\dot{C}) = \underline{\omega_1}$

Then $\mathcal{L}''$ is consistent.

proof.

We first note that Claim 1 of §3.6 holds. Let $\mathcal{L}'$ be as in that claim and let $\mathfrak{A}$ be a solid model of $\mathcal{L}'$. Set $\sigma' = \dot{\sigma}^{\mathfrak{A}}$.

Now let $k^* : V \prec W$ be the result of iterating $V$ by $U$ $\omega_1$ many times. Let $\mathcal{L}^* = k^*(\mathcal{L}'')$. It suffices to show that $\mathcal{L}^*$ is consistent. Set $\sigma^* = k^* \circ \sigma'$. Then $M^* = k^*(M^*)$ is the result of iterating $M$ $\omega_1$ many times with iteration map $k = k^* \upharpoonright M$. Moreover $\langle M, \sigma' \rangle$ is the liftup of $\langle \bar{M}, \sigma^* \upharpoonright H_{\omega_2}^{\bar{M}} \rangle$ and $\sigma^* \upharpoonright \bar{M} = k\sigma'$.

~~Last~~ $Q^* = k^*(Q')$, $\delta^* = \delta_{Q^*}$. Then $\mathcal{L}^*$ is a language on $L_{\delta^*}(Q^*)$ and $\sigma^*: \bar{Q} \prec Q^* \prec k^*(Q)$. It follows that

$\langle H_\mu^{V[g]}, \sigma^* \rangle$ models $\mathcal{L}^*$

where $\mu > \lambda$ is regular in $V[g]$ and $g$ generically collapses $\delta'$ to $\omega$. The verifications are left to the reader.

QED (Claim 1)

Now let $\bar{G}$ be $\mathbb{P}^*$-generic over $\bar{Q}$. Let $p \in \mathbb{P}^*$ s.t. $\bar{G}, \bar{Q} \in H_{\omega_1}^p$ and let $p$ conform to

$N^* = \langle H_\mu, M, N, Q, \bar{Q}, \sigma, \bar{s}, \bar{\lambda}_1, \ldots, \bar{\lambda}_n, \mathcal{L}'_m \rangle$.

Then $|p| = \omega_1^{\bar{Q}}$

Let $\bar{N}^* = \bar{N}^*(p, N^*) =$

$= \langle \tilde{H}, \tilde{M}, \tilde{N}, \tilde{Q}, \bar{Q}, \tilde{\sigma}, \bar{s}, \bar{\lambda}_1, \ldots, \bar{\lambda}_n, \tilde{\mathcal{L}}'_m \rangle$

Let $\tilde{Q}', \tilde{s}, \tilde{\lambda}_i, \tilde{\lambda}'_i$ be defined in $\bar{N}^*$ as $Q', s, \lambda_i, \lambda'_i$ are defined in $N^*$ $(i = 0, \ldots, m)$. Then $\tilde{\mathcal{L}}'$ is consistent, since $\bar{N}^*$, $N^*$ are elementarily equivalent.

Let $\mathfrak{A}$ be a solid model of $\tilde{\mathcal{L}}'$. Set $\sigma^* = \mathring{\sigma}\mathfrak{A}$. We define a new condition $q \in \mathbb{P}^*$ as follows: $|q| = |p|$, $M^q_{|q|} = M^p_{|p|}$, $F^q = F^p$. We set $M^q \upharpoonright (\alpha+1) = M^{\bar{G}}$, $\pi^q \upharpoonright (\alpha+1)^2 = \pi^{\bar{G}}$ where $\alpha = \omega_1^{\bar{Q}}$.

Let $\langle M', \sigma' \rangle$ be the liftup of $\langle \bar{M}, \sigma^* \upharpoonright H^{\bar{M}}_{\omega_2} \rangle$.

Let $X_\nu$ = the smallest $X \prec M'$ s.t. $\nu \cup \sigma'$ "$M \subset X$. Set $C = \{\nu < |p| \mid \nu = |p| \cap X_\nu\}$. Then $otp(C) = |p|$. Let $\langle \nu_i \mid \alpha \leq i \leq |p| \rangle$ enumerate $C \cup \{|p|\}$ monotonically. For $\alpha \leq i \leq |p|$ set $\sigma^\alpha_{i,|p|} : M^i_\alpha \overset{\sim}{\longleftrightarrow} X_{\nu_i}$. Then $M^{|p|}_\alpha = M'$ and $M^\alpha_\alpha = \bar{M}$. Moreover $\sigma^\alpha_{\alpha,|p|} = \sigma'$.

Set $\sigma^\alpha_{ij} = (\sigma^\alpha_{j,|p|})^{-1} \circ \sigma^\alpha_{i,|p|}$ for $\alpha \leq i \leq j \leq |p|$.

It follows easily that $\langle M^j_\alpha, \sigma^\alpha_{ij} \rangle$ is the liftup of $\langle M^i_\alpha, \sigma_{ij} \rangle$, where $\sigma_{ij} = \sigma^\alpha_{ij} \upharpoonright H^{M^i_\alpha}_{\omega_2}$. For $i \leq \alpha \leq j \leq |p|$ let $\langle M^j_i, \sigma^i_{\alpha j} \rangle$ be the liftup of $\langle M^\alpha_i, \sigma_{\alpha j} \rangle$. Then $\langle M^j_i \mid i \leq \alpha \rangle$ is an iteration with maps $k^j_{hi}$ $(h \leq i \leq \alpha)$. As usual, there are maps $\sigma^i_{jl} : M^j_i \prec M^l_i$ defined by: $\langle M^l_i, \sigma^i_{jl} \rangle$ is the liftup of $\langle M_i^j, \sigma_{jl}^i \rangle$. By Lemma 4.1 we get for $h \leq i \leq \alpha \leq j \leq l$:

(1) $k_{hi}^l \sigma_{jl}^h = \sigma_{jl}^i k_{hi}^j$

If we set: $M_i^j = (M_i^j)^{\bar{G}}$ $(i \leq j \leq \alpha)$

and $(\sigma_{jl}^i) = (\sigma_{jl}^i)^{\bar{G}}$ $(i \leq j \leq l \leq \alpha)$,

and $k_{hi}^j = (k_{hi}^j)^{\bar{G}}$ $(h \leq i \leq j \leq \alpha)$

and $\sigma_{jl}^i = \sigma_{\alpha l}^i \sigma_{j\alpha}^i$ $(j \leq \alpha \leq j \leq |p|)$,

then (1) continues to hold for $h \leq i \leq \alpha$, $h \leq i \leq j \leq l \leq |p|$. The characterization of $\sigma_{jl}^i$ as a liftup continues to hold. We note that $M_\alpha^{|p|} = M'$ is iterable and iterates to $M$ in $|p|$ many steps. Let $M_i^{|p|}$ be the $i$-th iterate of $M$ + let $k_{ij}^{|p|}$ $(\alpha \leq i \leq j \leq |p|)$ be the iteration maps. Set $k_{ij}^{|p|} = k_{\alpha j}^{|p|} \circ k_{i\alpha}^{|p|}$ for $i \leq \alpha \leq j \leq |p|$. Then $\langle M_i^{|p|} \mid i \leq |p| \rangle$ is an iteration of $M_0^{|p|}$ to $M'$ with maps $k_{ij}^{|p|}$ $(i \leq j \leq |p|)$. But $M_\alpha^j$ is iterable for $\alpha \leq j \leq |p|$, since $\sigma_{j,|p|}^\alpha : M_\alpha^j \prec M'$. Thus we can extend $\langle M_i^j \mid i \leq \alpha \rangle$ to $\langle M_i^j \mid i \leq j \rangle$ in exactly the same way for $\alpha \leq j \leq |p|$, $k_{hi}^j$ $(h \leq i \leq j)$ being the iteration maps.

If we set: $\langle M_i'', \sigma'' \rangle =$ the liftup of $\langle M_i^j, \sigma_{jl} \rangle$ for $\alpha \leq i \leq j \leq l$, then $\langle M_i'' \mid \alpha \leq i \leq j \rangle$ is an iteration of $M_\alpha'' = M_\alpha^l$. Thus $M_i'' = M_i^l$ for $\alpha \leq i \leq j$ & we have defined unique $\sigma_{jl}^i$ s.t. $\langle M_i^l, \sigma_{jl}^i \rangle$ is the liftup of $\langle M_i^j, \sigma_{jl} \rangle$ for $\alpha \leq j \leq l$. It is easily seen that (1) now holds for all $\alpha \leq i \leq j \leq l \leq |p|$. We set:

$M_i = M_i^i$, $\pi_{ij} = k_{ij}^j \circ \sigma_{ij}^i$ $(i \leq j \leq |p|)$.

(Hence $\pi_{\alpha,|p|} = \sigma^* \upharpoonright \overline{M}$, since $\mathfrak{A}$ models $\mathcal{L}$!)

We set: $M^q = \langle M_i \mid i \leq |p| \rangle$, $\pi^q = \langle \pi_{ij} \mid i \leq j \leq |p| \rangle$ and $F^q = F^p$. Then

(2) $q \in \mathbb{P}^*$

proof (sketch)

Let $\mathfrak{A}$ be a solid model of $\mathcal{L}^*(p)$. We can turn $\mathfrak{A}$ into a model $\mathfrak{A}'$ of $\mathcal{L}^*(q)$ by replacing $M^{\mathfrak{A}} \upharpoonright (|p|+1)$ with $M^q$ and $\pi^{\mathfrak{A}} \upharpoonright (|p|+1)^2$ with $\pi^q$. The verifications are left to the reader.

QED(2)

$q$ is our master condition. Let $G \ni q$ be $\mathbb{P}^*$-generic. $\pi^{G}_{|q|,\omega_1}$ extends uniquely to a $\pi^*: \bar{N}^* \prec N^*$ s.t. $F \psi \subset \pi^*$. Set $\sigma_0 = \pi^* \circ \sigma^*$.

Exactly as in §3.6 we get:

- $\sigma_0 : \bar{Q} \prec Q'$
- $\sigma_0(\bar{\mathbb{P}}, \bar{s}, \bar{\lambda}_i) = \mathbb{P}, s, \lambda_i \quad (i = 1, \ldots, m)$
- $\sup \sigma_0 {}'' \bar{\lambda}_i = \lambda'_i \quad (i = 0, \ldots, m)$
- $\bar{G} = \sigma_0^{-1} {}'' G$.

QED

## 3.8 Namba-prime Forcing

In [S] Shelah introduces a variant of Namba forcing which he calls Nm' and we labeled $\mathbb{N}'$ in [J] §6. $\mathbb{N}'$ is the set of subtrees T of $\omega_2^{<\omega}$ s.t. T has a finite stem s and above s every point has exactly $\omega_2$ many immediate successors. Magidor proved:

Lemma A Let b be an $\mathbb{N}'$-generic sequence in $\omega_2$ (i.e. $b = \bigcup\bigcap G$ where G is $\mathbb{N}'$-generic). Let $F \in V$ s.t. $F: \omega_2 \to \omega_2$. Then $\bigvee n \bigwedge i \geq n \ \delta_i > \sup_{h<i} F(\delta_h)$ where $b = \langle \delta_i \mid i < \omega \rangle$.

Lemma B Let $b = \langle \delta_i \mid i < \omega \rangle$ be Namba-generic. Then there is $F \in V$ s.t. $F: \omega_2 \to \omega_2$ and there are arbitrarily large $i < \omega$ s.t. $\delta_i \leq \sup_{h<i} F(\delta_h)$.

It follows easily that no $\mathbb{N}'$-generic sequence is $\mathbb{N}$ generic and conversely. It is known that if $2^\omega = \omega_1$, then $\mathbb{N}'$ adds no new reals. (These facts are reproven in [J] §6.)

We shall prove:

<u>Lemma 5</u> Let $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. Then $\mathbb{N}'$ is subproper.

Unfortunately, we do not know if $\mathbb{N}'$ is subcomplete. Our proof makes heavy use of [J] §6, which in turn draws on [J] §4. We first note that $\mathbb{N}'$ was there proven to be equivalent to the $\mathcal{L}$-forcing given in Example 6:

<u>Def</u> Let $\beta = \omega_2$ (hence $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$), $\mathcal{L}$ consists of the basic axioms of [J] §3 together with:

(a) $\dot{B} = \langle \dot{\gamma}_i \mid i < \underline{\omega} \rangle$ is cofinal in $\underline{\omega_2}$

(b) $\mathrm{rng}(\dot{\pi}_{\underline{i}, \omega_1}) =$ the smallest $X \prec \underline{M}$ s.t. $\dot{B} \cup \{\dot{\alpha}_h \mid h < \underline{i}\} \subset X$ for each $i < \omega_1$ (where $\dot{\alpha}_h = \omega_1^{\dot{M}_h}$)

(c) $\bigvee n < \underline{\omega}\ \bigwedge i \geq n\ (i < \underline{\omega} \rightarrow \dot{\gamma}_i > \sup_{h < i} \underline{F}(h))$ for each $F : \omega_2 \rightarrow \omega_2$.

(Note that by (b), $\dot{M}, \dot{\pi}$ are definable from $\dot{B}$.)

As in [J] §6 we prove:

<u>Lemma C</u> $BA(\mathbb{N}') = BA(\mathbb{P})$, where $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$.

More importantly:

<u>Def</u> Let $b = \langle \gamma_n \mid n < \omega \rangle$ be a sequence in $\omega_2^V$.
$b$ is $\mathbb{N}'$-<u>generic</u> iff $b = \bigcup \bigcap G$ for an $\mathbb{N}'$-generic $G$. (But in this case $G = G_b$ is definable from $b$ by:
$G = \{T \mid \bigwedge n \; b \restriction n \in T\}$.)
$b$ is $\mathbb{P}$-<u>generic</u> iff $b = B^G$ for a $\mathbb{P}$-generic $G$. (Hence $M^G, \pi^G$ are canonically definable from $b$ and $G$ is canonically definable from $B^G, M^G, \pi^G$ by [J] §4 Lemma 2.8.)

<u>Lemma D</u> $b$ is $\mathbb{N}'$-generic iff $b$ is $\mathbb{P}$-generic.
This follows from the proof of Lemma C in [J] §6.

Finally we need Lemma 4.6 of [J] §6 which reads:
<u>Lemma E</u> Let $W$ be a transitive ZFC$^-$ model s.t. $2^\omega = \omega_1 \wedge 2^{\omega_1} = \omega_2$ in $W$. Suppose moreover that $\alpha = (2^{\omega_2})^W$ exists and is countable in $V$. Let $\bar{\mathbb{N}} = \mathbb{N}'^W$. Let $F : \omega_2^W \to \omega_2^W$. For each $T \in \bar{\mathbb{N}}$ there is an $\bar{\mathbb{N}}$-generic $b = \langle \gamma_n \mid n < \omega \rangle$ s.t. $\bigvee n \bigwedge i \geq n \;\; \gamma_i > \sup\limits_{h < i} F(\gamma_h)$.

Using Lemma C we easily conclude:
Let $W$ be as above. Let $\mathbb{P} = \mathbb{P}^W$.
<u>Lemma F.</u> Let $W$ be as above. For each $p \in \mathbb{P}$ there
Let $F : \omega_2^W \to \omega_2^W$. For each $p \in \mathbb{P}$ there is a $\mathbb{P}$-generic $b = \langle \gamma_n \mid n < \omega \rangle$ s.t.
$\bigvee n \bigwedge i \geq n \;\; \gamma_i > \sup\limits_{h < i} F(\gamma_h)$.

Note We do not assume that $F \in W$.

Using these facts we shall prove the subproperness of $\mathbb{P}$ by a proof that is quite similar to the proof of subcompleteness in §3.6.

Let $\theta > 2^{2^{\beta}}$ be a cardinal and let $\lambda > \theta$ be regular. Let $Q = \langle L_\lambda[A], A, \ldots \rangle$. Let $\sigma : \bar{Q} \prec Q$ s.t. $\bar{Q}$ is countable and full. Let $\sigma(\bar{\theta}, \bar{\mathbb{P}}, \bar{\mathcal{L}}) = \theta, \mathbb{P}, \mathcal{L}$. We claim that $\sigma$ witnesses the subproperness of $\mathbb{P}$. Let $r \in \mathbb{P}$, $\sigma(\bar{r}) = r$. Let $\sigma(\bar{s}, \bar{\lambda}_i) = \lambda_i$ $(i = 1, \ldots, n)$ s.t. $\lambda_i < \theta$ is regular and $\mathbb{P} \in H_{\lambda_i}$. Set $\lambda_i' = \sup \sigma'' \bar{\lambda}_i$ for $i = 0, \ldots, n$, where $\bar{\lambda}_0 = On \cap \bar{Q}$; $\lambda_0 = \lambda = On \cap Q$.

Claim 1 of §3.6 holds just as before (now for our present $\mathbb{P}$). Let $p \in \mathbb{P}$ s.t. $\bar{Q} \in H_{\omega_1}^{M_{|p|}^p}$ and $p$ conforms to $N^* = \langle H_\mu, M, N, Q, \mathbb{P}; \mathcal{L}', \sigma, \lambda_0, \ldots, \lambda_n, \ldots \rangle$.
Let $\bar{N}^* = \bar{N}^*(p, N^*) =$
$= \langle \tilde{H}, \tilde{M}, \tilde{N}, \tilde{Q}, \tilde{\mathbb{P}}, \tilde{\mathcal{L}}, \tilde{\sigma}, \tilde{\lambda}_0, \ldots, \tilde{\lambda}_n, \ldots \rangle$

where $\mathcal{L}'$ is as in Claim 1.

Then $\tilde{\mathcal{L}}$ is consistent and countable. Let $\mathfrak{A}$ be a solid model of $\tilde{\mathcal{L}}$. Set $\sigma^* = \mathring{\sigma}^{\mathfrak{A}}$. Let $B^P = \langle \gamma_i^P \mid i < \omega \rangle$. Define $\bar{F} : \omega_2^{\bar{Q}} \to \omega_2^{\bar{Q}}$ by:

$\bar{F}(\xi) =$ the least $\mu > \xi$ s.t.
$\sigma^*(\xi) < \gamma_i^P < \gamma_{i+1}^P \leq \sigma^*(\mu)$ for some $i < \omega$.

Assume w.l.o.g. that $s$ was so chosen that it codes $r$. Then $\bar{r}$ codes $\bar{r}$, $\tilde{r}$ codes $\tilde{r} = \sigma^*(\bar{r})$ and $s$ codes $r$ by the same def. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{Q}$ s.t. $\bar{r} \in \bar{G}$ and there is $n < \omega$ s.t.

$$\bigwedge i \geq n \ \sup_{h<i} \bar{F}(\bar{\gamma}_h) < \bar{\gamma}_i ,$$

where $B^{\bar{G}} = \langle \bar{\gamma}_i \mid i < \omega \rangle$. We define a new condition $q \in \mathbb{P}$ as follows:

We first note:

(1) Let $\sigma^*(\bar{\gamma}_i) = \gamma_i^*$ $(i < \omega)$. Then for each $F \in M^P_{|P|}$ s.t. $F : \omega_2 \to \omega_2$ in $M^P_{|P|}$ we have:

$$\bigvee n \bigwedge i \geq n \ \sup_{h<i} F(\gamma_h^*) < \gamma_i^*$$

proof.
Assume w.l.o.g. that $F$ is monotone. Let $n < \omega$ s.t. $\bigwedge i \geq n \ \sup_{h<i} F(\gamma_h^P) < \gamma_i^P$

Then $\sup_{h<i} \gamma_h^P < \gamma_i^P$ for $i \geq n$.

Hence $\langle \gamma_i^P \mid i \geq n \rangle$ is monotone.

W.l.o.g. we also assume $n$ large enough that $\sup_{h<i} \bar{F}(\bar{\gamma}_h) < \bar{\gamma}_i$ for $i \geq n$. Since $\bar{F}(\xi) > \xi$ for all $\xi < \omega_2^{\bar{Q}}$, we conclude that $\sup_{h<n} \bar{\gamma}_h \leq \bar{\gamma}_n$, and $\langle \bar{\gamma}_i \mid i \geq n \rangle$, is monotone. The same holds for $\langle \gamma_i^* \rangle$, so it suffices to show that $F(\gamma_i^*) < F(\gamma_{i+1}^*)$ for sufficiently large $i \geq n$. Pick $i$ big enough that $\gamma_i^* > \gamma_n^p$. Then $\gamma_i^* < \gamma_j^p < \gamma_{j+1}^p \leq \sigma^*(\bar{F}(\bar{\gamma}_i)) < \gamma_{i+1}^*$, where $j \geq n$. Hence

$$F(\gamma_i^*) < F(\gamma_j^p) < \gamma_{j+1}^p < \gamma_{i+1}^* \ldots$$

QED(1)

We now define $q$. We set:

$|q| = |p|$, $M_{|q|}^q = M_{|p|}^p$, $F^q = F^p$

$M^q \restriction (\alpha+1) = M^{\bar{G}}$, $\pi^q \restriction (\alpha+1)^2 = \pi^{\bar{G}}$

Set: $X_\nu =$ the smallest $X \prec M_{|p|}^p$ s.t. $\nu \cup \sigma^{*}{}``\bar{M} \subset X$ for $\alpha \leq \nu \leq |p|$

Let $\langle \nu_i \mid \alpha \leq i \leq |p| \rangle$ enumerate the $\nu$ s.t. $\nu = |p| \cap X_\nu$ monotonically.

Set $\pi_{i,|p|}^q : M_i^q \leftrightarrow X_{\nu_i}$ $(\alpha \leq i \leq |p|)$

(Hence $M_\alpha^q = \bar{M}$, $\pi_{\alpha,|p|}^q = \sigma^*$, $M_{|p|}^q = M_{|p|}^p$.)

Set $\pi_{ij}^q = (\pi_{j,|p|}^q)^{-1} \circ (\pi_{i,|p|}^q)$ $(\alpha \leq i \leq j \leq |p|)$

Set $\pi^q_{ii'} = \pi^q_{\alpha i'} \circ \pi^q_{i\alpha}$ for $i \leq \alpha \leq i' \leq |p|$.

Finally set: $B^q = \sigma^{*}{}'' B^{\bar{G}} = \langle \sigma^*(\bar{\gamma}_i) \mid i < \omega \rangle$.

(2) $q \in \mathbb{P}$

Proof (sketch).

Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$.
We transform it into a solid model $\mathfrak{A}'$ of $\mathcal{L}(q)$ by replacing $M^{\mathfrak{A}} \restriction (|p|+1)$ with $M^q$ and $\mathfrak{A} \restriction (|p|+1)^2$ with $\pi^q_i$, we also set $B^{\mathfrak{A}'} = \pi^{\mathfrak{A}}_{|p|,\omega_1} \sigma^{*}{}'' B^{\bar{G}} = \langle \pi^{\mathfrak{A}'}_{\alpha\omega_1}(\bar{\gamma}_i) \mid i < \omega \rangle$. (It then follows by the argument of (1) that for every $F: \omega_1 \to \omega_2$ in $V$ we have:
$\bigvee n \bigwedge i \geq n \sup_{h<i} F(\gamma_h^{\mathfrak{A}'}) < \gamma_i^{\mathfrak{A}'}$.)

QED (2)

Now let $q \in G$ where $G$ is $\mathbb{P}$-generic.
Then $\pi^G_{|p|,\omega_1}$ extends uniquely to a $\pi^*: \bar{N}^* \prec N^*$ s.t. $F^p \subset \pi^*$. Set $\sigma_0 = \pi^* \circ \sigma^*$.
We can now literally repeat the argument in §3.6 to get:

- $\sigma_0 : \bar{Q} \prec Q' \prec Q$
- $\sigma_0(\bar{\mathbb{P}}, \bar{s}, \bar{\lambda}_i) = \mathbb{P}, s, \lambda_i \quad (i = 1, \ldots, m)$
- $\sup \sigma_0{}'' \bar{\lambda}_i = \lambda'_i \quad (i = $
- $\bar{G} = \sigma_0^{-1}{}'' G$. (where $\sigma_0(\tilde{s}) = \pi^*(\tilde{s}) = s$, since $\pi^*(\tilde{s}) = s$).

QED