§ 2 Liftups

Our intention is to prove that δ is Woodin in $J_\rho(N) =_{\text{Df}} J^{EN}_{\theta+\rho}$, where ρ is the least $\rho > 0$ s.t. $J_\rho(N)$ is admissible. By § 1 Lemma 1 it suffices to show that each $B \in \mathbb{P}(N) \cap J_\rho(N)$ is captured at some n. This is trivial for $B \in M_{b_0} \cap M_{b_1}$, but if one of the b_h is of type B, $M_{b_0} \cap M_{b_1}$ may be too small. A possible strategy for handling this is to pick a $\bar{\gamma} > 0$ s.t. $\bar{\pi}_{\xi_h b_h}$ "lifts" to

$$\pi^* : J_{\bar{\gamma}}(N_{\xi_h}) \longrightarrow J_\gamma(N)$$

and show that the $B \in \mathbb{P}(N) \cap J_\gamma(N)$ are captured at some n. This line of attack does, in fact, lead to a proof, (though we must deal with the fact that $J_{\bar{\gamma}}(N_{\xi_h})$ might be lifted to an ill founded structure). In this section we develop the techniques for handling such liftups.

Def Let $N = J^E_\theta =_{df} \langle |J^E_\theta|, \in, E \rangle$ be a ZFC^- model

Set: $J_\alpha(N) =_{df} J^{E^N}_{\theta+\alpha}$.

N is regular in $J_\alpha(N)$ iff $\langle N, A \rangle$ is a ZFC^- model for all $A \in J_\alpha(N)$ s.t. $A \subset N$

N is definably regular in $J_\alpha(N)$ iff iff $\langle N, A \rangle$ is a ZFC^- model for all $A \in \underline{\Sigma}_\omega(J_\alpha(N))$ s.t. $A \subset N$,

It follows easily that N is definably regular in $J_\alpha(N)$ iff it is regular in $J_{\alpha+1}(N)$,

Def Let $M = J_\alpha(N)$. M is grounded wrt. N iff there is $p \in M$ s.t. $h_M(\theta \cup \{p\}) = M$.

(Here $h_M(\theta \cup \{p\})$ = the closure of $\theta \cup \{p\}$ under Σ_1-functions in M, We have: $h_M(\theta \cup \{p\}) = h_M''(\omega \times (\theta \times \{p\}))$, where h_M is the Σ_1 Skolem function for M.)

In this case we also call p a grounding parameter for M,

Fact 1 If there is no ν s.t. $0 < \nu < \alpha$ and $J_\nu(N)$ is admissible, then $J_\alpha(N)$ is grounded.

<u>Fact 2</u> Let M be grounded. Then there is a $\underline{\Sigma}_1(M)$ set $B \subset N$ which codes the whole of M in such a way that every $A \in \underline{\Sigma}_\omega(M)$ with $A \subset N$ is $\underline{\Sigma}_\omega(\langle N, B\rangle)$. Moreover we can effectively assign to every first order formula φ a formula $\bar{\varphi}$ s.t.

$$M \models \varphi[x] \longleftrightarrow \langle N, B\rangle \models \bar{\varphi}[x]$$

for all $x \in N$.

Hence:

<u>Fact 3</u> Let N, M be as above. N is definably regular in M iff $\langle N, B\rangle$ is a ZFC$^-$ model for every $B \subset N$ which is $\underline{\Sigma}_1(M)$.

We also make use of the fact:

<u>Fact 4</u> Let $f : \bar{N} \longrightarrow_{\Sigma_0} N$ cofinally, where one of $\bar{N}, N$ is a ZFC$^-$ model. Then $f : \bar{N} \prec N$. (Hence both are ZFC$^-$ models.)

If $\mathfrak{A}$ is a (possibly ill founded) model of the extensionality axiom we define the <u>well founded core</u> of $\mathfrak{A}$ ($wfc(\mathfrak{A})$) to be the

set of x s.t. there is no infinite descending chain in $\in_{\mathfrak{A}}$ starting with x.
We call $\mathfrak{A}$ solid iff $wfc(\mathfrak{A})$ is transitive and : $x \in y \longleftrightarrow x \in_{\mathfrak{A}} y$ for $x, y \in wfc(\mathfrak{A})$.
Clearly, every model $\mathfrak{A}$ of the extensionality axiom is isomorphic to a solid model.

Fact 5 Let $\sigma : \bar{N} \prec N$ cofinally, where N is a ZFC^- model. Let $\bar{M} = J_{\bar{\delta}}(\bar{N})$. Let $\sigma' \supset \sigma$ s.t. $\sigma' : \bar{M} \to_{\Sigma_1} \mathfrak{A}$, $\sigma'(\bar{N}) = N$ and $\mathfrak{A}$ is solid. If $\mathfrak{A}$ is ill founded and $\delta = On \cap wfc(\mathfrak{A})$, then $J_\delta(N)$ is admissible.

proof.
Let $M = J_\delta(N)$, $u, z \in M$ s.t.

$$M \models \bigwedge x \in u \bigvee y\, \varphi(x, y z)$$

where φ is Σ_0

Claim $M \models \bigwedge x \in u \bigvee y \in v\, \varphi(x, y z)$
for some $v \in M$.
Let $\psi(u, v, z)$ be the Σ_0 formula
$\bigwedge x \in u \bigvee y \in v\, \varphi(x, y, z)$.
Recall that $\bar{M} = J^{\bar{E}}_{\bar{\theta}+\bar{\delta}}$, where
$\bar{N} = J^{\bar{E}}_{\bar{\theta}}$. We recall that for every β we have:

$J_\beta^{\bar{E}} = \bigcup_{\nu < \omega\beta} S_\nu^{\bar{E}}$, where $S_\nu^{\bar{E}}$ $(\nu < \infty)$ is a cumulative hierachy of transitive sets. Set $s = \langle S_\nu^{\bar{E}} \mid \nu < \infty \rangle$. We know that:

- $s \upharpoonright \nu \in J_\beta^{\bar{E}}$ for all $\nu < \omega\beta$
- The formula $y = s \upharpoonright \nu$ is a Σ_0 condition on y, ν (in the predicate $\bar{E}$).

The statement: $\bigwedge \nu \bigvee y \; y = s \upharpoonright \nu$ is then Π_1 and holds in $\bar{M}$. Hence it holds in $\mathfrak{A}$. But the statement:

$$\bigwedge u, z, \nu \, ((\bigvee \xi < \nu \; \psi(u, s(\xi), z)) \rightarrow$$
$$\rightarrow \bigvee \xi < \nu \, (\psi(u, s(\xi), z) \wedge$$
$$\wedge \bigwedge \xi' < \xi \; \neg \psi(u, s(\xi'), z)))$$

is also Π_1 & holds in $\bar{M}$. Hence it too holds in $\mathfrak{A}$. It is clear that if $a \in On_{\mathfrak{A}} \setminus \delta$, then

$$\mathfrak{A} \models \bigwedge x \in u \bigvee y \in s(a) \; \varphi(x, y, z).$$

But by the foregoing there must be a least such a in $\mathfrak{A}$. Hence $a \in M$ and

$$M \models \bigwedge x \in u \bigvee y \in S_a^{E^N} \varphi(x, y, z).$$

QED (Fact 5)

Note If σ' maps $\bar{M}$ cofinally to $\mathfrak{A}$, we also have: $\mathfrak{A} \models \bigwedge x \bigvee \nu\; x \in S(\nu)$. This will also be the case if σ' is Σ_2-preserving. Otherwise I don't see why it would hold.

. . . . . .

In this section we shall be very concerned with embeddings $\sigma' : \bar{M} \to \mathfrak{A}$ which, as in Fact 5, "lift up" an embedding $\sigma : \bar{N} \prec N$.

We shall, in fact, prove:

<u>Thm 1</u> Let $\sigma : \bar{N} \prec N$ cofinally, where N is a ZFC$^-$ model. Let $M = J_\alpha(N)$ s.t. $J_\nu(N)$ is not admissible for $0 < \nu < \alpha$. Then there exist $\sigma' \supset \sigma$ s.t. $\sigma' : \bar{M} \to_{\Sigma_1} \mathfrak{A}$, $\bar{M} = J_{\bar{\alpha}}(\bar{N})$, $\mathfrak{A}$ is solid, $\sigma'(\bar{N}) = N$, and $M \subset \mathfrak{A}$. (Hence $M \subset J_\delta(N)$, where $\delta = \mathrm{Onn\,wfc}(\mathfrak{A})$)

For later applications, however, we shall want more precise information about $\bar{\alpha}$ and σ'. (Fr. ins. we shall show that σ' is either cofinal into $\mathfrak{M}$ or is Σ_2-preserving.)

One type of liftup map is the Σ_0 - <u>liftup</u>:

<u>Def</u> Let $\sigma : \bar{N} \prec N$ where N is a ZFC$^-$ model. Let $\bar{M} = J_{\bar{\alpha}}(\bar{N})$. $\langle \mathfrak{A}, \sigma' \rangle$ is called a Σ_0 - <u>liftup</u> of $\langle \bar{M}, \sigma \rangle$ iff

(a) $\mathfrak{A}$ is solid

(b) $\sigma' \supset \sigma$ and $\sigma' : \bar{M} \to_{\Sigma_0} \mathfrak{A}$ cofinally

(c) $\mathfrak{A}$ is the Σ_0 - closure of $N \cup \operatorname{rng}(\sigma')$ in $\mathfrak{A}$.

Note: (c) is equivalent to

(c') Every $x \in \mathfrak{A}$ has the form $\sigma'(f)(\xi)$, where $f \in \bar{M}$, $f : \mu \to \bar{M}$, $\mu < \bar{\theta}$, and $\xi < \sigma(\mu)$.

It is also equivalent to:

(c'') If $x \in \mathfrak{A}$, then $x \in \sigma'(v)$ for a $v \in \bar{M}$ s.t. $\bar{\bar{v}} \in \bar{N}$ in $\bar{M}$.

<u>Fact 6</u> Let $\sigma, \bar{N}, N, \bar{M} = J_{\bar{\alpha}}(\bar{N})$ be as above. Then a Σ_0 liftup $\langle \mathfrak{A}, \sigma' \rangle$ of $\langle \bar{M}, \sigma \rangle$ exists. Moreover, $\sigma'(\bar{N}) = N$ (if $\bar{\alpha} > On_{\bar{N}}$) and $\langle \mathfrak{A}, \sigma' \rangle$ is unique up to isomorphism. (Hence it is unique if $\mathfrak{A}$ is fully transitive.)

Note The existence of $\langle \mathfrak{M}, \sigma' \rangle$ can be shown by an ultrapower-like construction.

Using the Σ_0-liftup alone will not suffice for the proof of Thm 1. Under certain circumstances we can form the Σ_1-liftup of $\langle \bar{M}, \sigma \rangle$, which exists whenever $\bar{M}$ is Σ_1-reflecting wrt. $\bar{N}$ in the following sense.

Def Let $N = J_\theta^E$ be a ZFC$^-$ model. Let $M = J_\alpha(N)$ with $\alpha > 0$. M is Σ_1-reflecting wrt. N iff the following hold:

- M is grounded wrt. N
- $M \models \theta$ is the largest cardinal
- N is regular in M
- Let p be a grounding parameter. Then there are arbitrarily large γ s.t. whenever $i < \omega$, $\xi < \gamma$, and $h_M(i, \langle \xi, p \rangle) < \theta$, then $h_M(i, \langle \xi, p \rangle) < \gamma$.

Def $C = C_M = C_{M,p}$ is the set of such γ.

(Note The choice of the grounding parameter p is not really important, since if p' is a second grounding parameter, then $C = C_{M,p}$ and $C' = C_{M,p'}$ coincide "on a tail" – i.e. $C \setminus \alpha = C' \setminus \alpha$ for some $\alpha < \theta$.)

Note $C = C_M$ is closed in θ. Hence it is club in θ if M is Σ_1-reflecting.

Lemma 1.1 Let M be grounded wrt. N, where $M \models \theta$ is the largest cardinal. Let N be definably regular in M. Then M is Σ_1 reflecting wrt. N.

proof.

Let p be a grounding parameter. Set:

$F = \langle h_M(i, \langle \xi, p \rangle) \mid i < \omega, \xi < \theta \wedge h_M(i, \langle \xi, p \rangle) < \theta \rangle$

Then $\langle N, F \rangle$ is a ZFC^- model. The conclusion follows easily.

QED (Lemma 1.1)

Lemma 1.2 Let M be Σ_1-reflecting wrt. N. Let B be $\Sigma_1(M)$ s.t. $B \subset N$. Then $\langle N; B \rangle$ is amenable.

prf. of Lemma 1.2

We make the following definition. Let p be a grounding parameter. For $\gamma \in C = C_{M,p}$ set:

$X_\gamma = h_M(\gamma \cup \{p\})$; Let $f_\gamma : M_\gamma \xrightarrow{\sim} X_\gamma$.

Then $M_\gamma = J_{\alpha_\gamma}(N|\gamma)$, where $N|\gamma = J_\gamma^{E^N}$.

Moreover $f_\gamma : M_\gamma \longrightarrow_{\Sigma_1} M$ with $\gamma = \mathrm{crit}(f_\gamma)$ and $f_\gamma(\gamma) = \delta$. We note, however, that $M_\gamma \in N$, since $\alpha_\gamma \leq \alpha$ and:

(a) $p_\gamma = f_\gamma^{-1}(p)$ is a grounding parameter for M_γ wrt $N|\gamma$

(b) $M_\gamma \models \gamma$ is the largest cardinal.

Thus θ would be collapsed to γ if $\alpha_\gamma \geq \theta$.

Now let B be $\Sigma_1(M)$ in the parameter q. Let $\gamma \in C$ s.t. $f_\gamma(\bar{q}) = q$. Then $B \cap (N|\gamma)$ is definable over M_γ in $\bar{q}$ as B was defined over M in q. Hence $B \cap (N|\gamma) \in N$. QED (Lemma 1.2)

<u>Note</u> The choice of p was not really important to the definition of $\langle M_\gamma \mid \gamma \in C \rangle$, since if p' is another grounding parameter yielding $\langle M'_\gamma \mid \gamma \in C' \rangle$, $\langle f'_\gamma \mid \gamma \in C' \rangle$, then:

the sequences again coincide on a tail – i.e. for some $\alpha < \theta$ we have:

$C \setminus \alpha = C' \setminus \alpha$, $M_\gamma = M'_\gamma$, $f_\gamma = f'_\gamma$ for $\gamma \in C \setminus \alpha$.

We also set:

Def $f_{\gamma\gamma'} = f_{\gamma'}^{-1} \circ f_\gamma$ for $\gamma \leq \gamma'$ in C.

Then $f_{\gamma\gamma'} \in N$, since $f_{\gamma\gamma'}$ is definable from $M_{\gamma'}, p_{\gamma'}, \gamma$ as f_γ was defined from M, p, γ.

Using this machinery we define the Σ_1-liftup:

Def Let $\sigma : \bar{N} \prec N$ cofinally, where N is a Σ_1-ZFC$^-$ model. Let $\bar{M} = J_{\bar{\alpha}}(\bar{N})$ be reflective wrt. $\bar{N}$. Define $\bar{C} = C_{\bar{M}}$, $\bar{M}_\gamma$, $\bar{f}_\gamma$ $(\gamma \in \bar{C})$, $\bar{f}_{\gamma\gamma'}$ $(\gamma \leq \gamma' \text{ in } \bar{C})$ as above. Set:

$$\tilde{M}_\gamma = \sigma(\bar{M}_\gamma),\ \tilde{f}_{\gamma\gamma'} = \sigma(\bar{f}_{\gamma\gamma'}).$$

Let $\langle \mathfrak{M}, \langle \tilde{f}_\gamma \mid \gamma \in \bar{C} \rangle\rangle$ be a direct limit of $\langle \tilde{M}_\gamma \mid \gamma \in \bar{C} \rangle$, $\langle \tilde{f}_{\gamma\gamma'} \mid \gamma \leq \gamma' \text{ in } \bar{C} \rangle$, where $\mathfrak{M}$ is solid.

Define $\sigma' : \bar{M} \to \mathfrak{M}$ by: $\tilde{\sigma} \bar{f}_\gamma = \tilde{f}_\gamma \sigma$.

$$\begin{array}{ccccc}
\tilde{M}_\gamma & \xrightarrow{\tilde{f}_{\gamma\gamma'}} & \tilde{M}_{\gamma'} & \xrightarrow{\tilde{f}_{\gamma'}} & \mathfrak{M} \\
\sigma \uparrow & & \sigma \uparrow & & \uparrow \sigma' \\
\bar{M}_\gamma & \xrightarrow[\bar{f}_{\gamma\gamma'}]{} & \bar{M}_{\gamma'} & \xrightarrow[\bar{f}_{\gamma'}]{} & \bar{M}
\end{array}$$

By our above remarks on "coincidence on a tail" it is clear that σ is defined independently of the choice of the grounding parameter $\bar{p}$ which gave $\bar{C} = \bar{C}_{\bar{M},\bar{p}}$.

Any such pair $\langle \mathfrak{M}, \sigma' \rangle$ is called a Σ_1-liftup of $\langle \bar{M}, \sigma \rangle$.

It is clear that $\langle \mathfrak{M}, \sigma' \rangle$ is unique up to isomorphism. Hence it is unique if $\mathfrak{M}$ is well founded (hence transitive).

We leave it to the reader to prove:

Lemma 1.3 Let $\langle \mathfrak{M}, \sigma' \rangle$ be a Σ_1-liftup of $\langle \bar{M}, \sigma \rangle$. Then

(a) $\sigma' : \bar{M} \xrightarrow[\Sigma_2]{} \mathfrak{M}$

(b) $\sigma'(\bar{N}) = N \in \mathrm{wfc}(\mathfrak{M})$

(c) Let $\bar{B}$ be $\Sigma_1(\bar{M})$ in $\bar{q}$ s.t. $\bar{B} \subset \bar{N}$. Set: $B = \bigcup_{u \in \bar{M}} \sigma(u \cap \bar{B})$. (Hence $\sigma : \langle \bar{N}, \bar{B} \rangle \xrightarrow[\Sigma_0]{} \langle N, B \rangle$ cofinally.) Then B is $\Sigma_1(\mathfrak{M})$ in $q = \sigma'(\bar{q})$ by the same definition.

Lemma 1.4 Let $\langle M, \sigma' \rangle$ be the well founded Σ_1-liftup of $\langle \bar{M}, \sigma \rangle$. Then

(a) M is Σ_1-reflective wrt N

(b) If $\bar{p} \in \bar{M}$ is a grounding parameter for $\bar{M}$, then $p = \sigma'(\bar{p})$ is a grounding parameter for M.

We now define:

Def Let $\sigma : \bar{N} \prec N$ cofinally, where $\bar{N}$ is a ZFC$^-$ model. Let $\bar{M} = J_{\bar{\alpha}}(\bar{N})$, where $\bar{N}$ is regular in $\bar{M}$.

$\langle M, \sigma' \rangle$ is a good liftup of $\langle \bar{M}, \sigma \rangle$ iff either $\bar{M}$ is Σ_1-reflective wrt $\bar{N}$ and $\langle M, \sigma' \rangle$ is a Σ_1-liftup, or else $\bar{M}$ is not Σ_1-reflective and $\langle M, \sigma' \rangle$ is a Σ_0-liftup.

Lemma 1.5 Let $\langle M, \sigma' \rangle$ be the well founded good liftup of $\langle \bar{M}, \sigma \rangle$. If M is Σ_1-reflective wrt. N, then $\bar{M}$ is Σ_1-reflective wrt $\bar{N}$. (Hence $\langle M, \sigma' \rangle$ is the Σ_1-liftup.)

proof

Suppose not. Then $\langle M, \sigma \rangle$ is the

Σ_0 - liftup. Since θ is the largest cardinal in M, it follows easily that $\bar{\theta}$ is the largest cardinal in $\bar{M}$.

Now let p be a grounding parameter for M. Then $p = \sigma'(f)(\xi)$, where $f \in \bar{M}$, $\xi < \theta$. Thus every $x \in M$ is $\Sigma_1(M)$ in $\sigma'(f)$ and parameters from θ. Hence $\sigma'(f)$ is a grounding parameter for M. Hence we may assume w.l.o.g. that $p = \sigma'(\bar{p})$ for a $\bar{p} \in \bar{M}$. But then for any $x \in \bar{M}$, we have:

$$\bigvee \xi < \theta \bigvee i < \omega \quad \sigma'(x) = h_M(i, \langle \xi, p \rangle)$$

Hence:

$$\bigvee \xi < \bar{\theta} \bigvee i < \omega \quad x = h_{\bar{M}}(i, \langle \xi, \bar{p} \rangle)$$

and $\bar{p}$ is a grounding predicate for $\bar{M}$. We must show:

<u>Claim</u> $\bar{C} = C_{\bar{M}, \bar{p}}$ is unbounded in $\bar{\theta}$.

Let $\eta < \bar{\theta}$. We must find $\bar{\gamma} > \eta$ s.t. $\bar{\gamma} \in \bar{C}$.

Let $C = C_{M,p}$. Pick $\gamma \in C$ s.t. $\sigma(\eta) < \gamma$. Set:

$\bar{\gamma}$ = the least $\bar{\gamma}$ s.t. $\sigma'(\bar{\gamma}) \geq \gamma$.

<u>Claim</u> $\bar{\gamma} \in \bar{C}$.

Let $h_{\bar{M}}(i, \langle \xi, \bar{p} \rangle) = \zeta < \bar{\theta}$,

where $\xi < \bar{\gamma}$. Then

$$h_M(i, \langle \sigma(\xi), p \rangle) = \sigma(\zeta) < \theta$$

where $\sigma(\xi) < \gamma$. Hence

$\sigma(\zeta) < \gamma \leq \sigma(\bar{\gamma})$, since $\gamma \in C$.

Hence $\zeta < \bar{\gamma}$. QED (Lemma 1.5)

We are now ready to prove Thm 1 in the stronger form:

Thm 2 Let $\sigma : \bar{N} \prec N$, where $N = J_\theta^E$ is a ZFC$^-$ model. Let $M = J_\gamma(N)$, where N is regular in M and $J_\nu(N)$ is not admissible for $0 < \nu < \gamma$. Let $\bar{M} = J_{\bar{\gamma}}(\bar{N})$ where $\bar{\gamma}$ is maximal s.t. $\bar{N}$ is regular in $\bar{\gamma}$ and no $J_\nu(\bar{N})$ is admissible for $0 < \nu < \bar{\gamma}$. Let $\langle \mathfrak{A}, \sigma' \rangle$ be a good liftup of $\langle \bar{M}, \sigma \rangle$. Then M is an initial segment of $\mathfrak{A}$.

proof:
Suppose not. Then $\mathfrak{A}$ is well founded, since otherwise, letting $\delta = On \cap wfc(\mathfrak{A})$, we know that $J_\delta(N)$ is admissible, hence contains M as a segment. Let $\mathfrak{A} = \tilde{M} = J_{\tilde{\gamma}}(N)$. Then $\tilde{\gamma} < \gamma$. Hence N is regular in $J_{\tilde{\gamma}+1}(N)$. Hence $\tilde{M}$ is Σ_1-reflecting wrt. N by Lemma 1.1. Hence $\bar{M}$ is Σ_1-reflecting wrt. $\bar{N}$ by Lemma 1.5 and $\langle \tilde{M}, \sigma' \rangle$ is the Σ_1-liftup of $\langle \bar{M}, \sigma \rangle$. Now let

$\bar{B}$ be $\Sigma_1(\bar{M})$ s.t. $\bar{B} \subset \bar{N}$. Set: $B = \bigcup_{u \in \bar{N}} \sigma(u \cap \bar{B})$.
By Lemma 1.3 (c), B is $\Sigma_1(\tilde{M})$; hence $B \in J_{\tilde{\gamma}+1}(N)$.
But then $\langle N, B \rangle$ is a ZFC$^-$ model and $\sigma : \langle \bar{N}, \bar{B} \rangle \to_{\Sigma_0} \langle N, B \rangle$. Hence $\langle \bar{N}, \bar{B} \rangle$ is a ZFC$^-$ model. By Fact 3 $\bar{N}$ is definably regular in $\bar{M}$, or in other words: $\bar{N}$ is regular in $J_{\bar{\gamma}+1}(\bar{N})$. Hence by the definition of $\bar{\gamma}$ we have: $\bar{M}$ is admissible. Now let $\bar{p}$ be a grounding parameter for $\bar{M}$ and let $\bar{B}$ be $\Sigma_1(\bar{M})$ in $\bar{p}$ s.t. $\bar{B}$ codes the whole of $\bar{M}$ with the properties given in Fact 2. Letting B be as above, B has the same $\Sigma_1(\tilde{M})$ definition over $p = \sigma^{\prime}(p)$, which is a grounding parameter for M. Hence B codes $\tilde{M}$ with the properties given in Fact 3. The statement "$\bar{M}$ is admissible" is a first order statement over $\bar{M}$, hence is expressible by a first order statement over $\langle \bar{N}, \bar{B} \rangle$. Hence $\langle N, B \rangle$ satisfies the same statement. Hence $\tilde{M}$ is admissible. Hence M is a segment of $\tilde{M}$. Contr! QED (Thm 2)

Now suppose that $\sigma : \bar{N} \prec N$ cofinally, where N is a ZFC$^-$ model, <u>and</u> $\sigma = \sigma_1 \circ \sigma_0$, where $\sigma_0 : \bar{N} \to N_0$, $\sigma_1 : N_0 \to N$ are cofinal Σ_0-preserving maps. (Hence, of course, all maps are elementary and all models are ZFC$^-$ models.)

Let $\bar{M} = J_{\bar{\gamma}}(\bar{N})$ be grounded wrt. $\bar{N}$ s.t. $\bar{M} \models \bar{\theta}$ is the largest cardinal, (where $\bar{\theta} = On \cap \bar{N}$, $\theta = On \cap N$, $\theta = On \cap N$).

Let $\langle M, \sigma' \rangle$ be a good liftup of $\langle \bar{M}, \sigma \rangle$ and $\langle M_1, \sigma_0' \rangle$ a good liftup of $\langle \bar{M}, \sigma_0 \rangle$. Then there is a <u>canonical</u> <u>bridge</u> $\sigma_1' : M_0 \to M$ s.t. $\sigma_1' \supset \sigma$ and $\sigma_1' \sigma_0' = \sigma'$. This is defined by cases as follows:

<u>Case 1</u>. $\bar{M}$ is not Σ_1-reflective wrt. $\bar{N}$. Then $\langle M, \sigma' \rangle$, $\langle M_0, \sigma_0' \rangle$ are Σ_0-liftups and every $x \in M_0$ has in M_0 the form $\sigma_0'(f)(\xi)$ where $f \in \bar{M}$ and $\xi < \theta_1$. We then set:

$$\sigma_1'(\sigma_0'(f)(\xi)) = \sigma'(f)(\sigma_1(\xi)).$$

<u>Case 2</u> Case 1 fails.

Then $\langle \mathfrak{M}, \sigma' \rangle$, $\langle \mathfrak{M}_0, \sigma_0' \rangle$ are Σ_1 - liftups.
Let $\bar{p}$ be a grounding parameter, $\bar{C} = C_{\bar{M},\bar{p}}$.
Set $\sigma'(\bar{M}_\gamma) = M_\gamma$, $\sigma'(\bar{f}_{\gamma\gamma'}) = f_{\gamma\gamma'}$
$\sigma_0(\bar{M}_\gamma) = M^0_\gamma$, $\sigma_0(\bar{f}_{\gamma\gamma'}) = f^0_{\gamma\gamma'}$
for $\gamma \leq \gamma'$ in $\bar{C}$. Then $\sigma'\bar{f}_\gamma = f_\gamma \sigma$ and
$\sigma_0'\bar{f}_\gamma = f^0_\gamma \sigma_0$, where:
$\langle \mathfrak{M}, \langle f_\gamma \mid \gamma \in \bar{C} \rangle \rangle$ = the limit of
$\langle M_\gamma \mid \gamma \in \bar{C} \rangle$, $\langle f_{\gamma\gamma'} \mid \gamma \leq \gamma'$ in $\bar{C} \rangle$,
and accordingly for $\langle \mathfrak{M}^0, \langle f^0_\gamma \mid \gamma \in \bar{C} \rangle \rangle$.
We define σ_1' by:
$$\sigma_1' \cdot f^0_\gamma = f_\gamma \circ \sigma \text{ for } \gamma \in \bar{C}.$$

In Case 1 we then have:
$$\sigma_1 : \mathfrak{M}_0 \xrightarrow[\Sigma_0]{} \mathfrak{M} \text{ cofinally}$$
and in Case 2:
$$\sigma_1 : \mathfrak{M}_0 \xrightarrow[\Sigma_2]{} \mathfrak{M}.$$
Now suppose that $\mathfrak{M}_0 = M_0$ is well founded. We leave it to the reader to prove:

<u>Lemma 2.1</u> Let $\mathfrak{M}_0 = M_0$ be well founded (hence transitive). Then:

- $\langle \mathfrak{M}, \sigma_1' \rangle$ is a Σ_0-liftup of $\langle M_0, \sigma_1 \rangle$ if $\bar{\mathfrak{M}}$ is not Σ_1-reflective wrt. $\bar{N}$
- $\langle \mathfrak{M}, \sigma_1' \rangle$ is a Σ_1-liftup of $\langle M_0, \sigma_1 \rangle$ if $\bar{\mathfrak{M}}$ is Σ_1-reflective wrt. $\bar{N}$.

But if $\bar{\mathfrak{M}}$ is not Σ_1-reflective, then neither is M_0 by Lemma 1.5. Hence: $\langle \mathfrak{M}, \sigma_1' \rangle$ is a good liftup of $\langle M_0, \sigma_1 \rangle$

Now suppose $\sigma_{ij} : N_i \prec N_j$ cofinally for $i \le j < \eta$, where each N_i is a ZFC^- model. Let $M_0 = J_{\gamma_0}(N_0)$ be grounded wrt. N_0 and s.t.

$M_0 \models \Theta_0$ is the largest cardinal,

where $\Theta_i = On \cap M_i$. Let $\langle \mathfrak{M}_j, \sigma_{0j}' \rangle$ be a good liftup of $\langle M_0, \sigma_{0j} \rangle$ for $j < \eta$. Let $\sigma_{ij}' : \mathfrak{M}_i \to \mathfrak{M}_j$ be the bridging map defined above. It follows from the definition that $\sigma_{jk}' \circ \sigma_{ij}' = \sigma_{ik}'$ for $i \le j \le k < \eta$.

Historical Note

Thm 1 has the corollary:

(*) Let $\sigma : \bar{N} \prec N$ cofinally, where $N = J_\theta^E$ is a ZFC^- model. Assume that N is regular in $J_\rho(N)$, where ρ is least s.t. $\rho > 0$ and $J_\rho(N)$ is admissible. Let $\bar{\rho}$ be least $\bar{\rho} > 0$ s.t. $J_{\bar{\rho}}(\bar{N})$ is admissible. Let $J_{\bar{\rho}}(\bar{N}) \models \varphi(x, \bar{N})$ where φ is a Π_1 formula and $x \in \bar{N}$. Then $J_\rho(N) \models \varphi(\sigma(x), N)$.

(To see this, note that we can w.l.o.g. assume the $\bar{\alpha}$ in Thm 1 to be $\leq \bar{\rho}$.)

In our note [SPSC] we made heavy use of a lemma which draws the same conclusion under the assumption that $\bar{N}$ is full. Hence (*) is a strengthening of that lemma. However, Theorem 1 and its corollary (*) predate the lemma by several decades. Our recollection is that we proved them in late 1975. The lemma on fullness was, of course, more suitable to [SPSC], since its proof involved no fine structure,