§5 A converse

In this section we prove converse theorems to §3 Thm 1, showing in particular that that theorem is best possible.

Def Let N be a premouse. Let $M = J_\gamma(N) =_{Df} J^{E^N}_{\delta+\gamma}$, where $\delta = ht(N)$, and γ is the least $\gamma > 0$ s.t. $J_\gamma(N)$ is admissible.

N is royal iff δ is E-Woodin in M and $\rho^1_M = \delta$.

If N is royal, then M is called the crown of M.

Def A premouse M is weakly iterable iff whenever $\mathcal{J} = \langle\langle M_i \mid i \leq n\rangle, \langle \nu_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ is a finite, truncation-free putative iteration of M by Σ_0 ultraproducts, then M_n is well founded (hence transitive).

Clearly, every mouse is weakly iterable.

Our first theorem in the converse direction reads:

<u>Thm 1</u> Let N be a countable royal premouse with crown M. Let M be weakly iterable. Then there is an iteration

$$\mathcal{I} = \langle \langle N_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$$

of N of length ω with distinct cofinal branches b_0, b_1 s.t.

$$N_{b_0} = N_{b_1} = \bigcup_{i<\omega} J^{E^{N_i}}_{\kappa_i} .$$

The proof will require a lengthy series of sublemmas. $\mathcal{I}$ will be an "alternating chain":

$$\begin{tikzcd}
N_0 \arrow[r] \arrow[rr, bend left] & N_1 \arrow[rr, bend right] & N_2 \arrow[rr, bend left] & N_3 \arrow[rr, bend right] & N_4 & \cdots & \begin{matrix} b_0 \\ b_1 \end{matrix}
\end{tikzcd}$$

Thus $T(i+1) = \begin{cases} 0 & \text{if } i = 0 \\ i-1 & \text{if } i > 0 \end{cases}$

In the following suppose that:

(*) • $N = J_\delta^E$ is a countable ZFC model.

• $M = J_\gamma(N) = J_{\delta+\gamma}^{E^N}$, where γ is the least $\gamma > 0$ s.t. $J_\gamma(N)$ is admissible

• N is regular in M

• $\rho_M^1 = \delta$.

Then:

<u>Lemma 1.1</u> Let $\bar{\pi} : N \to_{\Sigma_0} N'$ cofinally. Let $\pi : M \to M'$ be the Σ_0-liftup of $\bar{\pi}$, where M' is transitive. Then

(a) $\rho_{M'}^1 = \delta' =_{\text{Df}} On \cap N'$

(b) Let A be $\Sigma_1(M)$ in p and A' be $\Sigma_1(M')$ in $p' = \pi(p)$ by the same definition. Then
$$\pi(A \cap x) = A' \cap \pi(x) \quad \text{for } x \in N$$

(c) $\pi : M \to_{\Sigma_2} M'$.

prf.

Let $M' = J_{\gamma'}(N')$. Clearly $\rho_{M'}^1 \leq \delta'$, since if p is a grounding parameter for M, then $\pi(p)$ is a grounding parameter for M'.

Thus (a) follows from (b). We prove (b).

Let $a = A \cap x$, $a' = \pi(a)$.

<u>Claim</u> $a' = A' \cap \pi(x)$.

proof.

$(\supset)$ $\bigwedge z \in x\,(Az \to z \in a)$ is Π_1 in p.

Hence $\bigwedge z \in \pi(x)\,(A'z \to z \in a')$, since π is Σ_1 – preserving.

$(\subset)$ Let $Ax \leftrightarrow \bigvee z\, B(z, x)$

$A'x \leftrightarrow \bigvee z\, B'(z, x)$,

where B is $\Sigma_0(M)$ in p and B' is $\Sigma_0(M')$ in $p' = \pi(p)$ by the same definition.

Then $\bigwedge x \in a\, \bigvee z\, B(z, x)$. Hence, by admissibility, there is $u \in M$ s.t. $\bigwedge x \in a\, \bigvee z \in u\, B(z, x)$.

Hence $\bigwedge x \in a'\, \bigvee z \in \pi(u)\, B'(z, x)$.

Hence $a' \subset A'$. QED (b)

We now prove (c).

Let $M' \models \bigvee x \bigwedge y\, \varphi(x, y, p')$ where $p' = \pi(p)$ and φ is a Σ_0 formula.

<u>Claim</u> $M \models \bigvee x \bigwedge y\, \varphi(x, y, p)$.

prf.

Let $M' \models \bigwedge y\, \varphi(x, y, \pi(p))$ where $x = \pi(f)(z)$, $z \in N'$, $f \in M$, $f: u \to N$ where $u \in N$. (hence $f \in N$ by regularity of N in M).

Set $u' = \pi(u)$ and:

$$a = \{z \in u \mid \bigwedge y\, \varphi(f(z), y, p)\}$$

$$a' = \{z \in u' \mid \bigwedge y\, \varphi(\pi(f)(z), y, p')\}.$$

Then $\pi(a) = a'$ by (b). But $a' \neq \emptyset$, hence $a \neq \emptyset$. Hence

$M \models \bigwedge y\, \varphi(f(z), y, p)$ where $z \in a$,

QED (Lemma 1.1)

<u>Lemma 1.2</u> Let M be any admissible structure and let $\pi: \bar{M} \xrightarrow{\Sigma_2} M$. Then $\bar{M}$ is admissible.

prf.

Let $\bar{M} \models \bigwedge x \in u \bigvee y\, \varphi(x, y, p)$, where φ is Σ_0.

<u>Claim</u> $\bar{M} \models \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi(x, y, p)$.

By Σ_2 preservation we have:

$$M \models \bigwedge x \in \pi(u) \bigvee y \, \varphi(x, y, \pi(p)). \text{ Hence}$$

$$M \models \underbrace{\bigvee v \bigwedge x \in \pi(u) \bigvee y \in v \, \varphi(x, y, \pi(p))}_{\Sigma_1}$$

by admissibility. The conclusion is immediate. QED (Lemma 1.2)

However, we shall need a sort of converse to Lemma 1.2:

<u>Lemma 1.3</u> Let M, M' be as in Lemma 1.1. Then M' is admissible.

proof.

Unfortunately we don't know how to prove this without introducing extra machinery (which, however, we'll be needed later as well). By Barwise theory we can show that there is an ill founded model $\mathfrak{A}$ end extending $|M| = \{x \mid x \in M\}$ with the properties:

- $\mathfrak{A}$ is solid (i.e. $wfc(\mathfrak{A})$ is transitive and $x \in y \leftrightarrow x \in^{\mathfrak{A}} y$ for $x, y \in wfc(\mathfrak{A})$, where $wfc(\mathfrak{A})$ is the well founded core of $\mathfrak{A}$).

(a) $x \in M \to x \in wfc(\mathfrak{A})$ (Hence $N \in wfc(\mathfrak{A})$)

(b) $On \cap wfc(\mathfrak{A}) = \gamma$

(c) There is $\tilde{\gamma} \in On_{\mathfrak{A}}$ s.t.

- $\tilde{\gamma} > \nu$ in $\mathfrak{A}$ for all $\nu < \gamma$
- Let $\tilde{M} = J_{\tilde{\gamma}}(N)$ in $\mathfrak{A}$. Then $\tilde{M}$ satisfies the assumptions $(*)$ in $\mathfrak{A}$.

To see that $\mathfrak{A}$ exists, we consider the following infinitary language $\mathcal{L}$ on M:

Predicate: $\in$

Constants: $\underline{x}$ $(x \in M)$, $\mathring{\gamma}$, $\mathring{M}$, $\dot{c}$

Axioms: ZFC^-, $\bigwedge v\,(v \in \underline{x} \leftrightarrow \bigvee_{z \in x} v = \underline{z})$ for all $x \in M$, $\mathring{\gamma} \in On$, $\dot{c} \in On$,

$\mathring{M} = J_{\mathring{\gamma}}(\underline{N})$ satisfies $(*)$,

$\underline{\nu} < \dot{c} < \mathring{\gamma}$ for all $\nu < \gamma$.

Then $\mathcal{L}$ is consistent. To see this, let X be any M-finite subset of the axioms. Then $\langle H_{\omega_1}, \gamma, M, c \rangle$ models X for some $c < \gamma$.

Hence $\mathcal{L}$ has a solid model $\mathfrak{A}$.

Clearly $x = \underline{x}^{\mathfrak{A}} \in \mathrm{wfc}(\mathfrak{A})$ for $x \in M$.

Set: $\tilde{\gamma} = \mathring{\gamma}^{\mathfrak{A}}$, $\tilde{M} = \mathring{M}^{\mathfrak{A}}$. Then $\gamma \notin \mathrm{wfc}(\mathfrak{A})$, since otherwise $M = J_\gamma(N)$ is admissible, where $0 < \gamma \leq \dot{c}^{\mathfrak{A}} < \tilde{\gamma}$, contradicting the minimality of $\tilde{\gamma}$ in $\mathfrak{A}$. Hence $\gamma = \mathrm{On} \cap \mathrm{wfc}(\mathfrak{A})$.

We now prove Lemma 1.3. Let $\bar{\pi}, \pi$, N, N', M, M' be as in Lemma 1.1. Let $M' = J_{\gamma'}(N')$ where $N' = J^{E'}_{\delta'}$.

Then there is $\tilde{\pi}: \tilde{M} \to \tilde{M}'$ s.t. $\langle \tilde{M}, \tilde{\pi} \rangle$ is a Σ_0-liftup of $\langle \tilde{M}, \pi \rangle$ (or equivalently of $\langle \tilde{M}, \bar{\pi} \rangle$).

<u>Claim 1</u> $\gamma' = \mathrm{On} \cap \mathrm{wfc}(\tilde{M}')$

prf.
$(\subset)$ is given. We prove $(\supset)$.
It suffices to show:

Claim Let $\xi' \in On_{\tilde{M}'} \setminus \gamma'$. Then there is $\xi \in On_{\tilde{M}} \setminus \gamma$ s.t. $\tilde{\pi}(\xi) \leq \xi'$.

proof.

Since $\pi : \tilde{M} \to \tilde{M}'$ is the Σ_0-liftup of $\bar{\pi} : N \to N'$, there is $u \in \tilde{M}$ s.t. $\bar{\bar{u}} < \delta$ in $\tilde{M}$ and $\xi' \in \pi(u)$ in $\tilde{M}'$. Let $u \subset On \cap \tilde{M}$ (w.l.o.g.) and let $f : \mu \to u$ be the monotone enumeration of u in $\tilde{M}$.

Then $\xi' = \pi(f)(\eta)$ in $\tilde{M}'$ for an $\eta < \pi(\mu)$.

We note that $u = \operatorname{rng}(f) \not\subset \gamma$, since otherwise, letting $\zeta \in \tilde{M}$ be least s.t. $\operatorname{rng}(f) \subset \zeta$, we have $\zeta \in \gamma$.

But then $\operatorname{rng}(\pi(f)) \subset \pi(\zeta) \in \gamma'$.

Contradiction!

But then there is a least $\nu < \mu$ s.t. $f(\nu) \notin \gamma$. Letting $\tilde{\nu} = \sup f''\nu$, we then have $\tilde{\nu} \in \gamma$. (Otherwise $\gamma = \sup f''\nu \in \tilde{M}$.) Thus

$$\tilde{M} \models u \cap (\tilde{\nu}, f(\nu)) = \emptyset.$$

Hence

$$\tilde{M}' \models \pi(u) \cap (\pi(\tilde{\nu}), \pi(f(\nu))) = \emptyset.$$

Hence $\xi' \geq \pi(f(\nu))$. QED (Claim 1)

<u>Claim 2</u> M' is admissible.

proof

Let $M' \models \bigwedge x \in u \bigvee y \, \varphi(x, y, p)$, where $u, p \in M'$ and φ is Σ_0.

<u>Claim</u> $M' \models \bigvee v \bigwedge x \in u \bigvee y \in v \, \varphi(x, y, p)$.

Let $\langle J_\nu(N') \mid \nu \in On_{\tilde{M}'} \rangle$ have the same Σ_1 definition over $\tilde{M}'$ in the parameter N' as $\langle J_\nu(N) \mid \nu \in On_{\tilde{M}} \rangle$ over $\tilde{M}$ in the parameter N.

Set: $\tilde{M} | \nu = \{x \mid \tilde{M} \models x \in J_\nu(N)\}$ for $\nu \in On_{\tilde{M}}$

$\tilde{M}' | \nu = \{x \mid \tilde{M}' \models x \in J_\nu(N)\}$ for $\nu \in On_{\tilde{M}'}$.

Since $\tilde{\pi}$ is Σ_2 - preserving, we have:

- $\nu < \tau$ in $\tilde{M}' \rightarrow \tilde{M}' | \nu \subset \tilde{M}' | \tau$
- $\tilde{M}' = \bigcup\limits_{\nu \in On_{\tilde{M}'}} \tilde{M}' | \nu = \bigcup\limits_{\nu \in On_{\tilde{M}}} \tilde{M}' | \tilde{\pi}(\nu)$,
- $\tilde{\pi} \upharpoonright (\tilde{M} | \nu) : \tilde{M} | \nu \prec \tilde{M}' | \tilde{\pi}(\nu)$ for $\nu \in On_{\tilde{M}}$.

At $\nu \in On_{\tilde{M}'} \setminus \gamma'$, we obviously have:

$\tilde{M}' \models \bigwedge x \in u \bigvee y \in J_\nu(N') \, \varphi(x, y, p)$, since $M = \bigcup_{\nu \in \gamma'} \tilde{M}|\nu$. Now let $\lambda \in \tilde{M} \setminus \gamma'$ be p.r. closed. Then:

$$\langle J_\nu(N') \mid \nu < \pi(\lambda) \text{ in } \tilde{M}' \rangle$$

has the same Σ_1 definition over $\tilde{M}'$ as $\langle J_\nu(N) \mid \nu < \lambda \text{ in } \tilde{M} \rangle$ over $\tilde{M}$.

But $\tilde{M}'|\pi(\lambda)$ satisfies the axiom schema of foundation, since $\tilde{M}|\nu$ does. Hence:

$\tilde{M}'|\pi(\lambda) \models$ there is a least ζ s.t.

$$\bigwedge x \in u \bigvee y \in J_\zeta(N') \ \varphi(x, y, p).$$

Let ζ be the least such in $\tilde{M}'|\pi(\lambda)$.

Then clearly $\zeta \in \gamma'$. But then $J_\zeta(N')^{\tilde{M}'} = J_\zeta(N')^{M'} \in M'$ and:

$$M' \models \bigwedge x \in u \bigvee y \in J_\zeta(N') \ \varphi(x, y, p)$$

QED (Lemma 1.3)

From now on assume not only that M satisfies (*), but also that N is a premouse and δ is E-Woodin in M. (Hence N is royal and M is its crown.) In the following we write "Woodin" for "E-Woodin".

We first make some remarks on the internal structure of M.

<u>Def</u> S_δ = the set of $\tau \in M$ s.t.

$\tau > \delta$ is p.r. closed

More generally, we define S_α to be the set of $\tau > \alpha$ which look like an element of S_δ wrt. α instead of δ:

<u>Def</u> Let $\alpha \leq \delta$. Set $N|\alpha = J_\alpha^E$.

S_α = the set of τ s.t.

- $\tau > \alpha$ is p.r. closed
- $N|\alpha$ is Woodin in $Q_\tau =_{\mathrm{Df}} J_\tau(N|\alpha)$
- There is no ν s.t. $\alpha < \nu < \tau$ and $J_\nu(N|\alpha)$ is admissible.

<u>Note</u> If $\tau \in S_\alpha$, then $N|\alpha$ is a ZFC model and is regular in Q_τ

Note By the last clause, $Q_\tau = M|\tau$ if $\tau \in S_\alpha$.

But it then follows that there is at most one α s.t. $\tau \in S_\alpha$. We denote this by α_τ. We also set:

$$S = \bigcup_\alpha S_\alpha .$$

(Remark α does not have to be a cardinal in N in order that $S_\alpha \neq \emptyset$.)

Note that if $\tau \in S$, then every element of Q_τ is Q_τ-definable in parameters from α_τ. We set:

Def $a_\tau = \{\langle \varphi, \xi \rangle \mid \xi < \alpha_\tau \wedge Q_\tau \models \varphi(\xi)\}$.

$$\tau \lhd \tau' \longleftrightarrow (\tau, \tau' \in S \wedge \alpha_\tau \leq \alpha_{\tau'} \wedge$$
$$\wedge\, a_\tau = \alpha_\tau \cap a_{\tau'})$$

Then $\lhd$ is a partial ordering of S (in fact, it is a tree).

If $\tau \lhd \tau'$ there is a unique map $\sigma : Q_\tau \prec Q_{\tau'}$ s.t. $\sigma \restriction \alpha_\tau = \mathrm{id}$ and $\sigma(\alpha_\tau) = \alpha_{\tau'}$. We denote this map by $\sigma_{\tau, \tau'}$. Then $\langle \sigma_{\tau\tau'} \mid \tau \lhd \tau' \rangle$ is a commutative system of maps.

An alternative definition of $\prec$ is:

Def $\tau \prec \tau'$ iff, letting X = the smallest $X \prec Q_{\tau'}$ s.t. $\alpha_\tau \subset X$, we have:

- $\alpha_{\tau'} \cap X \subset \alpha_\tau$
- $Q_\tau \simeq X$.

$\sigma_{\tau\tau'}$ is then that σ s.t. $\sigma : Q_\tau \xrightarrow{\sim} X$.

Now set:

Def Let $\tau \in S$. $C_\tau = \{\alpha_{\bar{\tau}} \mid \bar{\tau} \prec \tau\}$.

Then C_τ is closed in α_τ. If $\tau \in S_\delta$, then C_τ is, in fact, club in δ.

The structure:

$\langle\langle Q_\tau \mid \tau \in S\rangle, \langle\sigma_{\bar{\tau}\tau'} \mid \bar{\tau} \prec \tau'\rangle\rangle$

is an example of a <u>coarse</u> <u>morass</u>.

(We could, of course, have used a fine morass, but saw no utility in doing so.)

We note the following facts:

By the regularity of N in M:

Fact 1 Let $f \in M$, $f : \kappa \to N$, where $\kappa < \delta$.
Then $f \in N$.

Hence:

Fact 2 Let $f \in M$, $f : \kappa \to J_\tau(N)$, where $\kappa \in N$ and $0 < \tau < \gamma$. Then $f \in J_{\tau+1}(N)$

prf.
There is $g \in J_{\tau+1}(N)$ s.t. $g : \delta \leftrightarrow J_\tau(N)$.
Then $g^{-1} \circ f \in N$ by Fact 1. Hence
$f = g \circ (g^{-1} f) \in J_{\tau+1}(N)$.

But then:

Fact 3 Let $\tau < \gamma$ s.t. $\mathrm{cf}(\tau) = \delta$ in M
Let $f \in M$, $f : \kappa \to J_\tau(N)$, $\kappa < \delta$.
Then $f \in J_\tau(N)$

Given that $N = J^E_\delta$ is a premouse satisfying ZFC, we have:

<u>Lemma 1.4</u> Let $\mathcal{J} = \langle\langle N_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a finite, truncation free putative iteration of N by Σ_0-ultrapowers. Then

(a) N_i is transitive and $On \cap N_i = \delta$.

(b) N_h $(h \leq i)$, π_{hi} $(h \leq_T i)$ are uniformly N-definable in $\langle\nu_h \mid h < i\rangle$

(c) ${}^\omega N_i \subset N_i$ in N

(d) $E^{N_i}_{\nu_i}$ is ω-complete in N and is close to $N_{T(i+1)}$.

The proof is by induction on i and is well known.

By the admissibility of M we know: If $\langle u, r\rangle \in M$ is a well founded structure satisfying the extensionality axiom, there are unique $v, \sigma \in M$ s.t. v is transitive and $\sigma : \langle u, r\rangle \xrightarrow{\sim} \langle v, \in \cap v^2\rangle$.

Hence, by Fact 3 we get:

<u>Lemma 1.5</u> Let $\tau < \gamma$ s.t. $cf(\tau) = \delta$ in M. Let $Q = J_\tau(N)$ be weakly iterable. Let $\mathcal{Y} = \langle\langle Q_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a finite, truncation-free putative iteration of Q by Σ_0-ultraproducts. Then

(a) Q_i is transitive and $Q_i, \pi_{hi} \in M$

(b) $\langle Q_h \mid h \leq i\rangle$, $\langle \pi_{hj} \mid h \leq_T j \leq_T i\rangle$ are uniformly $\Delta_1(M)$ in $\langle \nu_h \mid h < i\rangle, Q$.

(c) ${}^{\omega}Q_i \subset Q_i$ in M

(d) $E^{Q_i}_{\nu_i}$ is ω-complete in M and is close to $Q_{T(i+1)}$.

Now let $\Gamma = \{\tau < \gamma \mid cf(\tau) = \delta \text{ in } M\}$. For $\tau \in \Gamma$ let $\mathcal{Y}^\tau = \langle\langle Q_i^\tau\rangle, \langle\nu_i\rangle, \langle\pi_{ij}^\tau\rangle, T\rangle$ be the iteration described in Lemma 1.5 (determined by $\langle\nu_i\rangle \in N$). By Fact 3 it follows that $Q_i^\tau = Q_i^{\tau'} | \pi_{0i}^{\tau'}(\tau)$ for $\tau \leq \tau'$ in Γ. But then we can set:

$$M_i = \bigcup_\tau Q_i^\tau, \quad \pi_{ij} = \bigcup_\tau \pi_{ij}^\tau,$$

and $\langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ is easily seen to be a finite truncation-free iteration of M by Σ_0-ultra-powers.

Thus:

<u>Lemma 1.6</u> Let M be weakly iterable. Let $\mathcal{J} = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a finite, truncation free iteration of M by Σ_0-ultrapowers. Then:

(a) M_i is transitive and $On \cap M_i = \gamma$

(b) M_i, π_{hi} $(h \leq_T i)$ are uniformly M-definable in $\langle \nu_h \mid h < i\rangle$

(c) $M_i^{\omega} \subset M_i$ in M

(d) $E_{\nu_i}^{M_i}$ is ω-complete in M and close to $M_{T(i+1)}$.

We also note (using Fact 1) that if $\mathcal{J} = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ is as in Lemma 1.6 and $M_i = J_\gamma(N_i)$, and we set $\bar{\mathcal{J}} =_{Df} \langle\langle N_i\rangle, \langle\nu_i\rangle, \langle\bar{\pi}_{ij}\rangle, T\rangle$ with $\bar{\pi}_{ij} = \pi_{ij} \upharpoonright N_i$, then $\bar{\mathcal{J}}$ is an iteration of N as in Lemma 1.4. Conversely, we can obtain $\mathcal{J}$ from $\bar{\mathcal{J}}$ by setting:

$$\langle M_i, \pi_{hi}\rangle = \text{the } \Sigma_0\text{-liftup of } \langle M_h, \bar{\pi}_{hi}\rangle$$

for $h \leq_T i$.

Up until now we have made no real use of the fact that δ is E-Woodin in M. We restate the relevant definitions:

Def Let $a \in M$, $a \subset N$. Let κ be a cardinal in N. κ is a-<u>strong</u> (wrt. the sequence E)* iff whenever $\kappa \leq \eta < \delta$, there is $\nu < \delta$ s.t. $\kappa = \mathrm{crit}(E_\nu)$, $\eta^{+N} < \lambda_\nu =_{\mathrm{Df}} E_\nu(\kappa)$, and, letting $\pi : M \xrightarrow[E_\nu]{} M'$, we have: $a \cap \eta = \pi(a) \cap \eta$.

(We then also say: E_ν -<u>witnesses</u> a-strongness wrt. η)

δ is E-<u>Woodin</u> in M iff for every $a \in M$ s.t. $a \subset N$ there is $\kappa < \delta$ which is a-strong (wrt. the sequence E).

In the following we shall generally write "Woodin" for "E-Woodin".

*) In §4 we expressed this by saying: κ is E-strong in $\langle N, a \rangle$.

<u>Def</u> Let $\tau \in S_\delta$. κ is τ-strong iff iff κ is a_τ-strong.

Hence if $\bar{\tau} \prec \tau$ there is $\nu < \delta$ s.t. $\kappa = \mathrm{crit}(E_\nu)$, $\eta^{+N} < \lambda_\nu$, and letting $\pi : M \xrightarrow[E_\nu]{} M'$, we have: $\bar{\tau} \prec \pi(\tau)$ in M'. (Hence $a_{\bar{\tau}} = a_{\pi(\tau)} \cap a_{\bar{\tau}}$ in M').
This could be taken as the definition of "τ-strong".

<u>Lemma 1.7</u> Let κ be τ-strong. Then there is a $\bar{\tau} \in S_\kappa$ s.t. $\bar{\tau} \prec \tau$.

prf.
Suppose not. Then for some $\alpha < \kappa$ we have: $\bigwedge \bar{\tau} \in (\alpha, \kappa)\ \bar{\tau} \not\prec \tau$.
Let $\tau' \prec \tau$, $\kappa < \tau'$. Let $\kappa = \mathrm{crit}(E_\nu)$ s.t. if $\pi : M \xrightarrow[E_\nu]{} M'$, then $\tau' \prec \pi(\tau)$ in M', where $\tau' < \lambda_\nu = \pi(\kappa)$. Since $\pi : M \xrightarrow[E_\nu]{} M'$, we have $\bigwedge \bar{\tau} \in (\alpha, \pi(\kappa))\ \bar{\tau} \not\prec \pi(\tau)$. Hence $\tau' \not\prec \pi(\tau)$. Contr!

QED (Lemma 1.7)

The following fact will also be useful:

Lemma 1.8 Let F be close to N and let $\pi : N \xrightarrow[F]{} N'$. Let $\kappa = \mathrm{crit}(F)$. Let $\bar{\tau} \in S_\kappa$, $\tilde{\tau} = \pi(\bar{\tau})$. Then $\bar{\tau} \prec \tilde{\tau}$ in N' and $\sigma_{\bar{\tau}\tau} = \pi \upharpoonright Q_{\bar{\tau}}$.

proof.

Clearly $\tilde{\tau} \in S_{\pi(\kappa)}$ in N' and $Q_{\bar{\tau}}^N = Q_{\bar{\tau}}^{N'} = J_{\bar{\tau}}(N|\kappa)$, where $N|\kappa = N'|\kappa$. But then $a_{\bar{\tau}} = \kappa \cap a_{\tilde{\tau}}$ in N', since:

$$\langle \varphi, \xi \rangle \in a_{\bar{\tau}} \longleftrightarrow Q_{\bar{\tau}} \models \varphi[\xi]$$
$$\longleftrightarrow \underset{\pi(Q_{\bar{\tau}})}{\overset{\parallel}{Q_{\tilde{\tau}}}} \models \varphi[\xi]$$

Hence $\bar{\tau} \prec \tilde{\tau}$. But $\pi \upharpoonright Q_{\bar{\tau}}$ is then the unique $\sigma : Q_{\bar{\tau}} \prec Q_\tau$ s.t. $\sigma \upharpoonright \kappa = \mathrm{id}$. QED (Lemma 1.8)

(Note: If $F = E_\nu^N$, then $\tilde{\tau} < \nu = \pi(\kappa)^{+N'}$. Hence $\bar{\tau} \prec \tilde{\tau}$ in N, $Q_{\tilde{\tau}}^N = Q_{\tilde{\tau}}^{N'}$, and $\pi \upharpoonright Q_{\bar{\tau}} = \sigma_{\bar{\tau}\tilde{\tau}} \in N$.)

<u>Note</u> If $\pi : M \to_{F} M'$ and $\bar{\tau} < \tau$ in M, then $\bar{\tau} < \pi(\bar{\tau}) < \pi(\tau)$ in M' and

$\sigma_{\bar{\tau}, \pi(\tau)} = \sigma_{\pi(\bar{\tau}), \pi(\tau)} \circ \pi$ in M'.

We also define:

<u>Def</u> E_ν <u>witnesses</u> the τ-strongness of κ wrt τ' iff κ is τ-strong, $\tau' < \tau$, and E_ν witnesses the a_τ-strongness of κ wrt $\alpha_{\tau'}$.

(In other words: $\tau' < \tau$, $\kappa = \mathrm{crit}(E_\nu)$ and if $\pi : M \to_{E_\nu} M'$, then

. $(\alpha_{\tau'})^+ < \lambda = \pi(\kappa)$ in M

. $\tau' < \pi(\tau)$ in M'.)

<u>Lemma 1.9</u> Let E_ν witness the τ-strongness of κ wrt τ' in M. Let $\bar{\pi} : J^E_{\tau^+} \to_{E_\nu} J^E_\nu$. Then $\bar{\tau} < \tau' < \bar{\pi}(\bar{\tau})$ in M.

prf.
Let $\pi : M \to_{E_\nu} M'$. (Hence $\bar{\pi} = \pi \upharpoonright J^E_{\tau^+}$.)
Then $\tau' < \pi(\tau)$ and $\bar{\tau} < \bar{\pi}(\bar{\tau}) < \pi(\tau)$.

QED

(Hence, if $\pi' : M' \to_{E_\nu} M''$ and M' is a premouse with $J^{E^{M'}}_{\tau+M'} = J^{E^M}_{\tau+M}$, then $\bar{\tau} < \tau' < \pi'(\bar{\tau})$ in M'', since $\bar{\pi}(\bar{\tau}) = \pi'(\bar{\tau})$.)

We now turn to the proof of Thm 1. Essentially, the method is to produce an iteration of M s.t. the induced iteration on N with the same indices has the desired property. Basically we do this by iterating a version of the "one step lemma" described in [PD] ("A Proof of Projective Determinacy" by Martin and Steel). The authors remark that the one step lemma superficially appears inadequate for building infinite alternating chains, since that would seem to require an infinite descending chain of ordinals. But "we must sidestep the problem" they write. Accordingly, we shall iterate not M itself, but rather its ill founded end extension $\tilde{M}$, which does contain infinite descending chains of ordinals.

We may assume w.l.o.g. that all the things we have just proven about M are also true of $\tilde{M}$ in $\mathfrak{A}$. (To see this, simply adjoin the relevant statements about $\mathring{M}$ as additional axioms to $\mathcal{L}$. The consistency of the so enhanced version of $\mathcal{L}$ follows exactly as before.)

We construct an alternating chain

$$\tilde{\mathcal{Y}} = \langle \langle \tilde{M}_i \rangle, \langle \nu_i \rangle, \langle \tilde{\pi}_{ij} \rangle, T \rangle$$

$$\begin{tikzcd}
\tilde{M}_0 \arrow[r] \arrow[rr, bend left] & \tilde{M}_1 \arrow[rr, bend right] & \tilde{M}_2 \arrow[rr, bend left] & \tilde{M}_3 \arrow[rr, bend right] & \tilde{M}_4 & \cdots
\end{tikzcd} \quad \text{in} \quad \begin{matrix} b_0 \\ b_1 \end{matrix},$$

All finite stages of the construction take place in $\mathfrak{A}$. Thus, each initial segment $\tilde{\mathcal{Y}}|n$ lies in $\mathfrak{A}$, but the final iteration does not.

We will of course have:

$$T(i+1) = \begin{cases} 0 & \text{if } i = 0 \\ i-1 & \text{if not} \end{cases}$$

Let $\bar{\mathcal{I}} = \langle\langle N_i\rangle, \langle\nu_i\rangle, \langle\bar{\pi}_{ij}\rangle, T\rangle$ be the iteration of N by the same indices. By Lemma 1.4 we have: $On \cap N_i = \delta$.

Let $\mathcal{I} = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be the iteration of M. By Lemma 1.6 each M_i is transitive and $On \cap M_i = \gamma$.

Note that if $i \leq_T j$, then $\langle\tilde{M}_j, \tilde{\pi}_{ij}\rangle$ is the Σ_0-liftup of $\langle\tilde{M}_i, \pi_{ij}\rangle$. Hence by Claim 1 in the proof of Lemma 1.3, we have: $\gamma = On \cap wfc(\tilde{M}_i)$.

We note that by Lemma 1.3 each M_i is admissible. Since Lemma 1.3 holds in $\mathfrak{A}$, we also know that $\tilde{M}_i$ is admissible in $\mathfrak{A}$. We also have: $On \cap \tilde{M}_i = \tilde{\gamma}_i$ in $\mathfrak{A}$, where $\tilde{M} = J_{\tilde{\gamma}}(N)$ in $\mathfrak{A}$.

By induction on i we construct:

$\tilde{M}_i$, $\langle \nu_h \mid h < i \rangle$, $\langle \tilde{\pi}_{hi} \mid h \, T \, i \rangle$.

In preparation for the choice of ν_i we also construct $\kappa_i = \mathrm{crit}(E^{\tilde{M}_i}_{\nu_i})$, and points $\tau_i \in S^{\tilde{M}_i}_\delta$, $\tau_i' \in S^{\tilde{M}_{T(i+1)}}_\delta$.

We inductively verify:

(a) κ_i is strong wrt. τ_i in $\tilde{M}_i$

(b) Let $\bar{\tau}_i$ = that $\bar{\tau} \in S_{\kappa_i}$ s.t. $\bar{\tau} < \tau_i$ in $\tilde{M}_i$. Then κ_i is strong wrt. τ_i' and $\bar{\tau}_i < \tau_i'$ in $\tilde{M}_{T(i+1)}$

(c) If i is even, then $\tau_i \notin \gamma$.

(Note We cannot expect (c) to hold at odd i. In fact, we could sharpen our construction so that this is never so.)

Fix a sequence $\langle \delta_i \mid i < \omega \rangle$ which is monotone and cofinal in δ. We "diagonalize" N by requiring:

(d) $\kappa_i > \tilde{\pi}_{0i}(\delta_i)$

<u>Case 1</u> $i=0$. Set: $\tilde{M}_0 = \tilde{M}$. Pick any $\tau_0 \in S_\delta \setminus \gamma$ in $\tilde{M}$. Since $T(1)=0$, we set: $\tau_0' = \tau_0$. Pick $\varkappa_0 > \delta_0$ s.t. $\varkappa_0$ is strong w.r.t. τ_0.

(a)–(d) are trivially satisfied.

<u>Case 2</u> $i = j+1$, where j is even.

Then $\varkappa_j, \tau_j, \tau_j', \bar{\tau}_j$ satisfying (a)–(d) are given. Since $\tau_j \in S_\delta \setminus \gamma$, we can pick $\tau' \in S_\delta \setminus \gamma$ s.t. $\tau' < \tau_j$. We may also assume w.l.o.g. that τ' immediately succeeds a τ'' in S_δ. Then $\tau'' \in S_\delta \setminus \gamma$. Set: $\tau_{j+1}' = \tau'$.

Pick $\varkappa$ s.t.

- $\varkappa$ is τ_j strong in $\tilde{M}_j$
- $\varkappa > \lambda_h$ for $h < j$ and $\varkappa > \varkappa_j$
- $\tau' \in \operatorname{rng}(\sigma_{\hat{\tau}, \tau_j})$ where $\hat{\tau} \in S_\varkappa$ s.t. $\hat{\tau} \prec \tau_j$
- $\varkappa > \pi_{0, T(j+1)}(\delta_{j+1})$

Set: $\varkappa_{j+1} = \varkappa$, $\bar{\tau}_{j+1} = \sigma^{-1}_{\hat{\tau}, \tau_j}(\tau')$.

Then:

(1) $\bar{\tau}_{j+1} \in S_\kappa$ s.t. $\bar{\tau}_{j+1} \prec \tau'_{j+1}$ in $\tilde{M}_j$,

(2) κ_{j+1} is τ'_{j+1} - strong in $\tilde{M}_j$,

where $j = T(j+2)$.

Pick $\tilde{\tau}$ s.t.

$$\bar{\tau}_j \prec \tilde{\tau} \prec \tau_j \text{ and } \kappa < d_{\tilde{\tau}} < \delta \text{ in } \tilde{M}_j.$$

Pick ν_j s.t. E_{ν_j} witnesses the τ_j - - strength of κ_j wrt $\tilde{\tau}$. Set:

$$\pi = \pi_{T(j+1),\, j+1} : \tilde{M}_{T(j+1)} \xrightarrow[E_{\nu_j}]{} \tilde{M}_{j+1}.$$

By Lemma 1.9 we have:

(3) $\bar{\tau}_j \prec \tilde{\tau} \prec \pi(\bar{\tau}_j)$ in $\tilde{M}_j$

Note that $(J^E_{\nu_j})^{\tilde{M}_j} = (J^E_{\nu_j})^{\tilde{M}_{j+1}}$,

where $\tilde{\tau}^{+\tilde{M}_j} = \tilde{\tau}^{+\tilde{M}_{j+1}} < \lambda_j$. Hence the restriction of our coarse morass to $J^E_{\nu_j}$ is the same in $\tilde{M}_j$ and $\tilde{M}_{j+1}$.

In particular, (3) holds in $\tilde{M}_{j+1}$.

But since $\bar{\tau}_j \prec \tau'_j$ in $\tilde{M}_{T(j+1)}$, we have:

(4) $\bar{\tau}_j \prec \tilde{\tau} \prec \pi(\bar{\tau}_j) \prec \pi(\tau'_j)$ in $\tilde{M}_{j+1}$.

Let $\eta \in S_{\alpha_{\tilde{\tau}}}$ s.t. $\sigma_{\tilde{\tau},\tau_j}(\eta) = \tau'$ in $\tilde{M}_j$.

Set: $\tau_{j+1} =_{\mathrm{Df}} \sigma_{\tilde{\tau},\pi(\tau_j')}(\eta)$ in $\tilde{M}_{j+1}$.

Then:

(5) $\bar{\tau}_{j+1} < \eta < \tau_{j+1}$ in $\tilde{M}_{j+1}$,

Finally we show:

(6) κ_{j+1} is τ_{j+1}-strong in $\tilde{M}_{j+1}$.

proof.

κ_{j+1} is τ'-strong in $\tilde{M}_j$, hence in $Q^{\tilde{M}_j}_{\tau_j}$. But $\sigma_{\tilde{\tau}\tau_j}(\eta) = \tau_j'$. Hence:

κ_{j+1} is η-strong in $Q_{\tilde{\tau}}$.

But then κ_{j+1} is τ_{j+1}-strong in $Q^{\tilde{M}_{j+1}}_{\pi(\tau_j')}$, hence in $\tilde{M}_{j+1}$, since $\sigma_{\tilde{\tau},\pi(\tau_j')}(\eta) = \tau_{j+1}$.

QED (6)

Then (a) holds at $i = j+1$ by (6).

(b) holds by (2), (1), (5). (c) is vacuous at $i = j+1$. (d) holds, since:

$\kappa_{j+1} > \pi_{0,T(j+1)}(\delta_{j+1}) = \pi_{0,j+1}(\delta_{j+1})$,

since $\pi_{T(j+1),j+1} \upharpoonright \kappa_{j+1} = \mathrm{id}$.

This completes Case 2.

<u>Case 3</u> $i = j+2$, where j is even.

We have constructed $\kappa_{j+1}, \tau_{j+1}, \tau'_{j+1}, \bar{\tau}_{j+1}$ satisfying (a)–(d). By the construction in Case 2 we also know that τ'_{j+1} is an immediate successor of a τ'' in $S_\delta^{\tilde{M}_j}$, where $T(j+2) = j$. We set:

$\tau'_{j+2} = \tau''$. Then κ_{j+1} is τ'_{j+2}-strong in $\tilde{M}_j$, since it is τ'_{j+1}-strong.

Set: τ'_{j+2} = the immediate predecessor of τ_{j+1} in $S_\delta^{\tilde{M}_{j+1}}$. Then κ_{j+1} is τ'_{j+2}-strong in $\tilde{M}_{j+1}$, since it is τ_{j+1}-strong.

Pick κ s.t.

- $\kappa \geq \lambda_j$
- κ is τ_{j+1} strong in $\tilde{M}_{j+1}$
- $\kappa > \pi_{0,i+1}(\delta_{j+2})$

Set $\kappa_{j+2} = \kappa$. Let $\hat{\tau} \in S_\kappa^{\tilde{M}_{j+1}}$ s.t. $\hat{\tau} < \tau_{j+1}$. Then $\hat{\tau}$ immediately succeeds a $\bar{\tau}'$ in S_κ and

$\sigma_{\hat{\bar{\tau}},\tau_{j+1}}(\bar{\tau}') = \tau'_{j+2}$. We set:

$\bar{\tau}_{j+2} = \bar{\tau}'$. Then

(1) $\bar{\tau}_{j+2} \in S_\kappa$ and $\bar{\tau}_{j+2} < \tau'_{j+2}$ in $\tilde{M}_{j+1}$

(where $T(j+3) = j+1$)

(2) κ_{j+2} is τ'_{j+2}-strong in $\tilde{M}_{j+1}$

Pick $\tilde{\tau}$ s.t.

$\bar{\tau}_{j+1} < \tilde{\tau} < \tau_{j+1}$ and $\kappa < \alpha_{\tilde{\tau}} < \delta$.

in $\tilde{M}_{j+1}$.

Pick ν_{j+1} s.t. $E_{\nu_{j+1}}$ witnesses the

τ_{j+1}-strength of κ wrt. $\tilde{\tau}$ in $\tilde{M}_{j+1}$.

Set: $\pi = \pi_{j,j+2} : \tilde{M}_j \xrightarrow[E_{\nu_{j+1}}]{} \tilde{M}_{j+1}$.

As before, Lemma 1.4 gives:

(3) $\bar{\tau}_{j+1} < \tilde{\tau} < \pi(\bar{\tau}_{j+1})$ in $\tilde{M}_{j+2}$.

We again note that

$$(J^E_{\nu_{j+1}})^{\tilde{M}_{j+1}} = (J^E_{\nu_{j+1}})^{\tilde{M}_{j+2}},$$

where $\tilde{\tau}^{+\tilde{M}_{j+1}} = \tilde{\tau}^{+M_{j+2}} < \lambda_{j+1}$.

Hence the restriction of the coarse

mouse to $J^E_{\nu_{j+1}}$ is the same in

$\tilde{M}_{j+1}$ and $\tilde{M}_{j+2}$. In particular,

(3) holds in $\tilde{M}_{j+2}$. Hence, since $\bar{\tau}_{j+1} < \tau'_{j+1}$ in $\tilde{M}_j$.

(4) $\bar{\tau}_{j+1} < \tilde{\tau} < \pi(\bar{\tau}_{j+1}) < \pi(\tau'_{j+1})$ in $\tilde{M}_{j+2}$.

Let η be the immediate predecessor of $\tilde{\tau}$ in $S_{\alpha_{\tilde{\tau}}}$. Set $\tau_{j+2} = \eta_f\, \sigma_{\tilde{\tau}, \pi(\tau'_{j+1})}(\eta)$.

Then:

(5) $\bar{\tau}_{j+2} < \eta < \tau_{j+2}$, in $\tilde{M}_{j+2}$

By a virtual repetition of our previous proof we have:

(6) κ_{j+2} is τ_{j+2} - strong in $\tilde{M}_{j+2}$.

Note that $\tau_{j+2} = \pi(\tau'')$, where τ'' was the immediate predecessor of $\tau' = \tau'_{j+1}$ in $S_\delta^{\tilde{M}_j}$. Hence:

(7) $\tau_{j+2} \in S_\delta \setminus \gamma$ in $\tilde{M}_{j+2}$, since $\tau'' \in S_\delta \setminus \gamma$ in $\tilde{M}_j$.

(a), (b), and (d) then follow exactly as before. (c) follows by (7).

. . . . .

This completes the construction.

We now consider the iteration of N by the same indices:

$$\bar{\mathcal{J}} = \langle \langle N_i \rangle, \langle \nu_i \rangle, \langle \bar{\pi}_{ij} \rangle, T \rangle.$$

Let b_0, b_1 be the two infinite branches.

<u>Claim</u> $N_{b_h} = \bigcup_{i<\omega} J^{E^{N_i}}_{\kappa_i}$ $(h = 0,1)$

proof.

Let $x \in N_{b_h}$, $x = \bar{\pi}_{i b_h}(x')$. Then $x' \in J^{E^{N_0}}_{\bar{\pi}_{0i}(\delta_j)}$ for a $j \geq i$ s.t. $j \in b_h$. Hence:

$$\bar{\pi}_{ij}(x') \in J^{E^{N_j}}_{\bar{\pi}_{0j}(\delta_j)} \subset J^{E^{N_j}}_{\kappa_j}.$$

But $\bar{\pi}_{j b_h} \upharpoonright J^{E^{N_j}}_{\kappa_j} = \mathrm{id}$. Hence:

$$x = \bar{\pi}_{ij}(x') \in J^{E^{N_j}}_{\kappa_j}.$$

QED (Lemma 1)

Lemma 2 Let N be royal. Let $\pi : N \to_{\Sigma_0} N'$ cofinally, where N' is transitive.
Then N' is royal.

prf
Let $\langle \mathfrak{A}, \tilde{\pi} \rangle$ be the Σ_0-liftup of $\langle M, \pi \rangle$.
Case 1 $\mathfrak{A}$ is well founded (hence transitive)
Set $M' = \mathfrak{A}$. Clearly $\delta' = \tilde{\pi}(\delta)$ is Woodin in M'. M' is admissible by Lemma 1.3. $\rho^1_{M'} = \delta'$ by Lemma 1.1. As $M' = J_{\gamma'}(N')$, then there is no ν s.t. $0 < \nu < \gamma'$ and $J_\nu(N')$ is admissible, since otherwise there would be a ν s.t. $0 < \nu < \gamma$ and $J_\nu(N)$ is admissible. QED (Case 1)

Case 2 Case 1 fails.

Let $\gamma' = On \cap wfc(\mathfrak{A})$. We claim that $M' = J_{\gamma'}(N')$ verifies the royalty of N'.

Obviously δ' is Woodin in M' and δ' is admissible. Moreover δ' is least s.t. $J_{\gamma'}(N')$ is admissible by the previous argument. It then suffices to show:

Claim $\rho^1_{M'} \geq \delta'$.

proof.
Suppose not. Let $\rho^1_{M'} = \tau < \delta'$.
Then there is $p \in M'$ s.t.

$$h_{M'}(\tau \cup \{p\}) \supset \delta'.$$

But then for $\eta \in On_{\mathfrak{A}} \setminus \gamma'$ we have in $\mathfrak{A}$:

$$h_{J_\eta(N')}(\tau \cup \{p\}) \supset \delta',$$

violating the regularity of N' in $\mathfrak{A}$. Contr! QED (Lemma 2)

<u>Lemma 3</u> Let N be a royal mouse. Assume that there is no inner model with a Woodin cardinal. Then N has an iterate N' s.t. the crown M' of N' is a sound mouse.

proof

Coiterate N against K^c, getting N', K' s.t. N' is a non truncating iterate of N' and $N' = K' | \eta$ for some η. Note, however, that if a truncation occurred on the main branch on the K^c-side, then there is κ s.t. κ is inaccessible in K' and $\rho^{\omega}_{K'} \leq \kappa$.

<u>Case 1</u> There is $\nu \geq \delta'$ (where $N = J^E_{\delta'}$) s.t. $E_\nu \neq \emptyset$ in K'.

Then $\nu > \delta'$, since $N = J^E_\delta$ is a ZFC model and J^E_ν is not. But then $J_\nu(N')$ is a mouse, where $J_\nu(N') \models ZFC^-$, Hence M' is a proper segment of $J_\nu(N)$.

<u>Case 2</u> Case 1 fails.

Then $K' = J_\eta(N')$ for an η s.t

$\rho^{\omega}_{J_\eta(N')} \leq \delta'$. But then M' is a segment of K' since $\rho^{\omega}_{J_\nu(N')} > \delta'$ for $0 < \nu < \gamma'$; where $M' = J_{\gamma'}(N')$. QED (Lemma 3)

__Def__ By a __crown mouse__ we mean a mouse $M = J_\gamma(N)$ s.t. N is royal and M is the crown of N.

By §4.2 we know that if there is a mouse has an iteration with two distinct well founded cofinal branches, then there is a royal mouse (since we can certainly assume one of these branches b to be the one given by the iteration strategy; thus $N = \bigcup_{i < lh(\mathcal{Y})} J^{E^{M_i}}_{\kappa_i}$ is a segment of M_b and hence a mouse). But then, assuming no inner model with a Woodin, there is a crown mouse. But then there is a crown mouse $M = J_\gamma(N)$ which is __minimal__ in the sense that M is sound and no proper segment of M is a crown mouse. It is easily seen that $\omega\rho^2_M = \omega$ (and, in fact, that $\emptyset$ is the standard parameter of M).

We have thus found a specific mouse M s.t. M exists iff there is a mouse N which has an iteration with distinct cofinal well founded branches. (This does not mean that there is in any reasonable sense a minimal such N. Even if we fix the minimal height of such N, there will be 2^ω many such N of that height. However, any two of them will coiterate to the same thing.)

We now use our minimal crown mouse $M = J_\gamma(N)$ to show that §1 Theorem 1 was best possible. This follows from:

Lemma 4 Let $M = J_\gamma(N)$ be the minimal crown mouse, Then there exits an $\bar{N}$ and an iteration $\mathcal{J}$ with two distinct cofinal branches b_0, b_1 s.t. $\bar{N}_{b_0} = \bar{N}_{b_1} = N = \bigcup_{i < lh(\mathcal{J})} J^{E^{\bar{N}_i}}_{\kappa_i}$.

§ 1 Theorem 1 is then best possible since $\delta = On_N$ is not Woodin in in $J_{\gamma+1}(N)$.

To prove Lemma 4, we let $\mathcal{L}$ be the language on M with:

Predicate $\in$

Constants $\underline{x}$ $(x \in M)$, $\dot{N}$, $\dot{\mathcal{J}}$

Axioms ZFC^-, $\bigwedge u\,(u \in \underline{x} \leftrightarrow \bigvee_{z \in x} u = \underline{z})$ for $x \in M$, $\dot{N}$ is a premouse, $\dot{\mathcal{J}}$ is an alternating chain truncation free iteration of $\dot{N}$, $\dot{N}_{b_0} = \dot{N}_{b_1} = \underline{N} = \bigcup_{i < \omega} J^{\dot{N}_i}_{\kappa_i}$.

It suffices to show that $\mathcal{L}$ is consistent, since it then has a solid model $\mathfrak{A}$. If we set:

$\bar{N} = \mathring{N}^{\alpha}$, $y = \mathring{y}^{\alpha}$, then Lemma 4 is satisfied.

We prove the consistency of $\mathcal{L}$.

Let $\bar{\mathcal{J}} = \langle \langle N_i \rangle, \langle \nu_i \rangle, \langle \bar{\pi}_{ij} \rangle, T \rangle$ be the alternating chain constructed in the proof of Theorem 1. We recall that there is an iteration $\mathcal{J} = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \bar{\pi}_{ij} \rangle, T \rangle$ of M with the same indices by Σ_0-ultrapowers. In other words $\langle M_j, \pi_{ij} \rangle$ is the Σ_0-liftup of $\langle M_i, \bar{\pi}_{ij} \rangle$.

Since $\bar{\pi}_{ij}$ is Σ_2-preserving and $\omega\rho^2_{M_i} = \omega$, it follows that the maps are Σ^*-preserving and $\mathcal{J}$ is a $*$-iteration of M. Since M is a mouse, one of the branches will be well founded (in fact it is the branch b_0 of even integers). But then, letting $M' = M_{b_0}$, $\pi' = \pi_{0, b_0}$, we have: $\pi : M \xrightarrow[\Sigma_2]{} M'$. Let $\mathcal{L}'$ be the corresponding language on M' (with $\underline{N}'$ in place of $\underline{N}$, where $M' = J_{\gamma'}(N')$). $\mathcal{L}'$ is obviously consistent, since:

$\langle H_{\omega_1}, N', \gamma \rangle$ is a model.

But this is a $\Pi_1(M')$ statement in the parameter N'. Hence the same $\Pi_1(M)$ statement holds of N. Hence $\mathcal{L}$ is consistent.

QED (Lemma 4)

<u>Note</u> Lemma 3 also holds if e.g. N is a 1-small ω_1+1-iterable, countable royal mouse and V is closed under $\#$, since then, if the premise of Lemma 3 is false, then N can be coiterated with $M_1^\#$ to get the same result.