§2 <u>Namba*-Forcing</u>

Namba forcing is the set $\mathbb{N}$ of trees $T \subset (\omega_2)^{<\omega}$ s.t.

- $t \in T \rightarrow t|n \in T$ for $n < |t|$.
  (We write $|t| = \mathrm{dom}(t)$ for $t \in (\omega_2)^{<\omega}$)
- If $t \in T$, then there are ω_2 many $s \in T$ s.t. $t \subset s$.
- Each $t \in \mathbb{N}$ is a finite ascending sequence of ordinals in ω_2.

$\mathbb{N}$ is partially ordered by inclusion:

$T \leq T'$ in $\mathbb{N}$ $\leftrightarrow$: $T \subset T'$.

For $s \in T \in \mathbb{N}$ we set:

$T_{(s)} = \{t \in T \mid t \subset s \vee s \subset t\}$.

The <u>stem</u> of T is the largest s s.t. $T = T_{(s)}$.

Namba' forcing is the set $\mathbb{N}' \subset \mathbb{N}$ of trees T s.t. whenever $t \in T$ and $t \supset \mathrm{stem}(T)$, then t has ω_2 immediate successors in T. $\mathbb{N}'$ has been studied at length in [PIF] and elsewhere. In [DSC] we showed that $\mathbb{N}'$ is ω_1-subproper and dee-subcomplete.

Namba*-forcing is the set of $T \in \mathbb{N}'$ s.t. $\{\alpha \mid t^\frown\langle\alpha\rangle \in T\}$ is stationary in ω_2 whenever $t \supset \text{stem}(T)$, $t \in T$.

(Note It seems likely that $\mathbb{N}^*$ has also been treated in the literature, but at present we don't know where. We would be grateful for any references.)

We say that $b \in (\omega_2)^\omega$ is a branch through T iff $b \upharpoonright n \in T$ for $n < \omega$. If G is $\mathbb{N}^*$-generic, then $b = b_G =: \bigcup \bigcap G$ is a branch through $(\omega_2)^{<\omega}$. Conversely, G is recoverable from b by:

$$G = G_b =: \{T \in \mathbb{N}^* \mid b \text{ is a branch through } T\},$$

We call $b = \langle \gamma_i \mid i < \omega \rangle$ a $\mathbb{N}^*$-generic sequence iff G_b is $\mathbb{N}^*$-generic.

(Similarly for $\mathbb{N}, \mathbb{N}'$.)

$\mathbb{N}^*$ then has the following property:

<u>Lemma 1</u> Let b be $\mathbb{N}^*$-generic over V. Let $C \in V$ be club in ω_2. Then

(*) $\bigvee n \bigwedge i \geq n \; \gamma_i \in C$,

where $b = \langle \gamma_i \mid i < \omega \rangle$.

proof

Let $T \in \mathbb{N}^*$, ~~let~~ $n = \text{stem}(T)$. Set:

$T' = \{t \in T \mid \bigwedge i \geq n \; t(i) \in C\}$. Then

$T' \in \mathbb{N}^*$, item$(T') =$ item(T) and $T' \leq T$ in $\mathbb{N}^*$. But every branch through T' satisfies $(*)$ QED

An entirely similar proof shows:

Lemma 2 Let b be $\mathbb{N}'$-generic over V, Let $C \in V$ be club in ω_1 and let:

$$C(\gamma) = \min\{\beta > \gamma \mid \beta \in C\} \text{ for } \beta < \omega_2.$$

Then:

$$\bigvee n \bigwedge i \geq n \; C(\gamma_{i+1}) < \gamma_i, \text{ where } b = \langle \gamma_i \mid i < \omega \rangle.$$

We also mention that the properties of being $\mathbb{N}$-generic, $\mathbb{N}'$-generic, or $\mathbb{N}^*$-generic are mutually exclusive.

$\mathbb{N}^*$ satisfies the following weak amalgamation lemma:

Lemma 3 Let $\langle T_n \mid n < \omega \rangle$ be s.t. for $n < \omega$:

- $T_{n+1} \leq T_n$ in $\mathbb{N}^*$
- item$(T_n) =$ item(T_{n+1})
- $T_n | n = T_{n+1} | n+1$

(where $T|n =: \{s \in T \mid |s| \leq n\}$).

Then $T \in \mathbb{N}^*$ where:

$$T = \bigcap_n T_n = \bigcup_n T_n | n.$$

QED (Lemma 3)

It is easily seen that Lemma 3 also holds with $\mathbb{N}'$ in place of $\mathbb{N}^*$.

The following specialization lemma will be used to show that $\mathbb{N}^*$ adds no reals:

Lemma 4 (assume CH.) Let $T \in \mathbb{N}^*$, $f : T \to \omega_1$. There is $T' \leq T$ in $\mathbb{N}^*$ s.t.

$$|s| = |s'| \to f(s) = f(s') \quad \text{for all } s, s' \in T'.$$

(Hence there is $g : \omega \to \omega_1$ s.t. $f(s) = g(|s|)$ for $s \in T'$.)

proof.

For each $g : \omega \to \omega_1$ let G_g be the following game:

At stage i, I picks a club $C_i \subset \omega_2$ s.t. $C_i \subset \bigcap_{j<i} C_j$. If $i \leq m = |\text{stem}(T)|$, then $C_i = \omega_2$. II then picks δ_i s.t. $\langle \delta_0, \ldots, \delta_i \rangle \in T$, $\delta_i \in C_i$ and $f(\delta_0, \ldots, \delta_i) = g(i)$ if possible. If II has no move, then I wins. If I does not win at any finite stage, then II wins.

Claim II has a winning strategy for some G_g.

Suppose not. Then I has a winning strategy S_g for every g. $S_g(i)$ is then a club set in ω_2 and we set:

$$S(i) = \bigcap_{g : \omega \to \omega_1} S_g(i).$$

Then S is a strategy which wins

every game G_g. Now pick a branch $\langle \gamma_i \mid i<\omega \rangle$ through T s.t. $\gamma_i \in S(i)$ for $i<\omega$. Then $\langle \gamma_i \mid i<\omega \rangle$ is a play which defeats S. Contradiction! QED (Claim)

Now let II have a strategy S for G_g. Let T' be the set of $\langle \nu_0, \ldots, \nu_{n-1} \rangle \in T$ s.t. for some sequence $c_0, \ldots, c_{n-1}$ of possible plays for I we have: $\nu_i = S(c \upharpoonright i)$ $(i<n)$. It is easily seen that $T' \in \mathbb{N}^*$ and stem $(T') =$ stem (T).

QED (Lemma 4)

(Note Lemma 3 goes through for $\mathbb{N}'$ in place of $\mathbb{N}^*$. To prove it we alter the proof to have I pick a $\gamma_i < \omega_2$. The contradictory "universal" strategy S for I is then defined by:

$$S(i) = \sup_{g:\omega\to\omega_2} S_g(i) .$$

Everything goes through as before.)

Corollary 5 Assume CH. Then $\mathbb{N}^*$ adds no reals.

proof

Let $\Vdash \dot{f} : \check{\omega} \longrightarrow \check{\omega}_1$.

Claim $\dot{f}^G \in V$ for all $\mathbb{N}^*$-generic G.

proof:
Let $T \in \mathbb{N}^*$. It suffices to show:

Subclaim There is $T' \leq T$ s.t. $T' \Vdash \mathring{f} = \check{f}$ for some $f \in V$.

proof

We first apply the amalgamation theorem, forming T_n $(n < \omega)$ s.t.

- $\text{stem}(T_n) = \text{stem}(T)$
- $T_n | n = T_{n+1} | n$
- $T_{n+1} \leq T_n$.

Simultaneous we assign

$$g_n : (T_n | n) \longrightarrow \omega_1^{<\omega}$$

as follows:

Let $s = \text{stem}(T)$. For $n \leq |s|$, set:

$T_n = T$, $g_n(t) = \emptyset$.

Now let $m \geq |s|$. We form T_{m+1}, g_{m+1}.

For each $t \in T_m$ s.t. $|t| = m+1$

look at $T_m(t)$ and pick a maximal

$p \leq m$ s.t. there exist $T'_t \leq T_m(t)$

and $\langle \delta_0, \ldots, \delta_{p-1} \rangle$ s.t.

- $\text{stem}(T'_t) = t = \text{stem}\, T_m(t)$
- $T'_t \Vdash \mathring{f}(i) = \check{\delta}_i$ for $i < p$,

(We could have: $p = 0$, $T'_t = T_m(t)$)

Set: $T_{m+1} = \bigcup_{\substack{t \in T_m \\ |t| = m+1}} T_t'$.

$g_{m+1}(t) = \langle \gamma_0, \ldots, \gamma_{p-1} \rangle$

for $t \in T_m$, $|t| = m+1$,

For $t \in T_m$, $|t| \leq m$ we

let $g_{m+1}(t) = g_m(t)$.

Set $T'' = \bigcap_n T_m = \bigcup_n (T_n | n)$

Define $g : T'' \to \omega_1^{<\omega}$ by:

$g = \bigcup_n g_n$.

Then $T'' \in IN^*$, $T'' \leq T$, stem(T'') = stem(T).

We now apply specialization. Set $T' \leq T''$ s.t. stem(T') = stem(T) and $g(t) = g(t')$ for $|t| = |t'|$, $t, t' \in T'$.

Then $g(t) = \tilde{f}(|t|)$ for an $\tilde{f} : \omega \to (\omega_1)^{<\omega}$.

It is easily seen that $t \subset s \to g(t) \subset g(s)$. Hence $\tilde{f}(i) \subset \tilde{f}(j)$ for $i \leq j < \omega$.

Set: $f = \bigcup_i \tilde{f}(i)$. Then $\mathrm{dom}(f) \leq \omega$ and $T' \Vdash \dot{f}(i) = \check{f}(i)$ for $i \in \mathrm{dom}(f)$. Thus it suffices to show:

<u>Claim</u> $\mathrm{dom}(f) = \omega$.

Suppose not. Then for some n we have:

$i \geq n \to \tilde{f}(i) = f$.

Pick $t \in T'$, $|t| \geq n$. Let $d = \mathrm{dom}(f) < \omega$

Pick $T^* \leq T'$ in $\mathbb{N}^*$ s.t.

$T^* \Vdash \mathring{f}(\check{d}) = \check{\gamma}$ for some γ.

We assume w.l.o.g. that $|s| \geq m \geq d$ where $s = \mathrm{stem}\, T^*$. Then $s \in T''$. Hence $s \in T_{|s|-1}$ and there ~~is~~ $T^* \leq T_{|s|-1}(s)$ s.t. $T^* \Vdash \mathring{f}(\check{d}) = \check{\gamma}$. By our construction we have $g(s) = f$. But the existence of T^* shows that $d = \mathrm{dom}(f) \in \mathrm{dom}(g(s))$.

Contradiction!

QED (Lemma 5).

(<u>Note</u> Exactly the same proof shows that $\mathbb{N}'$ adds no reals.)