§4 An $\mathcal{L}$-Forcing Equivalent to $\mathbb{N}^*$

From now on assume CH. We apply the theory in §3, taking $\beta = \omega_2$, $N = H_{(2^{\omega_1})^+}$, and $M = L^A_{\omega_2} = \langle L_{\omega_2}[A], \in, A \rangle$, where $A \subset \omega_2$ is s.t. $L_{\omega_1}[A] = H_{\omega_1}$ and ω_1 is the largest cardinal in $L_{\omega_2}[A]$. $\mathcal{L}$ is the infinitary language on N which, in addition to the usual predicates, constants and basic axioms, has the further axioms:

- $\dot{B} = \langle \dot{\gamma}_i \mid i < \underline{\omega} \rangle$ is monotone and cofinal in $\underline{\omega_2}$
- $\bigvee m \bigwedge n \geq m \;\; \dot{\gamma}_n \in \underline{C}$, for every club $C \subset \omega_1$
- $\operatorname{rng}(\dot{\pi}_i, \underline{\omega_1}) =$ the smallest $X \prec \underline{M}$ s.t. $\dot{\alpha}_0 \cup \{\dot{\alpha}_j \mid j < i\} \cup \{\dot{\gamma}_m \mid m < \underline{\omega}\} \subset X$ for $i \leq \omega_1$.

Lemma 1 $\mathcal{L}$ is consistent.

proof.
Let $\langle \gamma_i \mid i < \omega \rangle$ be $\mathbb{N}^*$-generic over V. Set: $B = \langle \gamma_i \mid i < \omega \rangle$. Define M_i, π_i as follows: Define $\langle \alpha_i \mid i \leq \omega_1 \rangle$, $\langle X_i \mid i \leq \omega_1 \rangle$ by:

$X_i =:$ the smallest $X \prec M$ s.t.

$$\{\alpha_j \mid j < i\} \cup \{\gamma_h \mid h < \omega\} \subset X$$

$\alpha_i =: \omega_1 \cap X$.

Let $\pi_{i,\omega_1} : M_i \xrightarrow{\sim} M | X_i$, where M_i is transitive.

Set: $\pi_{ij} = \pi_{j\omega_1}^{-1} \cdot \pi_{i\omega_1}$ for $i \leq j < \omega_1$

Then $\langle H^V_{\omega_4}[B], \langle M_i | i \leq \omega_1 \rangle, \langle \pi_{ij} | i \leq j \leq \omega_1 \rangle, B \rangle$ models $\mathcal{L}$. QED (Lemma 1)

(Note There is a more elementary proof of Lemma 1 which makes no mention of $\mathbb{N}^*$.)

Set: $\mathbb{P} =: \mathbb{P}_{\mathcal{L}}$. If G is $\mathbb{P}$-generic, then it adds $M^G = \langle M_i^G | i \leq \omega_1 \rangle$, $\pi^G = \langle \pi_{ij}^G | i \leq j \leq \omega_1 \rangle$ and $B^G = \langle \gamma_i^G | i < \omega \rangle$ as defined in §3.

We call $B = \langle \gamma_i | i < \omega \rangle$ a $\mathbb{P}$-generic sequence iff $B = B^G$ for a $\mathbb{P}$-generic G. We shall show:

Theorem $B = \langle \gamma_i | i < \omega \rangle$ is $\mathbb{P}$-generic iff it is $\mathbb{N}^*$-generic.

The proof stretches over many lemmata. We shall, in fact, define a forcing $\mathbb{N}^{**}$ in which $\mathbb{N}^*$ lies dense. We then prove the Theorem with $\mathbb{N}^{**}$ in place of $\mathbb{N}^*$

We first note that we may have to deal with structures $\langle M, A \rangle$, where $A \subset M$ lies in V but $\langle M, A \rangle$ is not amenable.

a number of familiar arguments do not work. However, we do have:

<u>Lemma 2</u> Let $\langle a, \bar{a} \rangle \in F^p$, $\bar{M} = M^p_{|p|}$.

Assume (in any extension of V) that $\pi : \langle \bar{M}, \bar{a} \rangle \xrightarrow[\Sigma_0]{} \langle M, a \rangle$ cofinally.

Then $\pi : \langle \bar{M}, \bar{a} \rangle \prec \langle M, a \rangle$.

proof.

We first note that for any Σ_n (Π_n) formula φ there is a canonical Σ_n (Π_n) formula $\bar{\varphi}$ s.t.

(1) $\bigvee x \in u\, \varphi(x, \vec{y}) \longleftrightarrow \bar{\varphi}(u, \vec{y})$.

holds in $\langle M, a \rangle$. This follows by iterated application of:

(2) $\bigwedge x \in u \bigvee z\, \psi(x, z, \vec{y}) \longleftrightarrow$

$\longleftrightarrow \bigvee w \bigwedge x \in u \bigvee z \in w\; \psi(x, z, \vec{y})$

and:

(3) $\bigvee x \in u \bigvee z\; \psi(x, z, \vec{y}) \longleftrightarrow$

$\longleftrightarrow \bigvee w \bigvee x \in w \bigvee z \in w\; (z \in u \wedge \psi(x, z, \vec{y}))$

(3) is trivial, as is the direction ($\longleftarrow$) of (2)

To prove ($\rightarrow$), note that, by the regularity of ω_1, if the premise holds, then

$\bigwedge x \in u . \bigvee z \in L_\gamma[A]\; \psi(x, z, \vec{y})$

for a $\gamma < \omega_1$. QED(1)

But then:

(4) (1) holds uniformly in $\langle \bar{M}, \bar{a} \rangle$.

proof.

Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}(p)$.

Then $\pi^{\mathfrak{A}}_{|p|, \omega_1} : \langle \bar{M}, \bar{a} \rangle \prec \langle M, a \rangle$. QED

We now prove by induction on φ that

if φ is Σ_n, then:

(5) $\langle M, a \rangle \models \varphi[\pi(\vec{x})] \longleftrightarrow \langle \bar{M}, \bar{a} \rangle \models \varphi[\vec{x}]$.

We proceed by induction on n. The case $n = 0$ is trivial. Now let $n = m+1$, $\varphi = \bigvee z \psi(z, \vec{x})$, where ψ is Π_m.

It suffices to show:

<u>Claim</u> $\langle M, a \rangle \models \varphi[\pi(\vec{x})] \rightarrow \langle \bar{M}, \bar{a} \rangle \models \varphi[\vec{x}]$

proof.

Assume $\langle M, a \rangle \models \varphi[\pi(\vec{x})]$. Pick $\gamma < \beta \equiv \beta_{|p|}$ big enough that:

$\langle M, a \rangle \models \bigvee z \in \pi(u)\, \psi[z, \pi(\vec{x})]$

where $u = L^{\bar{a}}_{\gamma}$. Then by (1):

$\langle M, a \rangle \models \bar{\psi}[\pi(u), \pi(\vec{x})]$

where $\bar{\psi}$ is Π_m. Hence:

$\langle \bar{M}, \bar{a} \rangle \models \bar{\psi}[u, \vec{x}]$

Hence $\langle \bar{M}, \bar{a} \rangle \models \bigvee z \in u\, \psi[z, \vec{x}]$

QED (Lemma 2)

We now define:

Def $\mathbb{N}^{**} =:$ the set of $T \in (\omega_2)^{<\omega}$ s.t. for every $t \in T$ there is $T' \leq T$ s.t. $t \subset \mathrm{stem}(T')$ and $T' \in \mathbb{N}^*$.

Since $\mathbb{N}^*$ is dense in $\mathbb{N}^{**}$ we have:

$$BA(\mathbb{N}^*) \simeq BA(\mathbb{N}^{**}).$$

Def Let $p \in \mathbb{P}$.

$$T_p =: \{ s \in (\omega_2)^{<\omega} \mid \mathrm{con}(\mathcal{L} + \varphi_p + \bigwedge_{i < |s|} x_i = \gamma_i) \}$$

Lemma 3 $T_p \in \mathbb{N}^{**}$

proof

Let $s \in T_p$. For each $n \geq |s|$ define a game G_n as follows. In the i-th move:

I plays C_i where $C_i \subset \omega_2$ is club and $C_i = \omega_2$ if $i < n$.

II plays $\gamma_i \in C_i$ s.t. $\langle \gamma_0, \ldots, \gamma_i \rangle \in T_p$ and $\gamma_i = s(i)$ if $i < |s|$.

I then wins at i iff II has no play.

II wins iff I does not win at any i.

Claim II has a winning strategy for some game G_n.

proof

Suppose not. Then I has a winning strategy S^n for each $n \in [|s|, \omega)$.

Set $C = \{\lambda < \omega_2 \mid \bigwedge n < \omega \bigwedge h \in (\lambda)^{<\omega}\ \lambda \in S^n(h)\}$.

Then C is club in ω_2.

Hence there are $p' \leq p$, $m < \omega$ s.t.

$$p' \Vdash \bigwedge i \geq m\ \dot{\gamma}_i \in \underset{\sim}{C} \wedge \bigwedge i < |s|\ \dot{\gamma}_i = \underset{\sim}{s(i)}.$$

Successively choose γ_i $(i < \omega)$ s.t. $\mathrm{con}(\mathcal{L}(p') + \bigwedge\limits_{h < i} \dot{\gamma}_h = \underset{\sim}{\gamma_h})$. Then $\langle \gamma_i \mid i < \omega \rangle$ is a play by II in G_m which defeats S^m. QED (Claim)

Now let II have a winning strategy S for G_m. Let T' be the result of applying S to all possible plays of I. Then $T' \in \mathbb{N}^*$, $s \subset \mathrm{stem}(T')$, $m = |\mathrm{stem}(T')|$ and $T' \leq T_p$. QED (Lemma 3)

<u>Lemma 4</u> Let $\langle \bar\gamma_0, \dots, \bar\gamma_i \rangle = b_p \upharpoonright i+1$. Let $\langle \gamma_0, \dots, \gamma_i \rangle \in T_p$.
There is a unique $\pi^i : J^{A^p}_{\bar\gamma_i} \prec J^A_{\gamma_i}$ defined by:
Let f be M-least s.t. $f : \omega_1 \xrightarrow{\text{onto}} \gamma_i$
Let $\bar f$ be $M^p_{|p|}$-least s.t. $\bar f : \alpha^p_i \xrightarrow{\text{onto}} \bar\gamma_i$
Then $\pi^i(\bar f(\xi)) = f(\xi)$ for $\xi < \alpha_i$.

Moreover:

- $\pi^h = \pi^i \upharpoonright J^{A_p}_{\bar\gamma_h}$ for $h < i$ $\quad (M^p_{|p|} = J^{A^p}_{\beta_p})$
- If $\langle a, \bar a \rangle \in F^p$, then
$\pi^i : \langle J^{A^p}_{\bar\gamma_i}, \bar a \cap J^{A^p}_{\bar\gamma_i} \rangle \prec \langle J^A_{\gamma_i}, a \cap J^A_{\gamma_i} \rangle$.

proof.
It is clear that π^i, if it exists, is uniquely defined by: $\pi^i \circ \bar f = f$. To show existence let $\mathfrak{A}$ be a grounded model of $\mathcal{L} + \varphi_p$ s.t. $\pi^{\mathfrak{A}}_{i,\omega_1}(\bar\gamma_h) = \gamma_h$ for $h \le i$. Then $\pi^{\mathfrak{A}}_{i,\omega}(\bar f) = f$.
Hence $\pi^i = \pi^{\mathfrak{A}}_{i,\omega_1} \upharpoonright J^{A^p}_{\bar\gamma_i}$ has all of the above properties. QED (Lemma 4)

<u>Lemma 5.</u> Let $T' \le T_p$ in $\mathbb{N}^{**}$. There is $q \le p$ in $\mathbb{P}$ s.t. $q \Vdash (\langle \gamma_i \mid i < \omega \rangle$ is a branch in $T')$.

proof.
Let G be $\mathbb{N}^{**}$-generic, $T' \in G$. Let $\langle \gamma_i \mid i < \omega \rangle$ be the generic sequence given by G. Since $\langle \gamma_i \mid i < \omega \rangle$ is a

branch in $T' \subset T_p$, there is for each $i<\omega$ the unique $\pi^i : J^{A^p}_{\bar{\gamma}_i} \prec J^A_{\gamma_i}$ defined as in Lemma 3. Then:

(1) $\pi^i \subset \pi^j$ for $i \leq j < \omega$.

proof.

Let f_i be M-least s.t. $f : \omega_1 \xrightarrow{\text{onto}} \gamma_i$. Let $\bar{f}_i$ be M-least s.t. $f : \alpha_i^p \xrightarrow{\text{onto}} \bar{\gamma}_i$. Clearly, if $\bar{f}_i \in L^{A^p}_{\bar{\gamma}_j}$, then $f_i \in L^A_{\gamma_j}$, since by the definition of T_p there is (in a sufficient generic collapse) a grounded model $\mathfrak{A}$ of $\mathcal{L}(p) + \bigwedge_{n \leq j} \pi^{\mathfrak{A}}_{j\omega_1}(\bar{\gamma}_n) = \gamma_n$. But then, letting $\pi = \pi^{\mathfrak{A}}_{j\omega_1}$, we have $\pi(\bar{f}_i) = f_i$. Hence $\pi^i = \pi \restriction L^{A^p}_{\bar{\gamma}_i}$ and $\pi^j = \pi \restriction L^A_{\gamma_j} \supset \pi^i$. QED (1)

Set: $\pi = \bigcup_i \pi^i$. Then $\pi : \langle \bar{M}, \bar{a} \rangle \xrightarrow[\Sigma_0]{} \langle M, a \rangle$ whenever $\bar{M} = M^p_{|p|}$ and $\langle a, \bar{a} \rangle \in F^p$. Hence $\pi : \langle M, \bar{a} \rangle \prec \langle M, a \rangle$ by Lemma 2.
QED (1)

Let $\alpha = \alpha_p$. Clearly $\operatorname{rng}(\pi)$ is the smallest $X \prec M$ s.t. $\alpha \cup \{\gamma_i \mid i < \omega\} \subset X$.

(We set: $M^p_i = L^{A^p_i}_{\alpha^p_i}$, $\alpha_p = \alpha^p_{|p|}$, $A^p = A^p_{|p|}$)

Define $\langle \alpha_i \mid i \leq \omega_1 \rangle$, $\langle X_i \mid i \leq \omega_1 \rangle$ by:

$X_i =:$ the smallest $X \prec M$ s.t.
$\alpha \cup \{\alpha_j \mid j < i\} \cup \{\gamma_n \mid n < \omega\} \subset X$

$\alpha_i =: \omega_1 \cap X_i$.

Clearly $X_i = \operatorname{rng} \pi \circ \pi^p_{i,|p|}$ for $i \leq |p|$.

Define $\langle M_i \mid i \leq \omega_1 \rangle$, $\langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle$ by: $\pi_{i,\omega_1} : M_i \overset{\sim}{\longleftrightarrow} \langle X_i, \in, A \cap X_i \rangle$ when M_i is transitive

$\pi_{ij} = \pi^{-1}_{j\omega_1} \cdot \pi_{i\omega_1}$ for $i \leq j \leq \omega_1$.

Then $\pi = \pi_{|p|,\omega_1}$ and:
$M^p = \langle M_i \mid i \leq |p| \rangle$, $\pi^p = \langle \pi_{ij} \mid i \leq j \leq |p| \rangle$.

For $a \in R^p$ set: $a_i = (\pi^{-1}_{i,\omega_1})$ " a $(i \leq \omega_1)$.
Then $a_{|p|} = \bar{a}$, where $\langle \bar{a}, a \rangle \in R^p$.

Clearly $T' \subset M$. For $i \leq \omega_1$ set: $T'_i = (\pi_{i,\omega_1})^{-1}$ " T'. There is a least $\lambda \geq |p|$ s.t. $\pi_{\lambda,\omega_1} : \langle M_\lambda, a_\lambda \rangle \prec \langle M, a \rangle$ for all $a \in R^p$ and $\pi_{\lambda,\omega_1} : \langle M_\lambda, T'_\lambda \rangle \prec \langle M, T' \rangle$.

Set $q_0 = \langle M \restriction (\lambda+1), \pi \restriction (\lambda+1)^2, \langle \gamma^\lambda_n \mid n < \omega \rangle \rangle$

$q_1 = \{\langle a, a_\lambda \rangle \mid a \in R^p\} \cup \{\langle T', T'_\lambda \rangle\}$.

Then $q \in \mathbb{P}$, since:

$\langle H_\theta, \langle M_i \mid i \leq \omega_1\rangle, \langle \pi_{ii'} \mid i \leq i' \leq \omega_1\rangle, \langle \delta_n \mid n < \omega\rangle\rangle$

is a model of $\mathcal{L}(q)$ for sufficiently large regular θ in $V[G]$. Clearly, then, $q \leq p$ in $\mathbb{P}$.

<u>Claim</u> Let $G' \ni q$ be $\mathbb{P}$-generic. Set $\gamma_i' = \gamma_i^{G'}$ $(i \leq \omega)$. Then $\langle \gamma_i' \mid i < \omega\rangle$ is a branch through T'.

proof.

Since $\pi_{\lambda,\omega_1} : \langle M_\lambda, T_\lambda'\rangle \prec \langle M, T'\rangle$ and $\pi_{\lambda,\omega_1}(\gamma_i^\lambda) = \gamma_i$ for $i < \omega$, where $\langle \gamma_i \mid i < \omega\rangle$ is a branch through T', we know that $\langle \gamma_i^\lambda \mid i < \omega\rangle$ is a branch through T_λ' — i.e. $\gamma^\lambda \upharpoonright n \in T_\lambda'$ for $n < \omega$. Since $\pi^G_{\lambda\omega_1} : \langle M_\lambda, T_\lambda'\rangle \prec \langle M, T'\rangle$, it follows that:

$$\gamma' \upharpoonright n = \pi^G_{\lambda,\omega_1}(\gamma^\lambda \upharpoonright n) \in T'$$

for $n < \omega$. QED (Lemma 5)

Note that if q, T' are as in Lemma 5, then $T_q \leq T$. (Otherwise there is $s \in T_q$ s.t. $s \notin T'$. But then there is $q' \leq q$ s.t. $q' \Vdash \langle \dot{\gamma}_i \mid i \leq n \rangle = s$, where $n = |s|$. Let $G \ni q'$ be $\mathbb{P}$-generic. Then $q \in G$ and $\langle \gamma_i^G \mid i < \omega \rangle$ is not a branch in T'. Contradiction!)

<u>Corollary 6</u> Let Δ be strongly dense in $\mathbb{N}^{**}$. Then $\{p \mid T_p \in \Delta\}$ is dense in $\mathbb{P}$.

proof.
Let $T' \leq T_p$, $T' \in \Delta$. Pick $q \leq p$ s.t. $T_q \leq T'$. Then $T_q \in \Delta$. QED (Cor 6)

But then:

<u>Corollary 7</u> Let $B = \langle \gamma_i \mid i < \omega \rangle$ be $\mathbb{P}$-generic. Then B is $\mathbb{N}^{**}$-generic.

proof.
Let $B = \dot{B}^G$ where G is $\mathbb{P}$-generic. Then
$G' = \{T \in \mathbb{N}^{**} \mid \forall p \in G\ T_p \not\leq T\}$ is $\mathbb{N}^{**}$-generic. But B is a branch in every $T \in G'$. Hence B is the $\mathbb{N}^{*}$-generic sequence given by G'. QED (Cor 7)

It remains only to prove the converse:

By a virtual repetition of the proof of Lemma 5 we have:

<u>Lemma 8</u> Let $T \in \mathbb{N}^{**}$. There is $p \in \mathbb{P}$ s.t. $T_p \leq T$.

proof.

Let $G \ni T$ be $\mathbb{N}^{**}$-generic. Let $\langle \delta_i \mid i < \omega \rangle$ be the generic sequence given by G. Define $\alpha_i, X_i, M_i, \pi_{ij}$ $(i \leq j \leq \omega_1)$ exactly as before but without reference to a previously chosen $p \in \mathbb{P}$. (I.e. we set:

X_i = the smallest $X \prec M$ s.t.

$\{\alpha_j \mid j < i\} \cup \{\delta_n \mid n < \omega\} \subset X$.)

As before there is $\lambda < \omega_1$ s.t.

$\pi_{\lambda,\omega_1} : \langle M_\lambda, T_\lambda \rangle \prec \langle M, T \rangle$

where $T_\lambda = \pi^{-1}_{\lambda,\omega_1} {}'' T$. Define

$p_0 = \langle \langle M_i \mid i \leq \lambda \rangle, \langle \pi_{ij} \mid i \leq j \leq \lambda \rangle, B_\lambda \rangle$

where $B_\lambda = \pi_{\lambda,\omega_1} {}'' \{\delta_n \mid n < \omega\}$

$p_1 = \{\langle T, T_\lambda \rangle\}$.

The rest of the proof is as before.

QED (Lemma 8)

Lemma 9. Let $B = \langle \gamma_i \mid i<\omega \rangle$ be $\mathbb{N}^{**}$-generic. Then $\langle \gamma_i \mid i < \omega \rangle$ is $\mathbb{P}$-generic.

proof.

Suppose not. Then there is $T \in \mathbb{N}^{**}$ s.t. $T \Vdash \mathring{B}$ is not $\check{\mathbb{P}}$-generic. Let $p \in \mathbb{P}$ s.t. $T_p \leq T$. Let $G \ni p$ be $\mathbb{P}$-generic. Let $B = B^G$. Then B is $\mathbb{N}^{**}$-generic and, in fact, $B = \mathring{B}^{G'}$ where $G' = \{T \mid \bigvee p \in G\ T_p \leq T\}$ is $\mathbb{N}^{**}$-generic. But $T \in G'$ and B is $\mathbb{P}$-generic. Contradiction! QED (Lemma 9)

This proves the Theorem.

Note Carrying this further, we could show that $BA(\mathbb{N}^*) = BA(\mathbb{P})$.

Note If $\mathcal{L}'$ is like $\mathcal{L}$ except that the axiom:

$$\bigvee m \bigwedge n \geq m\ \mathring{\gamma}_n \in \bar{C} \quad \text{for all club } C \subset \omega_2$$

is replaced by:

$$\bigvee m \bigwedge n \geq m\ \mathring{\gamma}_{n+1} > F(\mathring{\gamma}_n) \quad \text{for all}$$

$$F : \omega_2 \longrightarrow \omega_2,$$

then a virtual repetition of our proof shows:

Theorem $B' = \langle \gamma_i \mid i<\omega \rangle$ is $\mathbb{P}_{\mathcal{L}'}$-generic iff it is $\mathbb{N}'$-generic.