§5 The conclusion

From now on assume: $CH + 2^{\omega_1} = \omega_2$

We now prove that $\mathbb{N}^*$ is subcomplete. The definition of subcompleteness can be found e.g. in [Sing]. The notion of subcompleteness involves the notion of fullness, which we first define:

Def Let $\mathfrak{A} = \langle |\mathfrak{A}|, \in^{\mathfrak{A}}, \ldots \rangle$ model the axiom of extensionality. $\mathfrak{A}$ is grounded iff its well founded core $A = wfc(\mathfrak{A})$ is transitive and $\in \cap A^2 = \in^{\mathfrak{A}} \cap A^2$.

Def Let M be a transitive ZFC^- model. M is full iff there is a grounded ZFC^- model $\mathfrak{A}$ s.t.

- $M \in wfc(\mathfrak{A})$
- If $B \in \mathfrak{A}$ s.t. $B \subset M$, then $\langle M, B \rangle$ is a ZFC^- model. (Another words, M looks like a 2nd order ZFC^- model in $\mathfrak{A}$.)

We then say that $\mathfrak{A}$ witnesses the fullness of M.

(Note This notion is sometimes called "almost full" with the word "full" being reserved for the case that $\mathfrak{A}$ is transitive.)

We note the following facts:

<u>Fact 1</u> If $\mathfrak{A}$ witnesses the fullness of M and ρ is the least ordinal s.t. $L_\rho(M)$ is admissible, then $L_\rho(M) \subset \mathrm{wfc}(\mathfrak{A})$.

<u>Fact 2</u> If $\bar{M}$ is full and $f: \bar{M} \prec M$ cofinally, then M is full. (In fact, if $\bar{\mathfrak{A}}$ witnesses the fullness of $\bar{M}$, then $\mathfrak{A}$ witness the fullness of M, where $f^*: \bar{\mathfrak{A}} \prec \mathfrak{A}$ is the liftup of f.

Combining these facts, it is easy to show:

<u>Fact 3</u> Let $\pi: \bar{M} \prec M$ cofinally, where $\bar{M}$ is full. Let $\bar{\rho}$ be least s.t. $L_{\bar{\rho}}(\bar{M})$ is admissible and ρ be least s.t. $L_\rho(M)$ is admissible. Let:

$$L_{\bar{\rho}}(\bar{M}) \models \varphi[\bar{M}, x_1, \ldots, x_n], \text{ where } x_1, \ldots, x_n \in \bar{M}$$

then

$$L_\rho(M) \models \varphi[M, \pi(x_1), \ldots, \pi(x_n)].$$

We also define:

<u>Def</u> Let $N = L_\tau^{A_1, \ldots, A_n} = \langle L_\tau[A_1, \ldots, A_n], \in, A_1, \ldots, A_n \rangle$ be a ZFC^- model. Let $X \subset N$ and let $\delta \in N$ be a cardinal in N.

$C_\delta^N(X) =$ the smallest $Y \prec N$ s.t. $(\delta + 1) \cup X \subset Y$.

<u>Theorem</u> $\mathbb{N}^*$ is subcomplete.

At the end of the proof we shall note the rather small changes needed to prove that $\mathbb{N}'$ is subcomplete.

Fix $\Omega > 2^{\overline{\overline{\mathbb{N}^*}}}$. Let $W = L^A_\tau = \langle L_\tau[A], \in, A\rangle$ be a transitive ZFC$^-$ model s.t. $H_\Omega \in W$. Let $\pi : \bar{W} \prec W$, where $\bar{W}$ is countable, transitive, and full. Assume that $\pi(\bar{\Omega}) = \Omega$ and $\pi(\bar{\mathbb{N}}^*) = \mathbb{N}^*$. In order to prove subcompleteness, it suffices to show that, given any pair $\langle \bar{s}, s\rangle$ with $\pi(\bar{s}) = s$ and any $\bar{G}$ which is $\bar{\mathbb{N}}^*$-generic over $\bar{W}$, there is a $T \in \mathbb{N}^*$ which forces the following statement:

If $G \ni T$ is $\mathbb{N}^*$-generic over V, there is $\sigma \in V[G]$ s.t.

(i) $\sigma : \bar{W} \prec W$

(ii) $\sigma(\bar{\Omega}, \bar{\mathbb{N}}^*, \bar{s}) = \Omega, \mathbb{N}^*, s$

(iii) $\sigma``\bar{G} \subset G$

(iv) $C^W_{\omega_2}(\operatorname{rng}\sigma) = C^W_{\omega_2}(\operatorname{rng}\pi)$.

The proof will stretch over many sublemmas.

Before proceeding further, we list some definitions and facts developed in [Sing].

Def Let M be transitive and $\mathfrak{A}$ a grounded model. We call an embedding $\pi : M \to \mathfrak{A}$ cofinal iff for each $x \in \mathfrak{A}$ there is $u \in M$ s.t. $\mathfrak{A} \models x \in \pi(u)$.

Fact 4 Let M be a transitive ZFC^- model and $\pi : M \xrightarrow[\Sigma_0]{} \mathfrak{A}$ cofinally. Then $\pi : M \prec \mathfrak{A}$.

Fact 5 Let $\mathfrak{A}$ be a ZFC^- model, M transitive and $\pi : M \xrightarrow[\Sigma_0]{} \mathfrak{A}$ cofinally. Then $\pi : M \prec \mathfrak{A}$.

Def Let M be a transitive ZFC^- model and $\tau \in M$ a regular cardinal in M.
$\pi : M \prec \mathfrak{A}$ τ-cofinally iff for each $x \in \mathfrak{A}$ there is $u \in M$ s.t. $\bar{\bar{u}} < \tau$ in M and $\mathfrak{A} \models x \in \pi(u)$.

Def Let $\bar{M}$ be a ZFC^- model and $\bar{\tau} \in M$ regular in M. Let $\bar{H} = H_{\bar{\tau}}^{\bar{M}}$ and let $\pi : \bar{H} \prec H$ cofinally, where H' is transitive. We say that $\langle \mathfrak{A}, \pi \rangle$ is a liftup of $\langle \bar{M}, \bar{\pi} \rangle$ iff
$\pi \supset \bar{\pi}$ and $\pi : \bar{M} \prec \mathfrak{A}$ $\bar{\tau}$-cofinally.

(Note $\bar{H}$ could be a proper class in $\bar{M}$ if GCH fails in $\bar{M}$.)

(In this case we also say that π is a liftup of $\bar{\pi}$.)

<u>Fact 6</u> The liftup $\langle \mathfrak{A}, \pi \rangle$ of $\langle \bar{M}, \bar{\pi} \rangle$ always exists and is determined up to isomorphism by $\langle \bar{M}, \bar{\pi} \rangle$. The liftup is unique if $\mathfrak{A}$ is transitive.

<u>Fact 7</u> ("Interpolation Lemma")
Let $\pi' : \bar{M} \prec M'$ and $\bar{\pi} = \pi' \upharpoonright \bar{H} : \bar{H} \prec H$ cofinally where $\bar{H} = H_{\bar{\tau}}^{\bar{M}}$ is as above. Then the transitive liftup $\langle M, \pi \rangle$ of $\langle \bar{M}, \bar{\pi} \rangle$ exists. Moreover, there is a unique $\sigma : M \prec M'$ s.t. $\sigma \pi = \pi'$ and $\sigma \upharpoonright H = \mathrm{id}$.

Another well known fact is:

<u>Fact 8</u>: If $\pi : \bar{M} \prec M$ $\bar{\tau}$-cofinally, then $\pi(\bar{\tau}) = \sup \pi '' \bar{\tau}$.

(<u>Note</u> To prove the theorem it will suffice to show that if $\bar{G}, \bar{\imath}, s$ are as stated, then in the generic collapse of a sufficient cardinal there exist G, g s.t. G is $\mathbb{N}^*$-generic over V, $g \in V[G]$ and (i)–(iv) hold. There will then be a $T \in G$ which forces this.)

We shall also make use of an auxiliary forcing $\mathbb{C}$ which is almost contained in every club set in the ground model: $\forall \alpha < \omega_2 \; C \setminus \alpha \subset A$ whenever $A \in V$ is club in ω_2.
The conditions are pairs $\langle \alpha, A \rangle$ s.t. A is club in ω_2 and $\alpha < \omega_2$. They are partially ordered by:

$$\langle \alpha', A' \rangle \leq \langle \alpha, A \rangle \longleftrightarrow : (\alpha' \geq \alpha \wedge A' \subset A \wedge A' \cap \alpha = A \cap \alpha)$$

These conditions are closed under ω_1-chains and hence do not add new subsets of the ground model of size ω_1. For this reason it is also complete in Shelah's sense.
Moreover $\mathbb{C}$ satisfies the ω_3-chain condition, since whenever $\langle \alpha, A \rangle, \langle \beta, B \rangle$ are incompatible, then $\langle \alpha, A \cap \alpha \rangle \neq \langle \beta, B \cap \beta \rangle$. (Otherwise $\langle \alpha, A \cap B \rangle \leq \langle \alpha, A \rangle, \langle \beta, B \rangle$.)
Hence $\mathbb{C}$ does not change cardinals or cofinalities. If G is $\mathbb{C}$-generic, we set:

$$C = C_G =: \bigcap \{ A \mid \forall \alpha \, \langle \alpha, A \rangle \in G \} = \bigcup \{ A \cap \alpha \mid \langle \alpha, A \rangle \in G \}.$$

C then has the above property. We call it the <u>generic club</u> added by G.
G is then recoverable from C by:

$$G = \{ \langle \alpha, A \rangle \in \mathbb{C} \mid C \subset A \wedge A \cap \alpha = C \cap \alpha \}$$

Since $\mathbb{C}$ is definable in W, there is $\bar{\mathbb{C}} \in \bar{W}$ s.t. $\pi(\bar{\mathbb{C}}) = \mathbb{C}$.

Def $\langle G, G' \rangle$ is a good pair for $\bar{W}$ iff

- G is $\bar{\mathbb{N}}^*$ - generic, adding the generic sequence $\langle \bar{\gamma}_n \mid n < \omega \rangle$
- G' is $\bar{\mathbb{C}}$ - generic, adding the generic club C
- $\bigvee n \bigwedge m \geq n \;\; \bar{\gamma}_m \in C$.

Lemma 1 Let $T \in \mathbb{N}^*$. There is a good pair $\langle G, G' \rangle$ s.t. $T \in G$.

proof.

Let $\langle \Delta_i \mid i < \omega \rangle$ enumerate the dense subsets of $\bar{\mathbb{N}}^*$ and $\langle \Delta'_i \mid i < \omega \rangle$ enumerate the dense subsets of $\bar{\mathbb{C}}$. We define:

$T_i \in \bar{\mathbb{N}}^*$, $\langle \alpha_i, A_i \rangle \in \bar{\mathbb{C}}$

by induction on $i < \omega$ as follows.

$T_0 = T$, $\langle \alpha_0, A_0 \rangle = \langle 0, \omega_2 \rangle$;

Now let $T_i, \langle \alpha_i, A_i \rangle$ be given.

Let $s_i = \text{stem}(T_i)$. Set:

$$T'_i = \{t \in T_i \mid \bigwedge_{j \geq |s_i|} t_j \in A_i \setminus \alpha_i\}.$$

Then $T'_i \leq T_i$ in $\bar{\mathbb{N}}^*$. Pick $T_{i+1} \leq T'_i$ s.t. $T_{i+1} \in \Delta_i$. Pick $\alpha'_i < \omega_2$ s.t. $\alpha'_i > s_{i+1}(j)$ for all $j < |s_{i+1}|$. Then $\langle \alpha'_i, A_i \rangle \leq \langle \alpha_i, A_i \rangle$ in $\bar{\mathbb{C}}$. Pick $\langle \alpha_{i+1}, A_{i+1} \rangle \leq \langle \alpha'_i, A_i \rangle$ s.t. $\langle \alpha_{i+1}, A_{i+1} \rangle \in \Delta'_i$.

$C = \bigcup_i (\alpha_i \cap A_i)$ is then $\bar{\mathbb{C}}$-generic and $b = \langle \gamma_n \mid n < \omega \rangle =: \bigcup \bigcap_{i<\omega} T_i$ is an $\bar{\mathbb{N}}^*$-generic sequence.

<u>Claim</u> $\bigwedge i \geq |\nu_0|\ \gamma_i \in C$.

proof.

Let $i < |\nu_j|$. Then $\gamma_i = \nu_j(i) < \alpha'_j$. Hence $\gamma_i \in \alpha'_j \cap A_j = \alpha'_j \cap C$. QED (Lemma 1)

But then:

<u>Corollary 2</u> Let G be $\mathbb{N}^*$-generic over $\bar{W}$. There is G' s.t. $\langle G, G'\rangle$ is a good pair.

proof.

Let ρ = the least ρ s.t. $L_\rho(\bar{W})$ is admissible. Since $\bar{W}$ is full, we know that $\mathcal{P}(\mathbb{N}^*) \cap L_\rho(\bar{W}) \subset \bar{W}$. Hence G is $\mathbb{N}^*$-generic over $L_\rho(\bar{W})$, and $L_\rho(\bar{W}[G])$ is admissible. Let $\mathcal{L}$ be the infinitary language over $L_\rho(\bar{W}, [G])$ with:

<u>Predicate</u>: $\dot{\in}$

<u>Constants</u>: $\underline{x}$ $(x \in L_\rho(\bar{W}[G]))$; $\mathring{G}$, $\mathring{C}$

<u>Axioms</u>: ZFC; $\bigwedge v (v \in \underline{x} \leftrightarrow \bigvee_{z \in x} v = \underline{z})$; $\dot{G}$ is $\underline{\mathbb{C}}$-generic over $\underline{\bar{W}}$, giving generic club set $\dot{C}$; $\bigvee n \bigwedge m \geq n\ \underline{\gamma}(m) \in \mathring{C}$, where $\gamma = \langle \gamma(i) \mid i < \omega\rangle$ is the generic sequence given by G.

It suffices to show:

<u>Claim</u> $\mathcal{L}$ is consistent,

since if $\mathfrak{A}$ is a grounded model of $\mathcal{L}$, then $\langle G, \mathring{G}^{\mathfrak{A}} \rangle$ is a good pair.

Suppose not. The consistency of $\mathcal{L}$ is a $\Pi_1(L_\rho[\bar{N}[G]])$ statement, so there is $T \in G$ which forces it to be false. Let $\langle G_0, G_1 \rangle$ be a good pair with $T \in G_1$. Then the corresponding language $\mathcal{L}'$ on $L_\rho[\bar{N}[G_0]]$ is inconsistent. But it is consistent, since $\langle H_{\omega_1}, G_1 \rangle$ is a model. Contradiction!

QED (Corollary 2)

Now let $\bar{G}$ be $\bar{\mathbb{N}}^*$-generic over $\bar{W}$ with generic sequence $\langle \delta_i \mid i<\omega \rangle$. Let $\langle \bar{G}, \bar{G}' \rangle$ be a good pair and let $\bar{C}$ be the generic club given by $\bar{G}'$.

Since $\mathbb{C}$ is a complete forcing, there is G' which is $\mathbb{C}$-generic over V s.t. $\pi``\bar{G}' \subset G'$. Let $C$ be the generic club given by $G'$. Then $\pi``\bar{C} \subset C$.

But then π extends uniquely to a $\pi' \supset \pi$ s.t. $\pi : \bar{W}[\bar{C}] \prec W[C]$, $\pi(\bar{C}) = C$. We now prove:

<u>Lemma 3</u> Assume the collapse of a sufficient cardinal to ω. Then there is σ s.t.

(i) $\sigma : \bar{W} \prec W$

(ii) σ takes $\omega_2^{\bar{W}}$ cofinally to $\omega_2^W = \omega_2$

(iii) $\sigma '' \bar{C} \subset C$

(iv) $C^W_{\omega_2}(\mathrm{rng}(\pi)) = C^W_{\omega_2}(\mathrm{rng}\,\sigma)$

Before proving Lemma 3 we note a consequence. Let $\delta_i = \sigma(\bar{\delta}_i)$. Then $\langle \delta_i \mid i < \omega \rangle$ is almost contained in C. But C is almost contained in every club $A \subset \omega_2$ in the ground model. Hence $\langle \delta_i \mid i < \omega \rangle$ is almost contained in every such A. If we knew that $\langle \delta_i \mid i < \omega \rangle$ were an $\mathbb{N}^*$-generic sequence given by G and that $\sigma \in V[G]$, we would be done. However, we do not know that and shall have to do a further argument, using ~~Lö~~wenheim - Skolem to apply Lemma 3 to a countable model instead of $V[C]$.

We now prove Lemma 3. Let $\bar{H}_0 = (H_{\omega_2})^{\bar{W}[\bar{C}]}$ and let: $\langle W_0[C_0], \pi_0' \rangle$ be the liftup of $\langle \bar{W}[\bar{C}], \pi' \upharpoonright \bar{H}_0 \rangle$ (noting that $\bar{W}$ is definable in $\bar{W}[\bar{C}]$ from the predicate in $\bar{W}$). Set $\pi_0 = \pi_0' \upharpoonright \bar{W}$. Then $\pi_0 : \bar{W} \prec W_0$ (noting that $\bar{W}$ is uniformly $\bar{W}[\bar{C}]$ definable from the predicates in $\bar{W}$.)

(1) $\pi_0 : \bar{W} \prec W_0$ $\omega_2^{\bar{W}}$ - cofinally.

proof.

Let $x \in W_0$. Then $x \in \pi_0'(u)$, where $u \in \bar{W}[\bar{C}]$, $u \subset \bar{W}$ and $\bar{\bar{u}} < \omega_2$ in $\bar{W}$. But the forcing $\bar{\mathbb{C}}$ adds no new subsets of $\bar{W}$ of size $< \omega_2$. Hence $u \in \bar{W}$ and $\pi_0'(u) = \pi_0(u)$.

QED (1)

By the same argument we have: $\bar{H}_0 = (H_{\omega_2})^{\bar{W}}$. Thus:

(2) $\langle W_0, \pi_0 \rangle$ is the liftup of $\langle \bar{W}, \pi \upharpoonright \bar{H}_0 \rangle$.

But if $x \in \pi(u)$, $u \in \bar{W}$, $\bar{\bar{u}} < \omega_2$ in $\bar{W}$, we have $x = \pi(f)(\nu)$ for a $\nu < \omega_1$, where $f \in \bar{W}$ maps ω_1 onto u. Thus

(3) $W_0 = C_{\omega_1}^{W_0}(\operatorname{rng} \pi_0)$.

Now let ρ_0 = the least ρ s.t. $L_\rho(W_0)$ is admissible. Then W_0 is a 2nd order ZFC^- model in $L_{\rho_0}(W_0)$, by the fullness of W_0.

It follows that C_0 is $\mathbb{C}_0$ - generic over $L_{\rho_0}(W_0)$ where $\pi_0(\bar{\mathbb{C}}) = \mathbb{C}_0$ and $\pi_0'(\bar{C}) = C_0$. Thus ρ_0 is minimal s.t. $L_{\rho_0}(W_0[C_0])$ is admissible. Let $\mathcal{L}_0$ be the following language on $L_{\rho_0}(W_0)$:

Predicate: $\in$

Constants: $\underline{x}$ $(x \in L_{\rho_0}(W_0))$, $\dot\sigma'$, $\dot\sigma$

Axioms: ZFC^-; $\bigwedge v\,(v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$;

$\dot\sigma' : \underline{\bar{W}_0[\bar{C}]} \prec \underline{W_0[C_0]}$; . ,

$\dot\sigma'(\underline{\bar\Omega}, \underline{\bar{\mathbb{B}}}, \underline{\bar{\mathbb{C}}}, \underline{\bar{C}}) = \Omega_0, \mathbb{B}_0, \mathbb{C}_0, C_0$

where $\pi_0'(\bar\Omega, \bar{\mathbb{B}}, \bar{\mathbb{C}}, \bar{C}) = \Omega_0, \mathbb{B}_0, \mathbb{C}_0, C_0$;

$\dot\sigma = \dot\sigma' \upharpoonright \bar{W}_0$;

$\dot\sigma : \underline{\bar{W}_0} \prec \underline{W_0}$ $\omega_2^{\bar{W}}$ - cofinally

Then $\mathcal{L}_0$ is consistent, since $\langle H_\theta, \pi', \pi \rangle$ is a model for sufficient regular θ.

The statement that $\mathcal{L}_0$ is consistent is $\Pi_1(L_{\rho_0}(W_0[C_0]))$ $\bar{W}, \bar\Omega, \bar{\mathbb{B}}, \bar{\mathbb{C}}, \bar{C}$ and $\Omega_0, \mathbb{B}_0, \mathbb{C}_0, C_0$.

Now let $\bar{H}_1' = (H_{\omega_3})^{\bar{W}[\bar{C}]}$. Let $\langle W_1[C_1], \pi_1' \rangle$ be the liftup of

$\langle \bar{W}[\bar{c}], \pi' \restriction \bar{H}_1 \rangle$. Let ρ be the least ρ s.t. $L_\rho(W_1[C_1])$ is admissible.

Note that $C_1 = C$, since $\bar{c} \in \bar{H}_1$ and $\pi' \restriction \bar{H}_1 = \pi_1' \restriction \bar{H}_1$. Let $\mathcal{L}_1$ be the language on $L_{\rho_1}(W_1[C_1])$ defined from $W_1, C_1, \Omega_1, \mathbb{B}_1, \mathbb{C}_1, \bar{W}, \bar{c}, \bar{\mathbb{B}}, \bar{\mathbb{C}}, \bar{C}$ as $\mathcal{L}_0$ was defined on $L_{\rho_0}(W_0[C_0])$ from $W_0, C_0, \Omega_0, \ldots$ etc. (where $\Omega_1, \mathbb{B}_1, \mathbb{C}_1 = \pi_1'(\bar{\Omega}, \bar{\mathbb{B}}, \bar{\mathbb{C}})$). By the interpolation lemma there is $\mu : W_0[C_0] \prec W_1[C_1]$ s.t. $\mu \restriction H_0 = id$ and $\mu\pi_0 = \pi_1$. Hence $\mu(C_0, \Omega_0, \mathbb{B}_0, \ldots) = C_1, \Omega_1, \mathbb{B}_1, \ldots$.

Applying Fact 3 we then conclude:

(1) $\mathcal{L}_1$ is consistent.

Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}_1$. Let $\sigma_1 = \mathring{\sigma}^{\mathfrak{A}}$. Then:

(1). $\sigma_1 : \bar{W} \prec W_1$ $\omega_2^{\bar{W}}$-cofinally.

- $\sigma_1(\bar{\Omega}, \bar{\mathbb{B}}, \bar{\mathbb{C}}) = \Omega_1, \mathbb{B}_1, \ldots$
- $\sigma_1 " \bar{C} \subset C$

On the other hand:

(2) $\overline{\pi}_1 : \overline{W} \prec W_1$ $\omega_3^{\overline{W}}$ - cofinally.

proof:

Let $x \in W_1$. Then $x \in \pi_0'(u)$ where $u \in \overline{W}[\overline{C}]$, $\overline{\overline{u}} \leq \omega_1$ in $\overline{W}[\overline{C}]$ and $u \subset \overline{W}$. But $\overline{\mathbb{C}}$ satisfies the ω_2-chain condition in $\overline{W}$, which implies that $u \subset v$ for a $v \in \overline{W}$ s.t. $\overline{\overline{v}} \leq \omega_1$ in $\overline{W}$. Thus $x \in \pi_0(v)$. QED (2)

But then:

(3) $C_\tau^{W_1}(\mathrm{rng}(\sigma_1)) = C_\tau^{W_1}(\mathrm{rng}(\pi_1)) = W_1$,

where $\tau = \omega_2^{W_1}$.

By (2) we also have:

(4) $\langle W_1, \pi_1 \rangle$ is the liftup of $\langle \overline{W}, \pi \upharpoonright \overline{H}_1 \rangle$,

where $\overline{H}_1 = (H_{\omega_3})^{\overline{W}}$.

But then there is $\delta : W_1 \prec W$ s.t.,

$\delta \upharpoonright (H_{\omega_3})^{W_1} = \mathrm{id}$ and $\delta \pi_1 = \pi$.

Set: $\sigma = \delta \circ \sigma_1$.

Then:

- $\sigma : \overline{W} \prec W$
- $\sigma '' \overline{C} \subset C$, since $\delta \upharpoonright C = id$
- $C^W_\tau(\mathrm{rng}\,\sigma) = \delta '' C^{W_1}_\tau(\mathrm{rng}\,\sigma) = \delta '' W_1$,
- $C^W_\tau(\mathrm{rng}\,\overline{\pi}) = \delta '' C^{W_1}_\tau(\mathrm{rng}\,\overline{\pi}_1) = \delta '' W_1$

for $\tau = \omega^{W_1} = \omega_2$, since $\delta \upharpoonright \tau = \tau$

Trivially we also have:

- $\sigma(\overline{\Omega}, \overline{\mathbb{B}}, \overline{\mathbb{C}}) = \pi(\overline{\Omega}, \overline{\mathbb{B}}, \overline{C}) = \Omega, \mathbb{B}, C,$

QED (Lemma 3)

Let $\delta = \langle \delta_i \mid i < \omega \rangle$ be a Namba-generic sequence over $V[C]$. Work in $V[C][\delta]$. Let θ be regular in V s.t. $W \in H_\theta$. Set $H = H_\theta$, $H' = H_\theta[C]$. Then $H' = (H_\theta)^{V[C]}$. Let $<$ be a well ordering of H. Let X be the smallest $X \prec \tilde{H} = \langle H', \in, <, W, \Omega, \mathbb{B}, \mathbb{C}, C, \overline{W}, \pi' \rangle$ s.t. $\{\delta_n \mid n < \omega\} \subset X$. Let

$\mu : H^* \overset{\sim}{\longleftrightarrow} \tilde{H} \mid X$, where

$H^* = \langle |H^*|, \in, <^*, W^*, \Omega^*, \mathbb{B}^*, \mathbb{C}^*, C^*, \overline{W}, \overline{\pi}^* \rangle$

Then $\mu \overline{\pi}^* = \pi'$. Clearly $\alpha = \mathrm{crit}(\mu)$ where $\alpha = \omega_1 \cap X$.

Now let $\langle \delta_n \mid n<\omega\rangle$ be a Namba-generic sequence over $V[C]$. Work in $V[C][\delta]$. Let θ be regular in V s.t. $W \in H_\theta$. Set:

$H =: \langle H_\theta, \in, <, W, \Omega, \mathbb{B}, \mathbb{C}, \bar{W}, \pi\rangle$,

where $<$ is a well ordering of H in V.

Then cardinals are not collapsed in $H[C]$ and $\pi' : \bar{W}[\bar{C}] \prec W[C]$ is $H[C]$-definable in C. Set:

$X =:$ the smallest $X \prec H[C]$ s.t.

$\{C\} \cup \{\delta_n \mid n<\omega\} \subset X$.

Let $\mu' : \tilde{H}[\tilde{C}] \overset{\sim}{\longleftrightarrow} H[C] | X$ s.t. $\mu'(\tilde{C}) = C$. Set $\mu = \mu'$. Then $\mu : \tilde{H} \overset{\sim}{\longleftrightarrow} H|X$. Let $\tilde{H} = \langle |\tilde{H}|, \in, \tilde{<}, \tilde{W}, \ldots, \bar{W}, \tilde{\pi}\rangle$.

Let $\tilde{\pi}'$ be defined from $\tilde{C}$ in $\tilde{H}[\tilde{C}]$ as π was defined in $H[C]$. Then

$\tilde{\pi}' : \bar{W}[\bar{C}] \prec \tilde{W}[\tilde{C}]$ and $\tilde{\pi}'(\bar{C}) = \tilde{C}$,

where $\tilde{\pi} = \tilde{\pi}' \upharpoonright \bar{W}$. Note that μ takes $\tilde{\omega}_1 = \omega_2^{\tilde{H}}$ cofinally to ω_2.

<u>Lemma 4</u> Every $x \in \tilde{H}$ is $\tilde{H}$-definable from parameters in $\omega_1^{\tilde{H}} \cup \{\tilde{\delta}_n \mid n<\omega\}$ where $\mu(\tilde{\delta}_n) = \delta_n$.

proof of Lemma 4.

Let $x \in \tilde{H}$. x is $\tilde{H}[\tilde{C}]$-definable from $\tilde{C}$ and a parameter $p = \langle \tilde{\delta}_{i_1}, \dots, \tilde{\delta}_{i_n} \rangle$. Let $x = t^{\tilde{H}[\tilde{C}]}(p, \tilde{C})$. Let u be the set of $z \in \tilde{H}$ s.t. $p \Vdash t(\check{p}, \mathring{C}) = \check{z}$ for some $p \in \mathbb{C}$. Then $u \in \tilde{H}$. Moreover, $\bar{\bar{u}} \leq \omega_2$ in $\tilde{H}$, since $\mathbb{C}$ satisfies the ω_3-chain condition. Clearly u is $\tilde{H}$-definable in p. Set f = the $\tilde{H}$-least $f : \omega_2 \xrightarrow{\text{onto}} u$. Then f is $\tilde{H}$-definable in p and $x = f(\xi)$ for a $\xi < \omega_2$ in $\tilde{H}$. Let $\xi < \tilde{\delta}_m$. Let g be $\tilde{H}$-least s.t. $g : \omega_1 \xrightarrow{\text{onto}} \tilde{\delta}_m$. Then $f \circ g$ is $\tilde{H}$ definable in $p, \tilde{\delta}_m$ and $x = f \circ g(\nu)$ for a $\nu < \omega_1^{\tilde{H}}$.

QED (Lemma 4)

Applying the argument of Lemma 3 to $\tilde{H}[\tilde{C}]$ instead of $V[C]$, we see that:

<u>Lemma 5</u> There is $\in H_{\omega_1}$ s.t.

(i) $\tilde{\sigma} : \bar{W} \prec \tilde{W}$

(ii) $\tilde{\sigma}$ takes $\omega_2^{\bar{W}}$ cofinally to $\omega_2^{\tilde{W}} = \tilde{\omega}_2$ (where $\tilde{\omega}_2 = \omega_2^{\tilde{H}}$.).

(iii) $\tilde{\sigma}$ "$\bar{C} \subset \tilde{C}$

(iv) $C^{\tilde{W}}_{\tilde{\omega}_2}(\text{rng}(\tilde{\sigma})) = C^{\tilde{W}}_{\tilde{\omega}_2}(\text{rng}(\tilde{\pi}))$.

We now use our construction to define a precondition in the forcing $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ defined in §4. We define $p = \langle p_0, p_1 \rangle$ by:

Def $p_0 = \langle M^p, \pi^p, b^p \rangle$ where:

- $\text{dom}(M^p) = \text{dom}(\pi^p) = \{\alpha_0\}$, where $\alpha_0 = \omega_1^{\tilde{H}}$
- $M_0^p = \tilde{\pi}(\bar{M}) = \mu^{-1}(m)$; $\pi_{00}^p = \text{id} \upharpoonright M_0^p$
- $b^p = \tilde{\pi}(\bar{B})$, where $\bar{B} = \mathbb{B}^{\bar{G}}$, $\bar{G}$ is the $\mathbb{B}$-generic set over $\bar{W}$ fixed at the outset.

To define p_1 we let $\langle \varphi_i(v_1, \ldots, v_{n_i}) \mid i < \omega \rangle$ enumerate the H-formulae and set:

$A_i = \{\langle x_1, \ldots, x_{n_i}\rangle \in m^{n_i} \mid H \models \varphi_i[x_1, \ldots, x_{n_i}]\}$

$\tilde{A}_i = \{\langle x_1, \ldots, x_{n_i}\rangle \in M_0^p \mid \tilde{H} \models \varphi_i[x_1, \ldots, x_{n_i}]\}$,

Then $\mu(\tilde{A}_i) = A_i$ for $i < \omega$.

We set: $p_0 =: \{\langle A_i, \tilde{A}_i \rangle \mid i < \omega\}$.

p is then a putative condition, i.e. $p \in \tilde{\mathbb{P}}$.

<u>Lemma 6</u> $p \in \mathbb{P}$

proof.

We show that $\mathcal{L}(p)$ is consistent by constructing a model for it in $V[C][\delta]$, using the fact that ω_1 is not collapsed. We construct $\langle \alpha_i \mid i \leq \omega_1 \rangle$, $\langle X_i \mid i \leq \omega_1 \rangle$ as follows:

$X_0 = \mu$ "M_0^p, $\alpha_0 =: \alpha_0^p = \omega_1^{M_0^p}$.

Set $B = \langle \delta_n \mid n < \omega \rangle$, where:

$\delta_n =: \mu\tilde{\sigma}(\bar{\delta}_n) = \mu(\delta_n^p)$ for $n < \omega$.

It is easily seen that X_0 is the smallest $X \prec M$ s.t. $\alpha_0 \cup \{\delta_n \mid n < \omega\} \subset X$.

For $j > 0$ set:

$X_j =$ the smallest $X \prec M$ s.t.

$$\bigcup_{i<j} X_i \cup \{\alpha_i\} \subset X$$

$\alpha_j = \omega_1 \cap X_j$.

Set: $\pi_{i,\omega_1} : M_i \overset{\sim}{\longleftrightarrow} M|X_i$, $\pi_{i,j} = \pi_{j,\omega_1}^{-1} \circ \pi_{i,\omega_1}$

for $i \leq j \leq \omega_1$. Note that since $\bar{\gamma}_n \in \bar{C}$ for sufficiently large n, we have:

$\tilde{\gamma}_n = \tilde{\sigma}(\bar{\gamma}_n) \in \tilde{C}$ for sufficient n, since $\tilde{\sigma}''\bar{C} \subset \tilde{C}$. But $\pi_{0,\omega_1} = \mu \upharpoonright M_0$ where

$\mu''\tilde{C} \subset C$. Hence $\gamma_n \in C$ for sufficiently large n. But whenever $A \subset \omega_2$, $A \in V$ is club in ω_2, then $\bigvee \alpha < \omega_2\ \alpha \cap C \subset A$. It follows that $\bigvee m \bigwedge n \geq m\ \gamma_n \in A$ for all such A.

Hence, letting κ be a sufficiently large regular cardinal in $V[C][\sigma]$, we see that

$$\mathfrak{A} = \langle H_\kappa, \langle M_i \mid i \leq \omega_1 \rangle, \langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle, B \rangle$$

models $\mathcal{L}$ (where $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ as in §4).

But $\pi_{0\omega_1} = \mu \upharpoonright M_0^p$ and $\mu(\tilde{A}_i) = A_i$ for $i < \omega$. Hence:

$$\pi_{0,\omega_1} : \langle M_0^p, \tilde{A}_i \rangle \prec \langle M, A_i \rangle \quad \text{for } i < \omega.$$

Thus $\mathfrak{A}$ models $\mathcal{L}(p)$.

QED (Lemma 6)

Now let G' be $\mathbb{P}$-generic with $p \in G'$. Let $B' = B^{G'}$, $\pi' = \pi^{G'}$, $M' = M^{G'}$.

Since p conforms to H, we know that π'_{0,ω_1} extends to $\pi^* : \tilde{H} \prec H$.

Set: $\sigma' = \pi^* \circ \tilde{\sigma}$. Then:

(i) $\sigma' : \bar{W} \prec W$

(ii) σ' takes $\omega_2^{\bar{W}}$ cofinally to ω_2.

Since $\pi^*(\tilde{\pi}) = \pi^* \circ \tilde{\pi} = \pi$, and

$$C^{\tilde{W}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\sigma}) = C^{\tilde{W}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\pi}),$$

we conclude:

(iii) $C^{W}_{\omega_2}(\mathrm{rng}\,\sigma') = C^{W}_{\omega_2}(\mathrm{rng}\,\pi)$.

Now set G = the set of $T \in \mathbb{N}^*$ s.t. $\bigvee p \in G'\ T_p \leq T$. Then G is $\mathbb{N}^*$-generic and B' is the sequence given by G. At $\bar{T} \in \bar{G}$, then $\bar{B}_i = \langle \bar{\delta}_n \mid n < \omega \rangle$ is a branch in $\bar{T}$. Hence $B' = \langle \sigma'(\bar{\delta}_n) \mid n < \omega \rangle$ is a branch in $\sigma'(\bar{T})$. Hence $\sigma'(\bar{T}) \in G$. Thus:

(iv) $\sigma'{}'' \bar{G} \subset G$.

QED (Theorem)

We now sketch the changes that must be made in order to prove the subcompleteness of IN'. If A is club in ω_2, set:

$$F_A(\alpha) =: \text{the least } \beta \in A \text{ s.t. } \alpha < \beta.$$

Then if C is $\mathbb{B}$-generic, the function F_C eventually majorizes every $F: \omega_2 \to \omega_2$ in the ground model.

We change the definition of good pair, requiring only $\bigvee n \bigwedge m \geq n \; F_C(\gamma_m) \leq \gamma_{m+1}$ instead of $\bigvee n \bigwedge m \geq n \; \gamma_m \in C$.

In the proof of Lemma 1 we alter the definition of T_i' to read:

$$T_i' = \{t \in T_i \mid \bigwedge j \geq |u_i| \, (\alpha < t_j \wedge F_{A_i}(t_j) \leq t_{j+1}\}.$$

The proofs of Lemma 1, Corollary 2 go through as before. The remaining proofs then go through with cosmetic changes