The Extended Namba Problem

§0 Introduction

Shortly after the discovery of forcing it was noticed that any regular $\tau > \omega_1$ can be collapsed to ω_1 in a generic extension without adding reals. Levy showed that, if κ is strongly inaccessible, then every smaller cardinal can be simultaneously collapsed to ω_1, so that κ becomes ω_2 in the extension. In these constructions, however, the collapsed cardinals always acquired cofinality ω_1 in the extension. Namba then showed that, assuming CH in the ground model, one can collapse ω_2 to ω_1 in such a way that it becomes ω-cofinal in the extension. A number of interesting results were obtained by combining these collapsing techniques with iterated forcing. Fr. ins. Shelah showed that one can iterate Namba forcing out to a strongly inaccessible κ in such a way that κ becomes ω_2 and every other regular $\tau < \kappa$ acquires cofinality ω_1. However, the answer to the following question – which we shall here call the "extended Namba problem" remained elusive;

Assume CH. Let κ be strongly inaccessible. ~~As~~ there a generic extension in which κ becomes ω_2 and in which every regular $\tau \in (\omega_1, \kappa)$ becomes ω-cofinal?

This seemed so difficult to us that for several years we conjectured that it was provably impossible in ZFC. Moti Gitik then refuted this conjecture by constructing a model of ZFC which admits such an extension. Gitik started with a cardinal θ which is super-compact up to a Mahlo cardinal κ. He then collapsed to make $\theta = \omega_1$

Over the resulting model he then found a generic extension in which $\kappa = \omega_2$ and every regular $\tau \in (\theta, \kappa)$ becomes ω-cofinal. The question remained open, however, whether the super Namba problem always has a positive solution. We now provide an affirmative answer:

<u>Theorem 1</u> Assume CH. Let κ be strongly inaccessible. There is a set of conditions $\mathbb{P}$ of size κ s.t. $\mathbb{P}$ is subcomplete and whenever G is $\mathbb{P}$-generic, then:

- $\kappa = \omega_2$ in $V[G]$
- Every regular $\tau \in (\omega_1, \kappa)$ becomes ω-cofinal in $V[G]$.
- Every stationary subset of κ remains stationary in $V[G]$

(<u>Note</u> By subcompleteness there are no new reals in $V[G]$.)

(<u>Note</u> $\mathbb{P}$ will satisfy κ-cc, which by itself gives the last property.)

Modifying our method slightly we can also get:

Theorem 2 Let κ be inaccessible and assume GCH below κ. Let $A_0 \subset \kappa$. There is a set of conditions $\mathbb{P}$ of size κ s.t. $\mathbb{P}$ is subcomplete and whenever G is $\mathbb{P}$-generic, then

- $\kappa = \omega_2$ in $V[G]$
- Let $\tau \in (\omega_1, \kappa)$ be regular. Then
$$cf(\tau) = \begin{cases} \omega_1 & \text{if } \tau \in A_0 \\ \omega & \text{if not} \end{cases} \text{ in } V[G].$$
- Every stationary subset of κ remains stationary in $V[G]$ (in fact, $\mathbb{P}$ satisfies κ-cc).

(Note Theorem 1 can be derived from Theorem 2, though that is not the way we do it in this paper: Using ω_1-complete conditions we can do an Easton type collapse below κ s.t GCH holds below κ and κ remains inaccessible in the extension. Since the conditions are ω_1-complete, every regular

$\tau < \kappa$ which is collapsed in this process aquires a cofinality $> \omega_1$. If we then apply Thm 2 with $A_0 = \emptyset$, it is clear that every $\tau \in (\omega_1, \kappa)$ which is regular in V will become ω-cofinal.)

As one might suspect, $\mathbb{P}$ is obtained as the result of an iteration — albeit an iteration in a very broad sense. The most general notion of iteration is the following: $\langle \mathbb{B}_i \mid i < \alpha \rangle$ is an iteration iff each $\mathbb{B}_i$ is a complete Boolean algebra, $\mathbb{B}_i \subseteq \mathbb{B}_j$ (i.e. $\mathbb{B}_i$ is completely contained in $\mathbb{B}_j$) for $i \leq j < \alpha$, and for all limit $\lambda < \alpha$, $\mathbb{B}_\lambda$ is generated by $\bigcup_{i<\lambda} \mathbb{B}_i$. Most often $\mathbb{B}_\lambda$ is taken as the direct limit or the inverse limit of $\langle \mathbb{B}_i \mid i < \lambda \rangle$, or as something intermediate between the two. In the present case, however, we shall often be forced to use a completion $\mathbb{B}_\lambda$ which has a much larger cardinality than either the direct or inverse limit. (The reasons for this are explained in §2.1.) Hence we are unable to use the iteration theorem of [SPSC]. In §1 we prove a new iteration theorem which we shall use to ensure that each $\mathbb{B}_i$ in our iteration $\langle \mathbb{B}_i \mid i \leq \kappa \rangle$ is subcomplete. In § 2 we open with a discussion of the constraints that the iteration must satisfy. We show fr. ins. that the first stage of the iteration, whose purpose is to give ω_2 the cofinality ω, must, in fact, make every regular $\tau \in (\omega_1, \omega_{\omega_1})$ ω-cofinal. Hence $\mathbb{B}_1$ must be much larger than ω_2. At the first limit point (and many others) there is a similar constraint on $\mathbb{B}_\lambda$, which is why $\mathbb{B}_\lambda$ must be larger than the direct or inverse limit. In § 2.2 we then introduce the notion of "proud forcing", which will will play a large role in our proof. In § 2.3 we describe the proposed iteration more precisely. The iteration is intended to prove both Theorem 1 and Theorem 2; hence an arbitrary $A_0 \subset \kappa$ enters into its definition. We assume GCH below κ only in the case $A_0 \neq \emptyset$.

In §3 and §4 we then fully define the iteration and verify its properties. All of the component forcings in the proof of Theorem 1 are $\mathcal{L}$-forcings and can be regarded as generalizations of Namba forcing. (The successor step will, in fact, be a variation on the forcing described in the appendix to [LF] §5.)

Bibliography

[LF] $\mathcal{L}$-Forcing

[SPSC] Subproper and Subcomplete Forcing

(Unfortunately, I must assume a knowledge of both papers, which exist in the form of handwritten notes and can be found on my website.)