§1 s - premice

A ppm (prepremouse) is a structure satisfying all of the premouse axioms other than the initial segment condition. There are, in fact, different versions of this condition. The minimal initial segment condition (MIS) says that if $F = E_\nu^M \neq \emptyset$ and $\tau_\nu < \lambda' < \lambda_\nu$ and $\nu' = \lambda'^{+M\|\nu}$, then $\langle J_{\nu'}^E, F|\lambda' \rangle$ is not a ppm. This condition does not, however, have the preservation properties we want. The full initial segment condition (IS) says that if ν, λ' are above, $J_{\lambda'}^{E^M} = J_{\lambda'}^{E}$, and E', ν' are s.t. $\langle J_{\nu'}^{E'}, F|\lambda' \rangle$ is a ppm satisfying MIS, then $F|\lambda' \in M\|\nu$. (Note that we then have $\pi : J_{\tau_\nu}^E \xrightarrow[F|\lambda']{} J_{\nu'}^{E'}$.)

[Note: Here $\kappa_\nu = \kappa_\nu^M = \text{crit}(E_\nu^M)$, $\lambda_\nu = \lambda_\nu^M = F(\kappa_\nu) = lh(F)$, $\tau_\nu = \tau_\nu^M = \kappa_\nu^{+M\|\nu}$. At $M = \langle J_\nu^E, F \rangle$ we also set: $\kappa(M) = \kappa(F) = \kappa_\nu^M$, $\lambda(M) = \lambda(F) = \lambda_\nu^M$, $\tau(M) = \tau(F) = \tau_\nu^M$.]

A premouse is a ppm satisfying IS.

Lemma 1.1 Every premouse satisfies MIS.

prf.
Suppose not. Let $M = \langle J_\nu^E, F \rangle$ be a counterexample of minimal length. Hence there is $\lambda' \in (\tau_\nu, \lambda_\nu)$ s.t.

$N = \langle J_{\nu'}^{E'}, F|\lambda' \rangle$ is a ppm, where $\nu' = \lambda'^{+M}$.
Hence λ' is a limit cardinal in M. Hence N satisfies IS. Hence N satisfies MIS by the minimality of M. Hence $N \in M$.
Hence $\bar{\bar{N}}^M \leq \lambda'$. Hence $\nu' < \lambda'^{+M}$. Contr!

QED (Lemma 1.1)

<u>Cor 1.2</u> The condition "$\langle J_{\nu'}^{E'}, F|\lambda' \rangle$ satisfies MIS" can be omitted from IS.

proof.
Let $\langle J_{\nu'}^{E'}, F|\lambda' \rangle$ be a ppm and let $\pi' : J_{\tau_\nu}^{E} \xrightarrow[F|\lambda']{} J_{\nu'}^{E'}$. There is $\sigma : J_{\nu'}^{E'} \xrightarrow[\Sigma_1]{} J_\nu^E$ defined by
$\sigma(\pi'(f)(\alpha)) = \pi(f)(\alpha)$ for $\alpha < \lambda'$,
$f : \kappa_\nu \to \kappa_\nu$ in $M\|\nu$. Then $\lambda' = \mathrm{crit}(\sigma)$.
Hence λ' is a limit cardinal in $M\|\nu$.
Hence $N = \langle J_{\nu'}^{E'}, F \rangle$ satisfies IS.
Hence N satisfies MIS. QED (1.2)

However, there are narrower classes of premice satisfying their own initial segment condition. An example is the class of s-premice.

Def Let $M = \langle J_\nu^E, F \rangle$ be an active ppm.

$s(M)$ = the natural length of F

$= \mathrm{lub}(\tau_\nu \cup \mathrm{gen}_F)$,

where gen_F = the set of generators of F = the set of $\xi < \lambda_\nu$ s.t.

$\xi \neq \pi(f)(\vec{\alpha})$ for any $f \in M \cap ({}^n\kappa)$ and any tuple $\alpha_1, \ldots, \alpha_m < \xi$.

Def $s(M)^+ = s(M)^{+M}$

Def For $\nu \in N$, $E_\nu^N = \emptyset$ set:

$s(\nu)^N = s(N\|\nu)$, $s^+(\nu)^N = s^+(N\|\nu)$

Def The minimal initial segment condition (s-MIS) says that if $F = E_\nu^N \neq \emptyset$ and $\tau_\nu \leq \xi < s_\nu$ in N, $\pi : J_{\tau_\nu}^{E^N} \xrightarrow[F|\xi]{} J_{\nu'}^{E'}$, and $\xi' = \xi^{+N\|\nu}$, then $J_{\xi'}^{E'} \neq J_\xi^{E^N}$.

Def Let $\nu = ht(N)$ and let $\xi, \pi, J_{\nu'}^{E'}$ be as above. $N_\xi =_{\mathrm{Df}} \langle J_{\nu'}^{E'}, F' \rangle$, where $F' = \pi \upharpoonright \mathbb{P}(\kappa_\nu)$.

Def The full s-initial segment condition (s-IS) says that whenever $E_\nu^M \neq \emptyset$ and $N = M\|\nu$, then if $\tau_\nu \leq \xi < s_\nu$ in N and $\xi = s(N_\xi)$, then:

(a) If N_ξ satisfies s-MIS, then $N_\xi \in N$

(b) If N_ξ satisfies s-MIS and $\tau_\nu \leq \xi' < \xi$ s.t. $N_{\xi'}$ satisfies s-MIS, then $N_{\xi'} \in N_\xi$.

Note Clearly $\xi = s(N_\xi)$ iff $\xi = lub(\tau_\nu \cup (g_F \cap \xi))$

Note (a) holds for any $\xi \in [\tau_\nu, s_\nu)$ if it holds for $\xi = s(N_\xi)$

Note If ξ, ξ' are as in (b), then $N_{\xi'} = (N_\xi)_{\xi'}$.

By this last remark we easily get:

<u>Lemma 2.1</u> Let N satisfy s-IS. Let $\tau_\nu \leq \zeta < s_\nu$ in N s.t. N_ζ satisfies s-MIS. Then N_ζ satisfies s-IS

Moreover:

<u>Lemma 2.2</u> Let N satisfy s-IS. Then N satisfies s-MIS.

prf. Suppose not.

Let $N = \langle J_\nu^E, F \rangle$ be a counterexample of minimal length. Then $N \| \xi$ satisfies s-MIS for all $\xi < \nu$. Thus there is ζ s.t. $\tau_\nu \leq \zeta < s(\nu)$ and $J_{\zeta'}^{E^{N_\zeta}} = J_{\zeta'}^{E^N}$, where $\zeta' = \zeta + N$

We suppose w.l.o.g. that $s(\nu)$ is minimal for counterexamples of lengths

For the given N we also suppose ζ chosen minimally.

(1) N_ζ satisfies s-IS.

proof. Assume w.l.o.g. $\zeta = \text{lub}(\tau_\nu \cap (\text{gen}_F \cap \zeta))$

Then $\zeta = s(N_\zeta)$. Let $\eta \in [\tau_\nu, \zeta)$ s.t. $(N_\zeta)_\eta$ satisfies s-MIS. Then $(N_\zeta)_\eta = N_\eta \in N$. But $\bar{\bar{N}}_\eta \leq \eta < \zeta' = \zeta$ in N. Hence $N_\eta \in J_{\zeta'}^{E^N} = J_{\zeta'}^{E^{N_\zeta}} \subset N_\zeta$.

QED(1)

There is $\sigma : N_\xi \xrightarrow[\Sigma_0]{} N$ defined by:
$\sigma(\pi'(f|(\vec{\alpha}))) = \pi(f|(\vec{\alpha}))$ for $f : \kappa_\nu^m \to \kappa_\nu$, $f \in N$, $\alpha_1, \ldots, \alpha_m < \xi$, where $\pi' : J_{\tau_\nu}^{E^N} \xrightarrow[F|\xi]{} J_{\nu_\xi'}^{E^{N_\xi}}$ and $\pi : J_{\tau_\nu}^{E^N} \xrightarrow[F]{} J_\nu^{E^N}$. Hence $ht(N_\xi) \leq \nu$. Moreover $s(N_\xi) = \xi < s(N)$. Hence N_ξ satisfies s-MIS by (1) and the minimality of M. Hence $N_\xi \in N$. But $\bar{N}_\xi \leq s(N_\xi) \leq \xi < \xi' = \xi^+$ in N. Hence $N_\xi \in J_{\xi'}^{E^N} = J_{\xi'}^{E^{N_\xi}} \subset N_\xi$. Contr!

QED (2.2)

<u>Def</u> An s-<u>premouse</u> is a ppm satisfying IS.

<u>Lemma 2.2</u> corresponds to Lemma 1.1. However, the proof of Lemma 1.2 does not go through. There can, in fact, be an s-premouse M and a $\xi < s(M)$ s.t. M_ξ does not satisfy s-MIS. By Lemmas 2.1, 2.2 however, we at least know that M_ξ satisfies s-MIS iff M_ξ is an s-premouse.

<u>Lemma 2.3</u> s-MIS $\to$ MIS

prf. Trivial

<u>Lemma 2.7</u> Let $\tau_\nu \le \eta < s(\nu)$ s.t. η is a cardinal in $N = \langle J_\nu^E, F\rangle$, where N is an active s-premouse. Then N_η is an s-premouse.

prf.

Let $\xi < \eta$ s.t. $N_\xi = (N_\eta)_\xi$ satisfies s-MIS and $\xi = \mathrm{lub}(\tau_\nu \cup \mathrm{gen}_F \cap \xi)$. Then $N_\xi \in N$. Hence $N_\xi \in J_\eta^E = J_\eta^{E^{N_\eta}} \subset N_\eta$.

Now let $\zeta < \xi$ s.t. $N_\zeta = (N_\eta)_\zeta$ satisfies s-MIS. Then $N_\zeta \in N_\xi$, since $N_\zeta = (N_\xi)_\zeta$ and N_ξ is an s-premou[se]

Hence $N_\zeta \in N_\xi \subset N_\eta$. QED (2.7)

<u>Lemma 2.8</u> Every s-premouse is a premous[e]

prf.

Let $N = \langle J_\nu^E, F\rangle$ be an active s-premouse. Let $\tau_\nu < \lambda' < \lambda_\nu$ s.t. $N_{\lambda'} = \langle J_{\nu'}^{E'}, F|\lambda\rangle$ and $J_{\lambda'}^E = J_{\lambda'}^{E'}$.

<u>Claim</u> $F|\lambda' \in N$.

Define $\sigma : N_{\lambda'} \xrightarrow[\Sigma_1]{} N$ by $\sigma(\pi_{\lambda'}(f)(\alpha)) = \pi(f)(\alpha)$ for $f \in N \cap ({}^{\kappa_\nu}\kappa_\nu)$, $\alpha < \lambda'$, where $\pi_{\lambda'} : J_{\tau_\nu}^E \xrightarrow[F|\lambda']{} |N_{\lambda'}|$, $\pi : J_{\tau_\nu}^E \xrightarrow[F]{} |N|$.

Then $\sigma \restriction \lambda' = \mathrm{id}$, $\sigma(\lambda') = \lambda$. Hence $\lambda' \in \mathrm{gen}_F$ and λ' is a limit cardinal in N. Hence $N_{\lambda'} \in N$ by Lemma 2.7.

QED (2.8)

Def Let N be a ppm.

$C_N = C_N^{s}$ = the set of $\xi < s(N)$ s.t.

$\xi = \mathrm{lub}(\tau_\nu \cup (\xi \cap \mathrm{gen}_N))$ and N_ξ satisfies

s-MIS.

$C_N(\nu) = C_N^{s}(\nu) = C_{N\|\nu}$ for $E_\nu^N \neq \emptyset$.

Lemma 3.1 Let N be an active s-premouse.

Then C_N is closed in $s = s(N)$.

proof.

Let $\eta < s$ be a limit pt. of $C = C_N$.

It suffices to show:

Claim N_η is an s-premouse.

Clearly $C_{N_\eta} = \eta \cap C_N$. For $\xi \in C_{N_\eta}$

let $\sigma_{\xi\eta} : N_\xi \xrightarrow{\Sigma_s} N_\eta$ be defined

by $\sigma_{\xi\eta}(\pi_\xi(f)(\vec{\alpha})) = \pi_\eta(f)(\vec{\alpha})$ for

$\vec{\alpha} < \xi$, $f \in N \cap ({}^{\kappa_\nu^m}\kappa_\nu)$, where

$\pi_\xi : J_{\tau_\nu}^E \xrightarrow[F|\xi]{} J_{\nu_\xi}^{E^{N_\xi}}$, where

$N = \langle J_\nu^E, F \rangle$, $N_\xi = \langle J_{\nu_\xi}^{E^{N_\xi}}, F_\xi \rangle$.

Then $\sigma_{\xi\eta} \upharpoonright \xi = \mathrm{id}$ and $N_\eta = \bigcup_{\xi \in C_{N_\eta}} \mathrm{rng}(\sigma_{\xi\eta})$

Let $\xi \in C_{N_\eta}$, $\xi' \in C_{N_\eta}$, $\xi < \xi'$.
Then $N_\xi \in N_{\xi'}$, since $N_\xi = (N_{\xi'})_\xi$ satisfies MIS. Hence $F|\xi \in N_{\xi'}$. Hence $F|\xi = \sigma_{\xi'\eta}(F|\xi) \in N_\eta$. Hence $N_\xi \in N_\eta$. Now let $\varsigma < \xi$, $\varsigma \in C_{N_\eta}$. Then $N_\varsigma = (N_\xi)_\varsigma \in N_\xi \subset N_\eta$.

QED (3.1)

<u>Lemma 3.2</u> Let $N = \langle J_\nu^E, F \rangle$ be an active 1-premouse. Let $\xi < s(\nu)$, $\xi = \mathrm{lub}(\tau_\nu \cup (\xi \cap \mathrm{gen}_F))$ s.t. N_ξ is not a premouse. Then $\xi = \eta + 1$ where $\eta \in \mathrm{gen}_F$ and $\eta = \mathrm{lub}(\tau_\nu \cup (\eta \cap \mathrm{gen}_F))$ is a limit cardinal in N,

proof. Suppose not.
Let ξ be a counterexample. There is a unique cardinal μ in N s.t. $\tau_\nu \le \mu \le \xi < \mu^{+N}$. Let $\sigma = \sigma_\xi$ where $\sigma_\xi : N_\xi \xrightarrow{\Sigma_1} N$ is the canonical map (i.e. $\sigma(\pi_\xi(f)(\vec{\alpha})) = \pi(f)(\vec{\alpha})$

for $f \in N \cap ({}^{\kappa_\nu^m}\kappa_\nu)$, $\vec{\alpha} < \xi$). Clearly $\sigma \upharpoonright \xi =$

(1) $\sigma(\mu) = \mu$,

since otherwise $\mu \in gen_F$ is a limit cardinal in N and $\mu = \xi$. Hence N_ξ is an s-premouse by Lemma 3.1. Contr!

Hence:

(2) $\sigma \upharpoonright \tilde{\xi} = id$, where $\tilde{\xi} = \mu^{+N_\xi}$.

(3) Let $\mu \leq \gamma < \xi$ s.t. $\sigma_\gamma(\mu) = \mu$. Set: $\tilde{\gamma} = \mu^{+N_\gamma}$. Then $\tilde{\gamma} < \xi$.

prf. $\sigma_\gamma \upharpoonright \tilde{\gamma} = id$ as before. But then $[\gamma, \tilde{\gamma}) \cap gen_F = \emptyset$, where $\gamma < \xi$ and $\xi = lub(\xi \cap gen_F)$. Hence $\tilde{\gamma} < \xi$.

(4) Let $\tau_\nu \leq \gamma < \mu$. Then $(J^E_{\gamma^+})^{N_\gamma} \neq (J^E_{\gamma^+})^N$

proof.

$(J^E_{\gamma^+})^{N_\gamma} \neq (J^E_{\gamma^+})^N$ since N satisfies MIS. But μ is a cardinal in N and $J^{E^{N_\mu}}_\mu = J^{E^N}_\mu$. N_μ is an s-premouse by Lemma 2.7. Hence $N_\mu \in N$ and μ is a cardinal in N_μ. Hence $(J^E_{\gamma^+})^N = (J^E_{\gamma^+})^{N_\mu}$.

QED (4)

But N_ξ does not satisfy MIS. Hence there is a $\varsigma < \xi$ s.t. $(J^E_{\varsigma+})^{N_\xi} = (J^E_\varsigma)^{N_\xi}$.

By (3), (4) there is only one possibility — i.e. $\varsigma = \mu$;

(5) $\tilde{\mu} = \tilde{\xi}$ and $J^{E\,N_\mu}_{\tilde{\mu}} = J^{E\,N_\xi}_{\tilde{\xi}}$,

where $\mu = \mathrm{crit}(\sigma_\mu)$.

But since ξ is a counterexample, we have: $\xi > \mu+1$. Hence by (3):

(6) $\tilde{\mu} > \widetilde{\mu+1} = \mu^{+N_{\mu+1}}$

But then $N_{\mu+1}$ satisfies s-MIS by (5). Hence $N_{\mu+1} \in N$ is an s-premouse. Since $N_\mu = (N_{\mu+1})_\mu$ is an s-premouse we conclude: $N_\mu \in N_{\mu+1}$. But $\bar{\bar{N}}_\mu = \mu$ in $N_{\mu+1}$. Hence $\tilde{\mu} = \mu^{+N_\mu} < \widetilde{\mu+1}$. Contr!

QED (Lemma 3.2)

(Thus N_ξ can only fail to be an s-premouse in an unlikely seeming case. Martin Zeman has shown, however that this case does occur at the level "strong past a measurable".)

Cor 3.3 Let η be a limit pt of gen_F where $N = \langle J_\nu^E, F \rangle$ is an active premouse. Then η is a limit point of C_N^1.

(Note $\min(\text{gen}_F) = \kappa_\nu$, but if $\xi > \kappa_\nu$, $\xi \in \text{gen}_F$, then $\xi > \tau_\nu$.)

Def An active premouse $N = \langle J_\nu^E, F \rangle$ is of Type 1 iff $s(\nu) = \tau_\nu$

N is of Type 2 iff $s(\nu) = \xi + 1$ for some ξ

N is of Typ 3 iff $s(\nu)$ is a limit ordinal $> \tau_\nu$.

[Note If N is of Type 2, there is a maximal $\eta \in C_N$ s.t. $\eta \leq \xi$, since if ξ is a limit point of gen_F, then $\xi \in C_N$. Moreover, if $\xi \notin C_N$ and $\eta = \sup(\xi \cap \text{gen}_F)$, then η is a limit cardinal in N, $\eta \in \text{gen}_F$ and $\xi = \eta^{+N_\xi}$, where $N_\xi = N_{\eta+1}$ and $J_\xi^{E^{N_\eta}} = J_\xi^{E^{N_\xi}} = J_\xi^{E^N}$.]

<u>Lemma 3.4</u> Let N be of type 3. Then $\omega\rho^1_N = \kappa(\nu)$.

prf. Let $\kappa = \kappa(\nu)$

$\omega\rho^1_N \le \kappa$, since $h_N(\kappa) = N$. We prove $\ge$. Let $\xi < \kappa$ and let $a \subset \xi$ be $\underline{\Sigma}_1(N)$. <u>Claim</u> $a \in N$.

Since $N_\kappa = N$, we have $N = \bigcup_{\eta<\kappa} \mathrm{rng}(\sigma_\eta)$ where $\sigma_\eta : N_\eta \xrightarrow[\Sigma_1]{} N$ is the canonical map (i.e. $\mathrm{rng}(\sigma_\eta) =$ the set of $\pi(f)(\vec{\alpha})$ s.t. $f \in N \cap ({}^{\kappa_\nu}\kappa_\nu)$, $\vec{\alpha} < \eta$, where $\pi : J^E_{\tau_\nu} \xrightarrow[F]{} J^E_\nu$ and $N = \langle J^E_\nu, F\rangle$),

Pick $\eta > \xi$ s.t. $q \in \mathrm{rng}\,\sigma_\eta$, where a is $\Sigma_1(N)$ in q. Then a is $\Sigma_1(N_\eta)$ in $\sigma_\eta^{-1}(q)$, where $N_\eta \in N$. Hence $a \in N$.

QED (3.4)

<u>Note</u> Steel codes $N = \langle J^E_\nu, F\rangle$ of type 3 by $\tilde{N} = \langle J^E_\kappa, \tilde{F}\rangle$, where $\tilde{F} = \{\langle X, \alpha\rangle \mid \alpha < \kappa \wedge \alpha \in F(X)\}$, where $\kappa = \kappa(N)$. $\tilde{N}$ is then an amenable structure and is essentially the same as the reduct $N^1 = N^{1,\emptyset}$, since e.g. $\Sigma_1(\tilde{N}) = \Sigma_1(N^1)$.

Lemma 3.5 Let $N = \langle J_\nu^E, F \rangle$ be an active ppm. Let $\sigma : N \xrightarrow[G]{} N' = \langle J_{\nu'}^{E'}, F' \rangle$, where $\mathrm{crit}(G) < \rho(N)$. Then $\rho(N') = \sup \sigma''\rho(N)$.

proof.

Set $\rho = \rho(N)$, $\rho' = \rho(N')$. Then $\sup \sigma''\rho \leq \rho'$, since $\sigma''(\tau_\nu \cup \mathrm{gen}_F) \subset$ $\subset (\tau_{\nu'} \cup \mathrm{gen}_{F'}) \subset \rho'$. We prove $\geq$.

(by Π_1 - preservation)

First note that:

(1) $\pi'\sigma(X) = \sigma\pi(X)$ for $X \in \mathbb{P}(\kappa_\nu) \cap N$,

where $\pi : J^E_{\tau_\nu} \xrightarrow[F]{} J^E_\nu$, $\pi' : J^{E'}_{\tau_{\nu'}} \xrightarrow[F']{} J^{E'}_{\nu'}$,

since:

$Y = \pi(X) \longleftrightarrow \sigma(Y) = \pi'\sigma(X)$.

But then if $\zeta \geq \sup \sigma''\rho$ and $\zeta = \sigma(f)(\alpha)$, $\alpha < lh(G)$, then, letting $f = \pi(g)(\vec{\beta})$, $\vec{\beta} < \rho$, we have:

$$\zeta = (\sigma\pi(g)(\sigma(\vec{\beta})))(\alpha)$$
$$= \sigma\pi(\tilde{g})(\sigma(\vec{\beta}))(\alpha)$$
$$= \pi'\sigma(\tilde{g})(\sigma(\vec{\beta}))(\alpha),$$

where $\tilde{g} \in ({}^{\kappa_\nu^m}\kappa_\nu) \cap N$ and $\alpha, \sigma(\vec{\beta}) < \rho'$, and $\sigma(\tilde{g}) \in ({}^{\kappa_{\nu'}^m}\kappa_{\nu'}) \cap N'$. Hence

$\zeta \notin \mathrm{gen}_{F'}$.

QED (3.5)

<u>Lemma 3.6</u> Let $\sigma : N \xrightarrow[\Sigma_2]{} N'$. Then
$\sup \sigma'' s(N) \leq s(N') \leq \sigma(s(N))$.

prf.
$\sup \sigma'' s(N) \leq s(N')$ follows as before.
But the condition "$\xi \geq s$" is uniformly
Π_2 in ξ ($\xi \geq s \leftrightarrow \bigwedge \delta \geq \xi \ \delta \notin gen_F$).
Hence $\sigma(s(N))$ satisfies this condition
in N'. QED (3.6)

<u>Lemma 4.1</u> Let $\pi : N \xrightarrow[G]{} N'$, where
$N = \langle J_\nu^E, F \rangle$ is of type 1 and $crit(G) <$
$< s(\nu)$. Then M' is of type 1.
prf. By Lemma 3.5

<u>Lemma 4.2</u> Let $\pi : N \xrightarrow[G]{*} N'$, where
N is of type 1 and $crit(G) < s(N)$.
Then N' is of type 1.

prf.
By 4.1 if $crit(G) \geq \omega\rho_N^1$. Otherwise
by 3.6 and $s(N) = \tau_\nu$ (hence
$\sigma(s(N)) = \tau_{\nu'}$ in N'). QED (4.2)

<u>Def</u> Let $N = \langle J_\nu^E, F \rangle$ be of type 2.

$q_N =_{Df} F | \max C_N^1$.

(Hence $q_N \in N$).

<u>Lemma 4.3</u> Let $\pi : N \longrightarrow_G N'$ where N is of type 2 and $\mathrm{crit}(G) < s_N$. Then N' is of type 2, $\pi(s_N) = s_{N'}$, and $\pi(q_N) = q_{N'}$.

prf.

Let $N = \langle J_\nu^E, F\rangle$, $N' = \langle J_{\nu'}^{E'}, F'\rangle$. Then $s = \xi + 1$ for $\xi = \max(\mathrm{gen}_F)$ and $s' = \pi(\xi) + 1 = \pi(s)$ by Lemma 3.5. We note:

<u>Lemma 4.3.1</u> Let $\gamma < s_N$. Then $\gamma^{+N_\gamma} \leq \gamma^{+N}$.

prf. Let $\xi < \gamma^{+N_\gamma}$. Let $a \subset \gamma$ code a well ordering of type ξ. Then $a = \sigma_\gamma(a) \cap \gamma \in N$

QED (4.3.1)

Now let $\gamma = \max C_N$. Then $\gamma' = \sigma(\gamma) =$
$= \mathrm{lub}(\mathrm{gen}_{\sigma(N_\gamma)})$. Moreover $E_{ht}^{\sigma(N_\gamma)} | \gamma' = F' |$ since the same Π_1 statement holds of γ, N_γ in N. Moreover, $\gamma' < s_N$ and $\sigma(N_\gamma)$ is an s-premouse. Hence:

(1) $\gamma' \in C_{N'}$ and $\sigma(N_\gamma) = N'_{\gamma'}$.

<u>Claim</u> $\gamma' = \max C_{N'}$.

Set $\delta = \mathrm{crit}(\sigma_\gamma)$, $\delta' = \pi(\delta)$. Then $\delta = \gamma$ or γ^{+N}

<u>Case 1</u> $N_{\gamma+1} \doteq N$

Then $\delta' = \gamma'$, $s' = \gamma' + 1$.

<u>Case 2</u> $N_{\gamma+1} = N_\gamma$ (i.e. $\sigma_\gamma(\gamma) = \gamma$),

Then $\eta = \pi_\xi(f)(\vec{\alpha})$ where $\vec{\alpha} < \eta$, $f \in ({}^n\kappa) \cap N$ $(\kappa = \kappa_N)$, and $\pi_\xi : J^E_\tau \longrightarrow_{F|\eta} J^{EN_\eta}_{\nu_{N_\eta}}$. Hence we have:

$\{\langle\vec{\xi},\mu\rangle \mid f(\vec{\xi}) = \mu\} \in F_{\langle\vec{\alpha},\eta\rangle}$. Hence the same Δ_1-statement holds of $\pi(\vec{\alpha})$, $\pi(f)$, η' in N'. Hence $N_{\eta'+1} = N'_{\eta'}$ and $\mathrm{crit}(\sigma^{N'}_{\eta'}) = \rho' = \eta'^{+N_{\eta'}}$, where $s_{N'} = \rho' + 1$.

QED (Case 2)

<u>Case 3</u> The above fail.

$N_{\eta+1}$ does not satisfy s-MIS. Hence $\eta \in \mathrm{gen}$, is a limit cardinal in N and $(J^E_{\eta^+})^{N_\eta} = (J^E_{\eta^+})^{N_{\eta}}$ $= J^{EN}_\xi$ for an $\xi < \eta^{+N}$. Set $\xi' = \pi(\xi)$.

Then $\xi = \mathrm{crit}(\sigma_{\eta+1})$, $s_N = \xi + 1$ and $s_{N'} = \xi' + 1$.

It suffices to show:

<u>Claim</u> $(J^E_{\eta'^+})^{N'_{\eta'}} = (J^E_{\eta'^+})^{N'_{\eta'+1}} = J^{EN'}_{\xi'}$.

$(J_{\eta'^+})^{N'_{\eta'}} = J^{EN'}_{\xi'}$ is immediate. Set:

$\delta = \eta'^{+N'_{\eta'+1}}$. Then $J^{EN}_\delta = J^{EN'_{\eta'+1}}_\delta$. But

$\xi' \leq \delta$ by Lemma 4.3.1, since $N'_{\eta'} = (N'_{\eta'+1})_\eta$

$\delta \leq \xi'$ " " , since $N'_{\eta'+1} = (N'_{\xi'})_\eta$

where $\xi' \in \mathrm{gen}_{N'}$. (Hence $\xi' = \eta'^{+N'_{\xi'}}$.)

QED (Lemma 4.3)

If $\mathrm{crit}(\sigma) < \omega\rho^1_N$ we Lemma 3.6 to get $\pi(s_N) = s_{N'}$. Otherwise the proof is the same.

Similarly:

<u>Lemma 4.4</u> Let $\pi : N \longrightarrow^*_G N'$, where N is of type 2 and $\mathrm{crit}(G) < s_N$. Then N' is of type 2, $\pi(s_N) = s_{N'}$, $\pi(q_N) = q_{N'}$.

<u>Lemma 4.5</u> Let $N = \langle J_\nu^E, F\rangle$ be of type 3. Let $\rho = \rho(N) = \omega\rho_N^1$. Let $\delta : J_\rho^E \to_G J_{\rho'}^{E'}$, where G is an extender on J_ρ^E. Let $\delta' : N \to_{\Sigma_0^{(1)}} N'$ be the canonical completion of δ. Then N' is an ρ-premouse of type 3 and $\rho' = \rho_{N'}$.

proof.

Since $N = h_N(\rho)$, we have $N' = h_{N'}(\rho')$

hence:

(1) $\rho' = \omega\rho_N^1 \leq \rho(N')$

(2) $\rho' = \rho(N')$

proof.

Let $x \in N'$. <u>Claim</u> $x = \pi'(f)(\alpha)$

for an $\alpha < \rho'$ and an $f \in ({}^{\kappa_{\nu'}}\kappa_{\nu'}) \cap N$

where $\pi' : J_{\tau_{\nu'}}^{E'} \to_{F'} J_{\nu'}^{E'}$ (and $N' = \langle J_{\nu'}^{E'}, F'\rangle$).

Let $x = h_{N'}(i, \gamma)$, where $\gamma < \rho'$.

Let $\xi < \rho$ be p.r. closed s.t. $\gamma < \delta(\xi)$. The statement: $E|\xi = E^{N_\xi}|\xi$ is

$\Pi_1(N)$ in $N_\xi | \xi$. Hence the corresponding statement holds for $\delta(N_\xi)$, $\delta(\xi)$.

Hence $N'_{\delta(\xi)} = \delta(N_\xi)$. But

$\sigma_\xi : N'_{\delta(\xi)} \xrightarrow[\Sigma_1]{} N'$ and $\sigma_\xi(\gamma) = \gamma$.

Let $\bar{x} = \sigma^{-1}_{\delta(\xi)}(x) = h_{N'_{\delta(\xi)}}(\gamma)$. Then

$\bar{x} = \pi'_{\delta(\xi)}(f)(\alpha)$ for some $f \in ({}^{\kappa}\kappa; \kappa_{\nu'}) \cap N$

and some $\alpha < \xi$, where

$\pi'_{\delta(\xi)} : J^{E'}_{\tau_{\nu'}} \xrightarrow[F'|\delta(\xi)]{} J^{E^{N'_{\delta(\xi)}}}_{\nu''}$.

Hence $x = \sigma_{\delta(\xi)}(\pi'_{\delta(\xi)}(f)(\alpha)) =$

$= \pi'(f)(\alpha)$. QED (2)

(3) N' is of type 3.

It suffices to show that $N'_\xi \in N'$ for arbitrarily large $\xi < s'$, since then $N'_\xi \in N'$ for all $\xi < s'$. But

$N'_{\delta(\xi)} = \delta(N_\xi) \in N'$ for $\xi < s$.

QED (Lemma 4.5)

(*Note* This uses only $\delta : J^E \to J^{E'}$ cofinally)

<u>Lemma 4.6</u> Let $N = \langle J_\nu^E, F \rangle$ be of type 3. Let $\sigma : N \longrightarrow_G^* N'$ where $\mathrm{crit}(G) < s(N)$. Then N' is of type 3. Moreover,

$$s(N') = \begin{cases} \sup \sigma'' s(N) & \text{if } \mathrm{crit}(G) \geq \omega\rho_N^2 \\ \sigma(s(N)) & \text{if not} \end{cases}$$

prf.
Set $s = s(N)$. If $\mathrm{crit}(G) \geq \omega\rho_N^2$, the result follows by Lemma 4.5, so assume $\mathrm{crit}(G) < \omega\rho_N^2$. Let $s' = \sigma(s)$. Then $s' = \omega\rho_N^1$. Hence $s' \leq s(N')$.

The statement: $\bigwedge \xi < s \; F \upharpoonright \xi \in N$ is $\Pi_2^{(1)}(N)$ in $\xi, \varkappa_\nu$, since it can be written as:

$$\bigwedge \xi^1 < s \; \bigvee e^1 \, [e^1 \text{ is a function} \wedge$$
$$\wedge \, \mathrm{dom}(e^1) \subset \mathbb{P}(\varkappa_\nu) \wedge$$
$$\wedge \underbrace{\bigwedge x^0 \bigwedge y^0 \, (y^0 = F(x^0) \rightarrow y^0 \cap \xi = e^1(x^0))}_{\Pi_1^0}]$$

But σ is $\Pi_2^{(1)}$-preserving; hence $F|\xi \in N'$ for $\xi < s'$. Hence $N_\xi \in N'$ for $\xi < s'$ and it follows that $s' = s(N')$. It also follows from this that N' is of type 3.

QED (Lemma 4.6)

Putting these together:

<u>Lemma 4.7</u> Let N be an active s-premouse. Let $\pi : N \xrightarrow[G]{*} N'$, where $\mathrm{crit}(G) < s(N)$. Then N' is an active s-premouse of the same type. Moreover:

(a) If N is of type 1 or 2, then $\pi(s(N)) = s(N')$

(b) If N is of type 2, then $\pi(q_N) = q_{N'}$

(c) If N is of type 3, then

$$s(N') = \begin{cases} \sup \pi''s(N) & \text{if } \omega\rho_N^2 \geq \mathrm{crit}(G) \\ \pi(s(N)) & \text{if not} \end{cases}$$

By the remark following the proof of Lemma 3.2, we know that it is possible to have: $\pi : \bar{N} \longrightarrow_{\Sigma_1} N$, where N is an active s-premouse and $\bar{N}$ is not. Nonetheless we do get:

<u>Lemma 5.1</u> Let N be an active s-premouse of type 1, where $\sigma : \bar{N} \longrightarrow_{\Sigma_1} N$. Then $\bar{N}$ is an s-premouse of type 1.

prf.
$\bar{N}$ is clearly a ppm. But $\xi \notin gen_{\bar{N}}$ for $\xi \geq \tau_{\bar{N}}$, since then $\sigma(\xi) \notin gen_N$.

QED (5.1)

<u>Lemma 5.2</u> Let N be of type 2, where $\sigma : \bar{N} \longrightarrow_{\Sigma_1} N$ and $q_N \in rng(\sigma)$. Then $\bar{N}$ is of type 2 and $\sigma(q_{\bar{N}}) = q_N$, $\sigma(s_{\bar{N}}) = s_N$.

proof.
Let $N = \langle J_\nu^E, F \rangle$, $\bar{N} = \langle J_{\bar{\nu}}^{\bar{E}}, \bar{F} \rangle$. $\bar{N}$ is then a ppm. Let $\sigma(\bar{q}) = q = q_N$. Let $s = s_N = \xi + 1$. We consider two cases:

<u>Case 1</u> N_ξ satisfies r-MIS.

Then $N_\xi \in N$ and $q = F|\xi$. Hence $\xi = q(\kappa_N) \in \text{rng}(\delta)$. Let $\delta(\bar\xi) = \xi$.

Then $\bar\xi \in \text{gen}_{\bar F}$, since $\xi \in \text{gen}_F$.

Hence $\bar\xi + 1 = \bar r = \sigma^{-1}(r) \leq r(\bar N)$. But if $\bar\zeta \geq \bar r$, then $\zeta \geq r^V$ + hence $\zeta \notin \text{gen}_F$. Hence $\bar\zeta \notin \text{gen}_{\bar F}$. Hence $\bar r = r(\bar N)$. N_ξ is coded by q; hence $N_\xi \in \text{rng}(\sigma)$. Let $\sigma(\bar N') = N_\xi$.

where $\zeta = \sigma(\bar\zeta)$

Then $\bar q = F^{\bar N'}|\bar\xi = F^{\bar N}|\bar\xi$ and

There is $\bar\pi' : J^{E^{\bar N'}}_{\bar\tau_V} \longrightarrow_{\bar F'|\bar\xi'} J^{\bar E'}_{\bar V'}$,

where $\bar N' = \langle J^{\bar E'}_{\bar V'}, \bar F' \rangle$. Hence $\bar N' = \bar N_{\bar\xi}$ and $\bar q = q_{\bar N}$. Hence $\bar N_\eta \in \bar N$ for $\eta \leq \bar\xi$, where $r(\bar N) = \bar\xi + 1$. Hence $\bar N$ is an r-premouse.

QED (Case 1)

<u>Case 2</u> Case 1 fails.

Let $\eta = \max C^r_N = \max(\text{gen}_F \cap \xi)$.

Then η is a limit cardinal in N

and $(J^E_{\eta+})^N = (J^E_{\eta+})^{N_\xi}$, where $\xi = \eta^{+N_\xi}$. Moreover $N_\xi = N_{\eta+1}$. Then $q = F|\eta = F^{N_\eta}|\eta$ and $\eta = r(N_\eta)$. Since $\xi = \eta^{+N_\eta}$ and N_η is constructed from q, we have $\xi \in rng(\sigma)$. Let $\sigma(\bar\eta, \bar\xi, \bar N', \bar q) = \eta, \xi, N_\eta, q$.

It follows exactly as in Case 1 that $\bar r = \bar\xi + 1 = r(\bar N)$, where $\sigma(\bar r) = r$. It also follows as before that $\bar N' = \bar N_{\bar\eta}$ and $\bar q = q_{\bar N}$. Hence $\bar N_\gamma \in \bar N$ for all $\gamma \leq \bar\eta$, where $\bar\xi \in gen_{\bar F}$ and $\bar\eta = \max(\bar\xi \cap gen_{\bar F})$.

In order to establish that $\bar N$ is an r-premouse, and that $\bar\eta = \max C^r_{\bar N}$, it remains only to show

<u>Claim</u> $\bar N_{\bar\xi}$ does not satisfy r-MIS

proof.

Since $\bar\eta < \bar\xi < \bar\eta^{+\bar N}$, we know that $\sigma_{\bar\xi} \upharpoonright \bar\xi = id$ (where $\sigma_{\bar\xi} : \bar N_{\bar\xi} \longrightarrow \bar N$ is the canonical map),

But $\bar{\xi} \in \mathrm{gen}_F$, hence $\bar{\xi} = \mathrm{crit}(\sigma_{\bar{\xi}})$.
Hence $\bar{\xi} = \bar{\eta} + \bar{N}_{\bar{\xi}}$. But $J_{\bar{\xi}}^{E^{\bar{N}_{\bar{\xi}}}} = J_{\bar{\xi}}^{E^{\bar{N}}} =$
$= J_{\bar{\xi}}^{E^{\bar{N}_{\bar{\eta}}}}$. Hence $\bar{N}_{\bar{\xi}}$ does not satisfy MIS. QED (Lemma 5.2)

For type 3 s-premice the best we can do is:

<u>Lemma 5.3</u> Let N be a type 3 s-premouse. Let $\sigma : \bar{N} \xrightarrow{\Sigma_1^{(1)}} N$. Then N is a type 3 s-premouse.

proof. Let $s = s(N) = \omega\rho_N^1$.
Then σ^{-1}"$s = \omega\rho_{\bar{N}}^1 \in s(\bar{N})$. But for $\bar{\xi} < \bar{s} = \sigma^{-1}$"$s$, we have: $F^N | \xi \in N$, where $\xi = \sigma(\bar{\xi})$. But "$F^N | \xi \in N$" is $\Sigma_1^{(1)}$ in ξ, κ_N as expressed by:

$$\vee e^1 (e^1 \text{ is a function} \wedge \bigwedge x^1 \in \mathrm{dom}(e)\, x^1 \subset \kappa$$
$$\wedge \underbrace{\bigwedge x^0 y^0 (y^0 = F(x^0) \to y^0 \cap \xi = e^1(x^0))}_{\Pi_1^0}$$

Hence $F^{\bar{N}} | \bar{\xi} \in \bar{N}$. It follows easily that $\bar{s} = s(\bar{N})$ and that $\bar{N}$ is a type 3 s-premouse.

QED (5.3)

In some contexts it is useful to augment the structure N, replacing N by $\langle N, \{q_N\} \rangle$ if N is of type 2. and by $\langle N, \emptyset \rangle$ if N is of type 1. For the so augmented s-premice, we then have:

- If $\sigma : \bar{N} \xrightarrow[\Sigma_1]{} N$ and N is of type 1 or 2, then $\bar{N}$ is of the same type.
- If $\sigma : N \xrightarrow[\Sigma_0]{} N'$ and N, N' are s-premice, then they are of the same type.

We adopt this convention when defining the notion "standard parameter", "witness" (to $\nu \in p_N$ where p_N is the standard parameter) and "solidity". The above lemmas then enable us to carry through the proof of solidity. (On the other hand, some versions of the condensation lemmas refer to the original non-augmented premice.)

In [CR] §1 ("Corrections and Remarks") we sketched a similar development for general premice. In this case the initial segment condition is simply the principle IS stated earlier. For a premouse $N = \langle J_\nu^E, F \rangle$ we then set:

$C_N =$ the set of λ' s.t. $\tau_\nu < \lambda' < \lambda_\nu$ and $\lambda' = \lambda_{N_\lambda}$ (i.e. N_λ has the form $\langle J_{\nu'}^{E'}, F|\lambda' \rangle$).

N has type 1 if $C_N = \emptyset$. N has type 2 if $\max C_N$ exists. (C_N is easily seen to be closed in λ_N). If $\sup C_N = \lambda_N$, then C_N has type 3.

For N of type 2 we set: $q_N = F|\lambda'$, where $\lambda' = \max C_N$.

We then get:

(1) If $\pi : N \xrightarrow[G]{*} N'$, $\mathrm{crit}(G) < \lambda_N$, and N is a premouse, then N' is a premouse of the same type. Moreover $\pi(q_N) = q_{N'}$, if N has type 2.

(2) If $\sigma : \bar{N} \longrightarrow_{\Sigma_1} N$ and N is of type 1, then $\bar{N}$ is of type 1.

(3) If $\sigma : \bar{N} \longrightarrow_{\Sigma_1} N$, N is of type 2 and $q_N \in \mathrm{rng}(\sigma)$, then $\bar{N}$ is of type 2 and $\sigma(q_{\bar{N}}) = q_N$.

(4) If $\sigma : \bar{N} \longrightarrow_{\Sigma_1^{(1)}} N$ and N is of type 3, then $\bar{N}$ is of type 3.

(<u>Note</u> We also have: $\lambda_N = \omega\rho_N^1$ iff $\bar{N}$ is of type 3.)

For the purpose of defining and proving solidity, we then augment the premice in the same way as above.

As these facts suggest, s-mice and general premice are special cases of a more general theory. There are in fact many interesting premouse classes lying between these two extremes. We develop this theory in the next section.