§4 $\mathbb{E}$ - premice

Def Let N be an active ppm.
N is special iff there is $\mu \in [\kappa_N, \lambda_N)$ which is Woodin in N.
If M is a ppm and $E_\nu^M \neq \emptyset$, we call ν special in M iff $M \| \nu$ is special.

Lemma 1.1 Let N be active. The following are equivalent:
(a) N is special
(b) $J_{\kappa_N}^{E^N} \models$ there are arbitrarily large Woodins
(c) $J_{\lambda_N}^{E^N} \models$ " " " "

The proof is left to the reader.

Def Let N be active.
$\mathbb{E}_N \subset (\tau_N, \lambda_N]$ is defined as follows:
$\mathbb{E}_N \subset (\tau_N, \lambda_N]$ is defined as follows:

- If N is special, set: $\mathbb{E}_N$ = the set of $\delta \in (\tau_N, \lambda_N)$ s.t. δ is a limit cardinal in N and $o(\mu) \leq \delta$ in N for all $\mu \in (\kappa_N, \delta)$.
- If N is not special, set: $\mathbb{E}_N$ = the set of $\delta \in (\tau_N, \lambda_N)$ s.t. δ is a limit cardinal in N.

If M is a ppm and $E_\nu^M \neq \emptyset$, set:
$\mathbb{E}_\nu^M = \mathbb{E}^M(\nu) = \mathbb{E}_{M\|\nu}$.

This gives us the class of $\mathbb{E}$-premice.
The iteration indices are, of course, defined by:

Def Let N be active.

$$e_N = \begin{cases} \text{the least } e \in \mathbb{E}_N \text{ s.t. } e \geq s_N, \\ \quad \text{if such } e \text{ exists;} \\ s_N \text{ if not} \end{cases}$$

Following §3 we then define:

Def $e^M(\nu) = e^M_\nu = e_{M\|\nu}$ if $E^M_\nu \neq \emptyset$

Def Let M be an $\mathbb{E}$-premouse.
M is an e-<u>premouse</u> iff
(a) $e_\nu \in \mathbb{E}_\nu$ in M whenever $E^M_\nu \neq \emptyset$
(b) Let $E^M_\nu \neq \emptyset$. Let β be a cardinal in $J^{E^M}_\nu$ s.t. $\tilde{\beta} = \beta^+ \leq s^M_\nu$ in $J^{E^M}_\nu$. Then there is $\bar{\nu} \in (\beta, \tilde{\beta})$ s.t. $E^M_{\bar{\nu}} \neq \emptyset$ and $\text{crit}(E^M_{\bar{\nu}}) = \text{crit}(E^M_\nu)$.

e-premice are closely related to the "domestic premice" studied by Neeman and Steel in [ ]. We define C_N, $\tilde{C}_N$ for e-premice as usual and get just as in §3:

<u>Lemma 1.2</u> Let N be an active e-premouse. Then $C_N = \tilde{C}_N = \{\xi \in \tilde{C}_N \mid N_\xi \in N\}$.

By an e-<u>iteration</u> we understand an $\mathbb{E}$-iteration of an e-premouse. If $\mathcal{I} = \langle\langle N_i \rangle, \ldots, T\rangle$ is such an iteration, we, as usual, set: $e_i = e(\nu_i)^{M_i}$, $e^+_i = e^+_i(\nu_i)^{M_i}$. We say "$i$ is special in $\mathcal{I}$" (or "ν_i is special") to mean: ν_i is special in M_i.

Lemma 2.1 Let $\mathcal{I} = \langle\langle N_i\rangle, \ldots, T\rangle$ be a normal e-iteration. If $i < j$, i is special and $\kappa_j \leq \kappa_i$, then j is special.

prf.

κ_i is a limit of Woodins in $J^{EN_i}_{\nu_i}$. If $\kappa_j = \kappa_i$, then κ_j is a limit of Woodins in $J^{EN_j}_{\nu_j}$. If $\kappa_j < \kappa_i$, then some $\mu \in$ $\in [\kappa_j, \kappa_i)$ is Woodin in $J^{EN_i}_{\nu_i}$, hence in $J^{EN_j}_{\nu_j}$. QED (2.1)

Lemma 2.2 Let $\mathcal{I}$ be as above. If $i < j$, j is special, and $\kappa_i \leq \kappa_j < e_i$, then i is special.

proof.

By Lemma 2.1 if $\kappa_i = \kappa_j$. Otherwise there is $\mu \in [\kappa_j, e_i)$ which is Woodin in $J^{EN_j}_{\nu_j}$, hence in $J^{EN_i}_{\nu_i}$. QED (2.2)

Lemma 2.3 Let $\mathcal{I}$ be as above. Let $i < j$ s.t. i is special. Then $\kappa_j \notin (\kappa_i, e_i)$

proof. Exactly like §3 Lemma 2.

Cor 2.4 Let $\mathcal{I}$ be as above. Let $i < j \leq k$ s.t. j is special. Then $\kappa_k \notin (\kappa_i, e_i)$.

proof of Cor 2.4

Suppose not. $\kappa_j \not= \kappa_i$, since otherwise $\kappa_k \in (\kappa_j, e_j)$. Hence $\kappa_j \geq e_i$, since otherwise i is special by Lemma 2.2. Hence $\kappa_i < \kappa_k < e_i \leq \kappa_j$. Hence k is special by Lemma 2.1. Hence i is special by Lemma 2.2.

QED(Cor 2.4)

<u>Def</u> $\Gamma(\mathcal{I})$ = the set of $i < lh(\mathcal{I})$ s.t. for all $j < lh(\mathcal{I})$, $i \leq j \to i \leq_T j$.

<u>Def</u> $\mathcal{I}$ is e-<u>linear</u> iff whenever $\lambda \leq lh(\mathcal{I})$ is a limit of special points, then $\Gamma(\mathcal{I}|\lambda)$ is cofinal in λ.

<u>Lemma 2.5</u> Let $\mathcal{I}$ be a normal e-iteration. The following are equivalent.

(a) $\mathcal{I}$ is e-linear

(b) For all special $\eta < lh(\mathcal{I})$ the set $\{i \mid T(i+1) \leq \eta < i\}$ is finite.

proof. Just like the proof of the corresponding Fact in §3.

<u>Lemma 2.6</u> Let $\mathcal{I}$ be a normal e-iteration of length $k+1$. Then $\mathcal{I}$ is e-linear.

proof. Like §3 Lemma 3.

Def Let $\mathcal{I} = \langle\langle N_i\rangle, \ldots, T\rangle$ be a putative normal e-iteration. i is prominent in $\mathcal{I}$ iff one of the following holds:

(a) i is special
(b) $i = l+1$ where l is special
(c) For all $j \in D \cap i$ there is $k \in D \cap (i+1)$ s.t. some $\kappa \geq e_j$ is Woodin in $J_{e_k}^{EN_k}$.

Def $pr(\mathcal{I})$ = the set of prominent $i < lh(\mathcal{I})$
$pr^*(\mathcal{I}) = \{ j \mid \forall i \in pr(\mathcal{I})\ j \leq_T i \}$

Lemma 2.7 $0 \in pr(\mathcal{I})$; $pr(\mathcal{I})$ is closed in $lh(\mathcal{I})$.

pf. (w.l.o.g. let $\mathcal{I}$ be direct) $0 \in pr(\mathcal{I})$ is trivial. Now let $\lambda < lh(\mathcal{I})$ be a limit pt. of $pr(\mathcal{I})$. Let $j < \lambda$. Let $j < \eta \in \Gamma(\mathcal{I}|\lambda)$. Let $\eta < i \in pr(\mathcal{I})$. Some $\kappa \geq e_j$ is Woodin in $J_{e_i}^{EN}$, since either i satisfies (c) above or some $k \in \{i-1, i\}$ is special and $\kappa_k \geq e_\eta$ is a limit of Woodins in $J_{e_i}^{EN_i}$. QED (2.7)

Lemma 2.8 If i is prominent, $j \geq i$, and $T(j+1) < i$, then j is special.

pf. Straightforward.

Cor 2.9 The following are equivalent:
(a) $\mathcal{I}$ is e-linear
(b) Whenever $\lambda \leq lh(\mathcal{I})$ is a limit of $pr(\mathcal{I})$, then $\Gamma(\mathcal{I}|\lambda)$ is cofinal in λ

pf. If $j < \lambda$ is prominent and no $k \in (j, \lambda)$ is special, then $j \in \Gamma(\mathcal{I}|\lambda)$ by Lemma 2.8. QED

Lemma 2.10 Let $i = \max pr(\mathcal{I})$. Then $i \in \Gamma(\mathcal{I})$

pf. Straightforward by Lemma 2.8.

Now let θ be a strong Mahlo cardinal (i.e. θ is strongly inaccessible and Mahlo.). We regard V_θ as our universe and construct an array $\langle N_i \mid i \leq \theta \rangle$ of e-premice. $N_\theta = K^e$ will then be the "weasel" resulting from this process. We shall prove that each N_i is weakly normally iterable. (This is the condition for N_i's existence. We again believe it can be shown that each N_i is a weak mouse, but haven't checked the details.) We define the sequence by specifying when a new extender is added.

Def Let $N = \langle J^E_\alpha, F \rangle$ be an active ppm. $\langle M, G \rangle$ is a background certificate for N iff

(i) M is a transitive ZFC^- model; $V_\kappa \in M$, where $\kappa = \mathrm{crit}(F)$

(ii) G is an extender at κ on M and $lh(G) > e_N$

(iii) If $\pi : M \xrightarrow[G]{} M'$, then $V_{e_N + 2} \subset M'$

(iv) $F(X) = G(X) \cap e_N$ for $X \in \mathbb{P}(\kappa) \cap N \cap M$.

Def N is certifiable iff for every $A \subset \kappa$ there is a certificate $\langle M, G \rangle$ s.t. $A \in M$.

<u>Note</u> This is a somewhat stronger notion of certifiability than is used in [MS]. It is taken from a set of handwritten notes written earlier by Steel, which formed the basis for [MS]. Under this definition certifiability implies that κ is regular in V. We adopted the stronger notion of certifiability for two reasons:

(a) It seems likely that the stronger notion will give us the full weak iterability of each N_i, though we don't prove it here.

(b) We need the stronger notion to prove the cardinal preservation properties of K^c at the end of this section.

We then complete the definition of $\langle N_i \mid i < \theta \rangle$ by setting:

Let $N_i = \langle J_\alpha^E, E_{\omega\alpha} \rangle$ be defined and weakly normally e-iterable.

If $E_{\omega\alpha} = \emptyset$ and there is $N' = \langle J_\beta^{E'}, F \rangle$ s.t. N' is an e-premouse, $J_\alpha^{E'} = J_\alpha^E$, $F \neq \emptyset$, and $\alpha = d_{N'}$ <u>and</u> N' is certifiable, set $N_{i+1} = N'$ for some such N', selecting N' to be of type B if possible.

If there is no such N', set $N_{i+1} = \langle J^E_{\alpha+1}, \emptyset \rangle$

If N_i is not weakly normally e-iterable, then N_{i+1} is undefined.

. . . .

We shall prove that N_i is defined for $i \leq \theta$.

<u>Def</u> Let $\delta : Q \longrightarrow N_\gamma$ where Q is a countable e-premouse. Let $\mathcal{I} = \langle\langle Q_i \rangle, \ldots, T\rangle$ be a countable putative normal e-iteration of Q. Let b be a branch in $\mathcal{I}$.

By a <u>realization</u> of b we mean $\langle \delta_i \mid i \in b\rangle$ s.t. for all $i \in b$:

(a) $\delta_i : Q \xrightarrow{\Sigma_0^{(n)}} N_{\gamma_i}$ is fine for all n s.t.

$$\underset{h \in b \cap i}{\mathrm{lub}}\, \kappa_h \leq \omega\rho^n_{Q_i}$$

(b) If i is simple above $h \in b$, then

$$\gamma_i = \gamma_h, \quad \delta_i \pi_{hi} = \delta_h$$

(c) Let $i = j+1$, $h = T(i)$. Let $\eta_i = \bar{\beta}_n(Q_h, \nu_h)$.

Set: $\gamma' = \eta_n[\gamma_h, \delta_h(\nu_h)]$ and $\sigma^{(n)} = \sigma^{(n)}[\gamma_h, \delta_h(\nu_h)]$. Then $\gamma_i = \gamma'$ and $\delta_i \pi_{hi} = \sigma^{(n)} \delta_h$.

(d) $\gamma_0 = \gamma$, $\delta_0 = \delta$.

<u>Note</u> It is clear that $\langle \gamma_i \mid i \in b\rangle$ is determined by $\langle \delta_i \mid i \in b\rangle$.

<u>Lemma 2.11</u> Let $\langle \sigma_i \mid i \in b \rangle$ realize a branch b in $\mathcal{T}$. Let $h < i$ in b. Let $h = T(j+1)$, $j+1 \leq_T i$. Let $\kappa = \mathrm{crit}(\pi_{hi}) = \kappa_j$. Let $\sigma^{(n)}$ be as in (c) above (hence $\sigma_{j+1} \pi_{h,j+1} = \sigma^{(n)} \sigma_h$). Then $\sigma_i(\kappa) < \sigma^{(n)} \sigma_h(\kappa)$, $\sigma_i \restriction \kappa = \sigma^{(n)} \sigma_h \restriction \kappa$, and $\sigma_i(X) = \sigma_i(\kappa) \cap \sigma^{(n)} \sigma_h(X)$ for $X \in \mathcal{P}(\kappa) \cap Q_i$.

proof Exactly like (1) in the proof of §3 Lemma 4.1

Def Let $\sigma, Q, \mathcal{I}, N_\gamma, b$ be as above.
b is *realizable* wrt $\sigma: Q \to N_\gamma$ iff b has a realization.
We call $\mathcal{I}$ *realizable* wrt. σ iff one of the following holds:

- $lh(\mathcal{I}) = k+1$ and $\{i \mid i \leq_T k\}$ is realizable.
- $lh(\mathcal{I})$ is a limit ordinal and $\mathcal{I}$ has a cofinal realizable branch.

The major lemma on realizability is due to Steel and is proven in [S]. Adapted to our circumstances it reads:

Lemma 3 Let $\sigma: Q \to N_\gamma$ be fine where Q is a countable e-premouse. Let $\mathcal{I}$ be a countable putative normal e-iteration of Q. Assume that σ is $\Sigma_0^{(n)}$-preserving whenever $T(i+1) = 0$, $Q_i^* = Q$, and $\kappa_i < \omega\rho_Q^n$.

Either $\mathcal{I}$ is realizable or else $\mathcal{I}$ has a maximal realizable branch b, where b is of limit length.

(Hence in the latter case, b is ~~cofinal~~ in a $\lambda < lh(\mathcal{I})$ and $b \neq \{i \mid i \leq_T \lambda\}$.)

__Def__ Let $\sigma : Q \to N_\gamma$ where Q is a countable e-premouse. Let $\mathcal{T}$ be a countable putative normal e-iteration of Q. $\mathcal{T}$ has the __uniqueness property__ wrt $\sigma : Q \to N_\gamma$ iff for all limit $\lambda < lh(\mathcal{T})$, $\{ i \mid i <_T \lambda \}$ is the unique realizable branch cofinal in λ.

__Corollary 3.1__ Let $Q, \sigma, N_\gamma, \mathcal{T}$ be as in Lemma 3. If $\mathcal{T}$ has the uniqueness property, then $\mathcal{T}$ is realizable wrt. σ.

This is the foundation upon which we shall build.

__Def__ Let $\mathcal{T}$ be a putative normal e-iteration of Q. Let $\sigma : Q \xrightarrow[\Sigma^*]{} N_\gamma$ be fine. By a __support__ for $\mathcal{T}$ wrt σ, γ we mean a sequence $\vec{\sigma} = \langle \sigma_i \mid i \in pr^*(\mathcal{T}) \rangle$ s.t.

(a) $\langle \sigma_j \mid j \leq_T i \rangle$ is a realization of $\mathcal{T}|i+1$

(b) If i is special, $i < j$, $\kappa_i = \kappa_j$, $h = T(i+1)$, then $\sigma_{j+1}(\kappa_j) < \sigma_{i+1}(\kappa_i)$.

The triple $\langle \mathcal{T}, \vec{\sigma}, \gamma \rangle$ is then called a __supported iteration__. . .

<u>Lemma 4.1</u> Let $\langle \mathcal{T}, \vec{\sigma}, \gamma \rangle$ be supported. Then $\mathcal{T}$ is e-linear.

prf.

Let $\eta < lh(\mathcal{T})$ be special. Let $\eta \le i < j$ s.t. $i, j \in X = \{i \mid T(i+1) \le \eta < i\}$. Then $\kappa_i \le \kappa_\eta$ since $\kappa_i \notin (\kappa_\eta, \alpha_\eta]$. Hence i is special and, by the same argument, $\kappa_j \le \kappa_i$. Hence if X is infinite there must be κ s.t. $X' = \{i \in X \mid \kappa_i = \kappa\}$ is infinite. Let $i < j$, $i, j \in X'$. Then $\sigma_{j+1}(\kappa) < \sigma_{i+1}(\kappa)$. Contr!

QED (4.1)

<u>Def</u> Let $\mathbb{T} = \langle \mathcal{T}, \vec{\sigma}, \gamma \rangle$ be supported.

Let b be a branch in $\mathcal{T}$.

b is <u>realizable</u> in $\mathbb{T}$ iff b has a realization $\langle \sigma_i' \mid i \in b \rangle$ wrt $\sigma_0 : Q_0 \longrightarrow N_\gamma$ s.t. $\sigma_i' = \sigma_i$ whenever $i \in b \cap pr^*(\mathcal{T})$.

$\mathbb{T}$ is <u>realizable</u> iff one of the following hold:

- $lh(\mathcal{T}) = k+1$ and $\{i \mid i \le_T k\}$ is realizabl
- $\mathcal{T}$ is of limit length and some cofinal branch b is realizable.

Def $\mathcal{J} = \langle \mathcal{I}, \vec{\delta}, \gamma \rangle$ is a fine iteration iff

(a) $\mathcal{J}|\xi$ is a realizable supported iteration for $1 \le \xi \le lh(\mathcal{I})$,

(b) Let $\lambda < lh(\mathcal{I})$, Lim$(\lambda)$. Suppose there is a cofinal branch b in $\mathcal{I}|\lambda$ s.t. $b \neq$ $\neq \{i \mid i \le_T \lambda\}$ and b is realizable in $\mathcal{J}|\lambda$. Let k = the least $k > \lambda$ s.t. $k \in D$. Then $\tilde{\kappa}_\lambda = \sup_{i<\lambda} \kappa_i$ is Woodin in $J^{EQ_i}_{\nu_i}$.

(Note The formulation of (b) is simpler if $\mathcal{I}$ is direct.)

Lemma 4.2 Let $\mathcal{J}$ be a supported iteration of limit length λ and let $\mathcal{J}|\xi$ be fine for $\xi < \lambda$. Then $\mathcal{I}$ is fine. Moreover $\mathcal{J}$ extends to a fine iteration $\mathcal{J}'$ of length $\lambda+1$.

proof.

Case 1 $\eta = \sup pr(\mathcal{I}) < \lambda$.

Then $\mathcal{I}|[\eta, \lambda)$ has the uniqueness property wrt $\delta_\eta : Q_\eta \to N_{\gamma_\eta}$. By Cor 3.2 $\mathcal{I}|[\eta, \lambda)$ has a realizable branch b. Let $\langle \delta'_i \mid \eta \le_T i \in b \rangle$ be a realization wrt. $\delta_\eta : Q_\eta \to N_{\gamma_\eta}$.

Set: $\delta'_i = \delta_i$ for $i \le_T \eta$.

Then $\langle \delta_i' \mid i \in b \rangle$ is a realization of b in $\mathbb{J}$. Finally we note that if we set $Q_\lambda' = Q_b$, $T'``\{\lambda\} = b$, then $\mathcal{J}' = \langle\langle Q_i' \rangle, \ldots, T' \rangle$ is an extension of $\mathcal{J}$ and $\mathbb{J}' = \langle \mathcal{J}', \vec{\delta}', \gamma \rangle$ is a fine extension of $\mathbb{J}$, where $\vec{\delta}' = \vec{\delta}$ if λ is not prominent in $\mathcal{J}'$ and otherwise has an obvious definition.

QED (Case 1)

<u>Case 2</u> Case 1 fails.

Then $\Gamma(\mathcal{J})$ is cofinal in λ by Lemma 4.1. Set: $b = \{i \mid \bigvee j \in \Gamma(\mathcal{J})\ i \leq_T j\}$. b is the unique cofinal branch. For $j \in \Gamma(\mathcal{J})$ there is $k \geq j$ s.t. $k \in pr(\mathcal{J})$. Hence $j \leq_T k$ & $j \in pr^*(\mathcal{J})$. Thus $b \subset pr^*(\mathcal{J})$. It is easily verified that $i < j \rightarrow \gamma_j \leq \gamma_i$ for $i, j \in b$. Hence there is $i_0 \in b$ s.t. $\gamma_i = \gamma_{i_0}$ for $i \in b \setminus i_0$. Thus $\delta_j \pi_{ji} = \delta_i$ for $i \leq j$, $i, j \in b \setminus i_0$. Define $\delta' : Q_b \longrightarrow N_{\gamma_{i_0}}$ by: $\delta' \pi_{i\lambda} = \delta_i$ for $i_0 \leq i \in b$. Set $Q_\lambda' = Q_b$, $\gamma_\lambda = \gamma_{i_0}$. Then $\mathcal{J}' = \langle\langle Q_i' \rangle, \langle \nu_i \rangle, T' \rangle$ is an extension of $\mathcal{J}$, where $T'``\{\lambda\} = b$. Clearly $\lambda \in pr(\mathcal{J}')$, so we set: $\delta_\lambda' = \delta'$. $\mathbb{J}' = \langle \mathcal{J}', \vec{\delta}', \gamma \rangle$ is then a fine extension of $\mathbb{J}$.

QED (Lemma 4.2)

Def Let $\mathbb{I} = \langle \mathcal{I}, \vec{\sigma}, \gamma \rangle$ be a fine iteration of length $k+1$. Let $\mathcal{I}'$ be a putative normal e-iteration of length $k+2$ extending $\mathcal{I}$. $\mathcal{I}'$ is acceptable to $\mathbb{I}$ (or ν_k' is acceptable to $\mathbb{I}$) if ν_k' does not violate (b) in the def of "fine iteration" (or if $k \notin D$).

Lemma 4.3 Let $\mathbb{I} = \langle \mathcal{I}, \vec{\sigma}, \gamma \rangle$ be a fine iteration of length $k+1$ and let $\mathcal{I}'$ be an acceptable putative extension of length $k+2$. Then $\mathbb{I}$ has a fine extension $\mathbb{I}' = \langle \mathcal{I}', \vec{\sigma}', \gamma \rangle$.

proof. (w.l.o.g. let $\mathcal{I}'$ be direct)

Case 1 k is not prominent in $\mathcal{I}'$.

Let $\eta = \max(\mathrm{pr}(\mathcal{I})) = \max(\mathrm{pr}(\mathcal{I}'))$. By the fineness of $\mathbb{I}$ and the acceptability of $\mathcal{I}'$, $\mathcal{I}' | [\eta, k+2)$ is a normal e-iteration of length $(k+2) - \eta$ which has the uniqueness property w.r.t. $\sigma_\eta : Q_\eta \longrightarrow N_{\gamma_\eta}$. It follows straightforwardly by Lemma 3.1 that $\mathbb{I}' = \langle \mathcal{I}', \vec{\sigma}, \gamma \rangle$ is fine. QED (Case 1)

Case 2 Case 1 fails.

If $k \notin pr(\mathcal{T})$, extend $\vec{\delta}$ by adding the maps δ_h', where $\langle \delta_h' \mid h \leq_T k \rangle$ is a realization of $\mathcal{T}$. Otherwise leave $\vec{\delta}$ unchanged.

Then:

<u>Case 2.1</u> $T'(k+1) = k$

Let $\eta_k = \bar{\beta}_n(Q_k, \nu_k)$. Set:

$\gamma_{k+1} = \eta_m[\gamma_k, \delta_k(\nu_k)]$; $\sigma^{(n)} = \sigma^{(n)}[\gamma_k, \delta_k(\nu_k)]$.

Set: $F = E_{\nu_k}^{Q_k}$, $F^* = \sigma^{(n)}\delta_k(F)$.

Then F^* is ω-complete, since $\sigma^{(n)}\delta_k(\tau_k)$ is a cardinal in $N_{\gamma_{k+1}}$.

Let $g: \lambda_k \to \sigma^{(n)}\delta_k(\kappa_k)$ s.t.

$$\langle g(\vec{\alpha})\rangle \in \sigma^{(n)}\delta_k(X) \longleftrightarrow$$
$$\longleftrightarrow \langle \sigma^{(n)}\delta_k(\vec{\alpha})\rangle \in F^*(\sigma^{(n)}\delta_k(X))$$
$$\longleftrightarrow \langle \vec{\alpha}\rangle \in F(X).$$

Setting: $\delta'(\pi_{k,k+1}(f)(\alpha)) = \sigma^{(n)}\delta_k(f)(g(\alpha))$

for $g \in \Gamma(\kappa_k, Q_k^*)$, $\alpha < \lambda_k$, we get $\delta_{k+1}: Q_{k+1} \to N_{\gamma_{k+1}}$ with the right preservation properties.

Moreover, $\delta_{k+1}\pi_{k,k+1} = \sigma^{(n)}\delta_k$.

Then:

(A) $\langle \delta_h \mid h \leq_{T'} k+1 \rangle$ realizes $\mathcal{J}'$

(B) Extend $\vec{\sigma}$ by adding δ_{k+1} if $k+1 \in pr(\mathcal{J}')$, otherwise not. Then $\mathcal{J}' = \langle \mathcal{J}', \vec{\sigma}, \delta \rangle$ is a supported iteration.

(C) $\mathcal{J}'$ is fine.

<u>Note</u> (C) is immediate from (A), (B) and the acceptability of ν_k. To prove (B) we observe that the assumption of (b) in the def. of "support" cannot hold for $j = k$, since $T(k+1) = k$. QED (Case 2.1)

<u>Case 2.2</u> The above cases fail.
Then k is special in $\mathcal{J}'$. We extend $\vec{\sigma}$ (if necessary) as before. Let
$h = T'(k+1)$. Let $\eta_k = \bar{\beta}_m(Q_h, \nu_h)$. Set:
$\delta_{k+1} = \eta_m[\delta_h, \delta_h(\nu_h)]$, $\sigma^{(m)} = \sigma^{(m)}[\delta_h, \delta_h(\nu_h)]$.
Set: $F = E^{Q_h}_{\nu_k}$; $F^* = \delta_k(F)$. Since $\tau_k \leq e^+_h$ and $J^{E Q_h}_{e^+_h} = J^{E Q_k}_{e^+_h}$ and e^+_h is a cardinal in Q_k, we know that τ_k is a cardinal in Q_k. Hence F^* is ω-complete.

<u>Claim</u> $\delta_k(\kappa_k) \leq \sigma^{(m)}\delta_h(\kappa_k)$; $\delta_k \restriction \kappa_k = \sigma^{(m)}\delta_h \restriction \kappa_k$;
$\delta_k(X) = \delta_k(\kappa_k) \cap \sigma^{(m)}\delta_h(X)$ for $X \in \mathbb{P}(\kappa_k) \cap Q_k$.

proof of Claim.

(This is exactly like the proof of the Claim in Case 2 of the proof of § Lemma 4.1.)

$\kappa_i \geq \kappa_k$ for $h \leq i < k$, since otherwise $\kappa_k \in (\kappa_i, e_i)$. Hence $T(i+1) \geq h$ for $h < i < k$. Hence $h \leq_T k$. Let $\kappa = \mathrm{crit}(\pi_{hk})$. Then $\kappa = \kappa_i$ where $h = T(i+1)$, $i+1 \leq_T k$.

But $\eta_i \leq \eta_k$ since $\kappa_k \leq \kappa_i$. Let $\eta_i = \bar\beta_{n+m}[Q_h, \nu_h]$. Set: $\sigma^{(n+m)} = \sigma^{(n+m)}[\gamma_h, \delta_h(\nu_h)]$.

Then by Lemma 2.11:

(*) $\delta_k(\kappa) < \sigma^{(n+m)}\delta_h(\kappa)$,

$\delta_k \restriction \kappa = \sigma^{(n+m)}\delta_h \restriction \kappa$, and

$\delta_k(X) = \delta_k(\kappa) \cap \sigma^{(n+m)}\delta_h(X)$ for $X \in \mathcal{P}(\kappa) \cap Q_k$

But $\sigma^{(n+m)} = \tilde\sigma\sigma^{(m)}$, where $\tilde\sigma = \sigma^{(m)}[\gamma_{k+1}, \sigma^{(n)}\delta_h(\nu_h)]$. Since $\tau' = \sigma^{(n)}\delta_h(\tau_k)$ is a cardinal in $N_{\gamma_{k+1}}$, we conclude: $\tilde\sigma \restriction \tau'+1 = \mathrm{id}$.

The conclusion follows easily.

QED (Claim)

Since F^* is ω-complete we can choose $g : \lambda_k \longrightarrow \delta_k(\kappa_k)$ s.t. for all $X \in \mathbb{P}(\kappa_k) \cap Q_k$ and all $\alpha_1, \ldots, \alpha_n < \lambda_k$ we have:

$$\langle \vec{\alpha} \rangle \in F(X) \leftrightarrow \langle \delta_h(\vec{\alpha}) \rangle \in F^*(\delta_h(X))$$
$$\leftrightarrow \langle g(\vec{\alpha}) \rangle \in \delta_k(X)$$
$$\leftrightarrow \langle g(\vec{\alpha}) \rangle \in \sigma^{(n)} \delta_h(X).$$

Setting: $\delta_{k+1}(\pi_{h,k+1}(f)(\alpha)) = \sigma^{(n)} \delta_h(f)(g(\alpha))$ for $\alpha < \lambda_k$, $f \in \Gamma^*(\kappa_k, Q_k^*)$ we get $\delta_{k+1} : Q_k^* \longrightarrow N_{\gamma_{k+1}}$ with the right preservation properties s.t. $\delta_{k+1} \pi_{h,k+1} = \sigma^{(n)} \delta_h$. (This is like the corresponding proof in §3.)

Thus we have:

(A) $\langle \delta_i \mid i \leq_T k+1 \rangle$ realizes $\mathcal{I}'$

(B) Extend $\vec{\delta}$ by adding δ_{k+1}. Then $\mathbb{I}' = \langle \mathcal{I}', \vec{\delta}, \gamma \rangle$ is a supported iteration.

(C) $\mathbb{I}$ is fine.

(C) follows again by (A), (B).

In proving (B) we must show that if $h \leq j < k$ and $\kappa_j = \kappa_k$ (hence j is special), then $\pi_{h,j+1}(\kappa_j) > \pi_{h,k+1}(\kappa_k)$.

Let j be the largest such. It suffices to prove it for this j.

<u>Claim</u> $\delta_k(\kappa_k) = \delta_{j+1}(\kappa_k)$.

If $k = j+1$ this is immediate. Otherwise let $\kappa = \mathrm{crit}(\pi_{j+1,k}) > \kappa_j$. By Lemma 2.11 $\delta_k \upharpoonright \kappa = \sigma^{(m)} \delta_{j+1} \upharpoonright \kappa$ where $\sigma^{(m)} = \sigma^{(m)}[\gamma_{j+1}, \nu_{j+1}]$ for some m.

But τ_j a cardinal in Q_{j+1}. Hence $\delta_{j+1}(\tau_j)$ is a cardinal in $N_{\gamma_{j+1}}$. Hence $\sigma^{(m)} \upharpoonright \delta_{j+1}(\tau_j) = \mathrm{id}$. Hence $\delta_k(\kappa_j) = \sigma^{(m)}\delta_{j+1}(\kappa_j) = \delta_{j+1}(\kappa_j)$.

QED(Claim)

By our construction, however, $\delta_{k+1}(\kappa_k) = g(\kappa_k)$, where $g : \lambda_k \to \delta_k(\kappa_k)$. QED (Lemma 4.3)

Thus countable fine iterations can always be continued, as long as the indices chosen are acceptable.

Moreover:

Lemma 4.4 Let $\mathcal{I} = \langle \mathcal{T}, \vec{\delta}, \gamma \rangle$ be a fine iteration of a countable e-premouse with $lh(\mathcal{I}) = \omega_1$. Then $\mathcal{I}$ has a cofinal branch.

(Note By the regularity of ω_1 it follows that this branch is unique and that its limit model is well founded.)

proof.

Claim 1 Let g be a generic collapse of ω_1 to ω (over V). Then $\mathcal{I}$ is realizable in $V[g]$ (w.r.t. $\langle N_i \rangle$ as defined in V).

proof (sketch)

If F is the top extender of N_γ and $\langle M, F^* \rangle$ is a background certificate for F, there is a canonical F^*_g s.t. $\langle M[g], F^*_g \rangle$ is a background certificate for F in $V[g]$. Moreover, if $A \in V^{\mathrm{Coll}(\omega_1, \omega)}$ s.t. $\Vdash A \subset \check{\kappa}$, $\kappa = \mathrm{crit}(F)$, there is $A' \subset \kappa$ s.t. $A' \in M \rightarrow A^g \in M[g]$ for $M \models ZFC$ s.t. $V_\kappa \in M$. Hence $\langle N_i \rangle$ has the relevant properties in $V[g]$ and $\mathcal{I}$ is still a supported iteration in $V[g]$. We need only show

<u>Claim</u> $\mathbb{T}$ is a fine iteration in $V[g]$.

proof.

Suppose not. Then for some $\lambda < \omega_1$, $\mathbb{T}|\lambda$ has a realizable cofinal branch b in $V[g]$ s.t. $b \neq b_\lambda = \{i \mid i \leq_T \lambda\}$, but has no such branch in V. We derive a contradiction. Let $\lambda \in X \prec H_{\theta^+}$ be countable. Let $\sigma : \bar{H} \overset{\sim}{\longleftrightarrow} X$. Then $\alpha = \mathrm{crit}(\sigma)$, where $\sigma(\alpha) = \omega_1$. Let $\bar{g}$ be $\mathrm{coll}(\alpha, \omega_1)$ - generic over $\bar{H}$, $g \in V$. In $\bar{H}$, $\sigma^{-1}(\mathbb{T})|\lambda = \sigma^{-1}(\mathbb{T}|\lambda)$ has a realizable cofinal branch b s.t. $b \neq b_\lambda = \sigma^{-1}(b_\lambda)$. But then b is realizable in V wrt $\mathbb{T}|\lambda$, since if $\langle \bar{\sigma}_i \mid i \in b \rangle$ is a realization in $\bar{H}$, $\langle \sigma\bar{\sigma}_i \mid i \in b \rangle$ is a realization of $\mathbb{T}|\lambda$ and $\sigma\bar{\sigma}_i = \sigma \circ \sigma^{-1}(\sigma_i) = \sigma_i$ for $i \in p\pi^*(\mathbb{T}|\lambda)$

Contr! QED (Claim 1)

We now prove the lemma. Suppose not. Then $pr(\mathcal{Y})$ is not cofinal in ω_1 by e-linearity. Let $\gamma = \sup pr(\mathcal{Y})$. Then for each limit $\lambda \in (\gamma, \omega_1)$, $b_\lambda = \{i \mid i <_T \lambda\}$ is the unique cofinal branch in $\mathcal{Y}|\lambda$ which is realizable wrt $\mathbb{Y}|\lambda$. Let $X \prec H_{\theta^+}$ be countable and let $\sigma : \bar{H} \hookleftarrow\!\!\!\to X$. Let $\bar{\mathbb{Y}} = \langle \bar{T}, \langle \bar{\delta}_i \mid i \in pr(\bar{T}) \rangle, \bar{\gamma} \rangle = \sigma^{-1}(\mathbb{Y})$. Then $\sigma\bar{\delta}_i = \delta_i$ and $pr(\bar{T}) = pr(\mathcal{Y}) \subset \gamma < \alpha = \sigma^{-1}(\omega_1)$. Let $\langle g_0, g_1 \rangle \in V$ be a generic pair of collapsing functions over $\bar{H}$. Then $\bar{\mathbb{Y}}$ has a realizable cofinal branch b in $\bar{H}[g_0]$. ~~Let~~ $\langle \bar{\delta}_i \mid i \in b \rangle$ realize b wrt. $\bar{\mathbb{Y}}$ in $\bar{H}[g_0]$. Then $\langle \sigma\bar{\delta}_i \mid i \in b \rangle$ realizes b wrt $\mathbb{Y}|\alpha$ in V. Hence $b = b_\lambda$. By the same argument $b_\lambda \in \bar{H}[g_1]$. But then $b_\lambda \in \bar{H}$, since $b_\lambda \subset \bar{H}$ and $\langle g_0, g_1 \rangle$ is a generic pair. Hence $\sigma(b_\lambda)$ is a cofinal branch in $\mathbb{Y}$. QED (Lemma 4.4)

Call an e-premouse Q <u>realizable</u> iff there are δ, γ s.t. $\delta : Q \to N_\gamma$. We can attempt to coiterate two countable realizable premice via fine iterations. Lemma 4.4 should then give us what we need to show that the coiteration terminates at a countable stage. However, because of the acceptability restriction on the iteration indices it is not prima facie clear that such a coiteration is possible. We show that this is the case.

<u>Def</u> For Q an e-premouse and $e \leq ht(Q)$ set: $\nu_e \simeq$ that ν s.t. $e = e_\nu^Q$

$$\tilde{E}_e = \tilde{E}_e^Q = \begin{cases} E_{\nu_e}^Q | e & \text{if } \nu_e \text{ exists,} \\ \emptyset & \text{if not} \end{cases}$$

Thus, if $Q \neq Q'$, there is a least $e \leq \min(ht(Q), ht(Q'))$ s.t. $\tilde{E}_e^Q \neq \tilde{E}_e^{Q'}$.

<u>Def</u> If $\langle \mathfrak{I}^0, \mathfrak{I}^1 \rangle$ is a pair of fine iterations of limit length λ and b^h is a realizable cofinal branch in $\mathfrak{I}^h$ wrt $\mathfrak{I}^h$ $(h = 0, 1)$, then $e(b^0, b^1) \simeq_{\text{Df}}$ the least $e \leq ht(Q_{b^h})$ $(h = 0, 1)$ s.t. $\tilde{E}_e^{Q_{b^0}} \neq \tilde{E}_e^{Q_{b^1}}$.

<u>Def</u> Let Q^0, Q^1 be realizable. By a <u>fine coiteration</u> of Q^0, Q^1 with <u>coiteration indices</u> $\langle e_i \rangle$, we mean a pair of fine iterations $\mathfrak{I} = \langle \mathfrak{I}^0, \mathfrak{I}^1 \rangle$ of common length $\theta = lh(\mathfrak{I})$ s.t.

(a) $\mathfrak{I}^h$ is a fine iteration of Q^h.

(b) e_i that $e \leq ht(Q_i^h)$ $(h=0,1)$ s.t.

$$J_e^{Q_i^0} = J_e^{Q_i^1} \text{ but } \tilde{E}^{Q_i^0} \neq \tilde{E}^{Q_i^1}, \text{ for } i+1<$$

(c) $\nu_i^h = \nu_{e_i}^{Q_i^h}$ if — exists

otherwise $i \notin D_\theta^h$

(d) If $\lim(\lambda)$, $\lambda < \theta$, then

$b^h = \{ i \mid i \leq_{T^h} \lambda \}$ $(h=0,1)$ are chosen s.t. $e(b^0, b^1)$. is minimal,

<u>Def</u> A fine coiteration is <u>terminal</u> iff it cannot be extended to a coiteration of greater length.

By the previous lemmas we have:

<u>Lemma 4.5</u> Let Q^0, Q^1 be realizable countable e-premice. Let $\mathfrak{I} = \langle \mathfrak{I}^0, \mathfrak{I}^1 \rangle$ be a terminal coiteration which is countable. Then $lh(\mathfrak{I}) = k+1$ and e_k is either undefined or $\nu_{e_k}^{Q_k^h}$ is not acceptable to $\mathfrak{I}^h$ for an $h=0$ or 1.

We now show that the second alternative cannot occur.

Lemma 4.6 Let $\mathcal{J} = \langle \mathcal{J}^0, \mathcal{J}^1 \rangle$, Q^0, Q^1 be as above. Then e_k does not exist.

proof. Suppose not.

Assume not. Then $k = \lambda$ is a limit ordinal and $\tilde{E}_{e_\lambda}^{Q_\lambda^h} \neq \emptyset$ but $\tilde{\kappa} = \sup_{i<\lambda} \kappa_i^h$ is not Woodin in $J_{e_\lambda}^{E^{Q_\lambda^h}}$, although there is a $b \neq b_\lambda^h = \{i \mid i <_{T^h} \lambda\}$ which is cofinal and realizable wrt $\mathcal{J}^h$. For such a b we know that $\tilde{\kappa}$ is Woodin in $(J_{\tilde{\kappa}^+}^E)^{Q_\lambda^h}$ wrt. all $A \subset \tilde{\kappa}$ s.t. $A \in Q_\lambda^h \cap Q_b$. Hence there is a b s.t. $J_{e_\lambda}^{E Q_\lambda^h} \neq J_{e_\lambda}^{E Q_b^h}$. Hence there is $\bar{e} < e_\lambda$ s.t. $J_{\bar{e}}^{E Q_\lambda^h} = J_{\bar{e}}^{E Q_b}$ but $\tilde{E}_{\bar{e}}^{Q_\lambda^h} \neq E_{\bar{e}}^{Q_b}$. Hence $J_{\bar{e}}^{E Q_b} = J_{\bar{e}}^{E Q_\lambda^{h-1}}$ and $\tilde{E}_{\bar{e}}^{Q_b} \neq \tilde{E}_{\bar{e}}^{Q_\lambda^{h-1}}$, violating the minimal choice of b_λ^0, b_λ^1. Contr!

QED (4.6)

A coiteration of countable Q^0, Q^1 can be extended either to a countable terminal coiteration or to a coiteration of length ω_1. The usual proofe then shows:

<u>Lemma 4.7</u> Let Q^0, Q^1 be countable realizable e-premice. There is no coiteration of length ω_1.

Thus coiterations must terminate at a countable stage $k+1$, which means that e_k is not defined; hence Q^0_k is a segment of Q^1_k or conversely

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The appropriate version for "double rooted iterations" will also follow. This should be enough to prove the soundness and the condensation properties for N_i. That, in turn, enables us to define $M_i = \mathrm{core}(N_i)$ and continue the array construction. Unfortunately, however, we extravagantly made weak mousehood the official criterion for proceeding to M_i and N_{i+1} in the definition of "array".

Building upon what we have done here, it is, indeed, possible to prove that each N_i is a weak mouse. The proof requires a bit of extra work, however, and will, therefore, be relegated to an appendix. For the moment we adopt a "quick fix" by amending the definition of "array" to provide that M_i, N_{i+1} are defined whenever:

(a) N_i is solid

(b) If $Q = \mathrm{core}_\rho(N_i)$ and $\rho = \rho^n$, then $(J_{\rho^+}^E)^Q = (J_{\rho^+}^E)^{N_i}$

(c) If $\bar{\alpha}$ is a cardinal in N_i and $\eta < ht(N_i)$, $\omega\rho^\omega_{N_i\|\eta} = \alpha$, $\sigma : \bar{N} \xrightarrow{\Sigma^*} N_i\|\eta$, $\sigma \upharpoonright \bar{\alpha} = id$, where $\sigma(\bar{\alpha}) = \alpha$, then $\bar{N} = N_i\|\bar{\eta}$ for some $\bar{\eta}$.

((b), (c) are the only condensation properties we have made use of.)

Under this definition we have already shown that $K^c = N_\Theta$ exists and can proceed to the study of its large cardinal properties.

Exactly as before (§ 3 Lemma 6.1) we get:

<u>Lemma 5.1</u> Let $M_i = \mathrm{core}(N_i) = \langle J_\alpha^E, \emptyset \rangle$. There is at most one candidate $N = \langle J_\beta^{E'}, F \rangle$, $F \neq \emptyset$ for N_{i+1}.

<u>Cor 5.1.1</u> The def. of $\langle N_i \mid i < \gamma \rangle$ is uniform over every V_γ s.t. $\overline{\overline{V_\gamma}} = \gamma$ and γ is a limit of inaccessibles.

<u>Lemma 5.2</u> Let $i \leq \infty$ and let γ be a cardinal in N_i. There is at most one $\bar{N}$ s.t. $\bar{N} = \langle J_\beta^{E'}, F \rangle$, F is certified, $J_\alpha^{E^{N_i}} = J_\alpha^{E'}$ and $\bar{N}$ is an e-premouse with $\alpha = \alpha_{\bar{N}}^{+}$ where $\gamma < \alpha < \gamma^{+}$ in N_i.

An e-premouse is an $\mathbb{E}$-premouse satisfying the conditions (a), (b). As before (§3 Lemma 6.3) we get:

Lemma 5.3 Let $M_i = \langle J_\alpha^E, \emptyset \rangle$. Let $N = \langle J_\beta^{E'}, F \rangle$ be certifiable s.t. $J_\alpha^E = J_\alpha^{E'}$, $\alpha = e_N^+$, and N is an $\mathbb{E}$-premouse satisfying (a). Then N is an e-premouse.

proof.

The proof is as before with one change: In Case 1 it is not enough to show that $\bar{F}$ is ω-complete; we must show that it is certified. Clearly $\bar{F}|\gamma = F|\gamma$. It follows easily that if $\langle Q, F^* \rangle$ is a certificate for N, with $A \in Q$, $A \subset \kappa$, then $\langle Q, F^* \rangle$ is also a certificate for $\bar{N}$.

In Case 2 we have $\bar{N} = \langle J_{\bar{\beta}}^{\bar{E}}, F \rangle$, $\gamma = e_{\bar{N}}$, where $\bar{F}|\gamma = \tilde{F}|\gamma$ and $\tilde{F}$ is defined by: $\tilde{F} = \pi_1 \pi_0 \upharpoonright \mathbb{P}(\kappa)$,
$\pi_0 : J_{\tau}^{E} \longrightarrow_{F} J_{\beta}^{E'}$, $\pi_1 : J_{\beta}^{E} \longrightarrow_{E_\nu} J_{\bar{\beta}}^{\tilde{E}}$.
Let $A \subset \kappa$. We must find a certificate $\langle Q, F^* \rangle$ for $\bar{N}$ with $A \in Q$. Let $\langle Q, F^* \rangle$ be a certificate for N with $A \in Q$. Let $\sigma = \sigma^*[i, \nu]$, $\eta = \eta^*[i, \nu]$. Then $\sigma : N_i \| \nu \longrightarrow_{E^*} N_\eta$.
Let F_1 be the top extender of N_η. Since $\nu \in N_i$ and $\gamma < \nu$ is a cardinal in N_i, we have: $\sigma \upharpoonright \gamma = \mathrm{id}$. Let $\langle Q_1, F_1^* \rangle$ be a certificate for N_η s.t. $B \in Q_1$, where B codes the set $\langle F(X) \cap \kappa_\nu \mid X \in \mathbb{P}(\kappa) \cap N \rangle$. (Since $\kappa_\nu = \mathrm{crit}(F_1) = \mathrm{crit}(E_\nu)$ is inaccessible, we may assume $\bar{\bar{Q}} < \kappa_\nu$.) It follows easily that $\langle Q, F_1^* F^* \rangle$ is a certificate for $\bar{N}$. QED (5.3)

. . . . .

Virtually as before we get:

<u>Lemma 6.1</u> Let $K^e = J^E_\theta$. Let κ be Σ_2-strong. Then κ is E-strong.

proof like §3 Lemma 7.1 using Cor 5.2.

<u>Lemma 6.2</u> Let κ be E-strong. Then $o(\kappa) = \infty$ in K^e. (Hence κ is E-strong in K^e.)

proof. Exactly like §3 Lemma 7.2 using the method of Lemma 5.3 to provide the required background certificates.

Similarly:

<u>Lemma 6.3</u> Let κ be Σ_2-strong at level $\alpha < \theta$. Then κ is E-strong at level α, where $K^e = J^E_\theta$.

<u>Lemma 6.4</u> Let κ be E-strong at level α where $K^e = J^E_\theta$. Then κ has the same property in K^e.

. . . .

In addition to these results we get:

<u>Lemma 7</u> Let $\alpha < \kappa$ be a limit of Woodin cardinals. Then α is a limit of Woodin cardinals in K^e.

proof of Lemma 7.

Suppose not. Set:

$\tilde{\mu} = \text{lub}\{\mu < \alpha \mid \mu$ is Woodin in $K^e\}$.

Let $\tilde{\mu} < \mu < \alpha$ s.t. μ is Woodin. Since μ is not Woodin in K^e, there is $A \in \mathbb{P}(\mu) \cap K^e$ s.t. no $\kappa < \mu$ is A-strong in J^E_μ. ($J^E_\mu = K^e | \mu = V^{K^e}_\mu$, since μ is a limit of inaccessibles.)

We assume w.l.o.g. that A also codes $E \cap J^E_\mu$ in some natural way. Let $\kappa < \mu$ be A-strong (hence E-strong) in V_μ, where $\tilde{\mu} < \kappa < \mu$. Let

$\gamma = \text{lub}\{\gamma < \mu \mid$ There is $F \in K^e$ s.t. $lh(F) < \mu$ and F is an A-strong extender up to γ in $K^e\}$.

Then $\gamma < \mu$. Let $\gamma < \beta < \mu$ where β is inaccessible (in V). Let $F^* \in V_\mu$ be an extender on κ s.t. if $\pi : V \xrightarrow[F^*]{} W$, then $V_\beta \subset W$ and $\pi(A) \cap \beta = A \cap \beta$. (Hence $\pi(E) \cap V_\beta = E \cap V_\beta$.)

We now imitate the proof of §3 Lemma 7.2 (or Lemma 6.2 of this section). We operate as in Case 1 of that proof. (The question of whether there is $\eta \in (\kappa, \beta)$ s.t. $o(\eta) > \beta$ in K^e is uninteresting, since κ is not a limit of Woodins in K^e.) Set:
$F = (F^* | \beta) \upharpoonright N_\beta$. (Note, as before, that $N_\beta = \langle J_\beta^E, \emptyset \rangle$.) Let:
$\pi' : J_\tau^E \xrightarrow[F]{} J_\gamma^{E'}$ $(\tau = \kappa^{+K^e})$.
There is $\sigma : J_\gamma^{E'} \xrightarrow[\Sigma_0]{} \pi(N_\beta)$ defined by $\sigma(\pi'(f | (\alpha))) = \pi(f | (\alpha))$ $(\alpha < \beta)$. Then $\sigma \upharpoonright \beta = \mathrm{id}$. Set: $F' = \pi' \upharpoonright \mathbb{P}(\kappa)$. Set: $Q = \langle J_\gamma^{E'}, F' \rangle$. We then essentially repeat the proof in §3 Lemma 7.2, to show that $Q = K^e \| \nu$ for a $\nu > \beta$. (The fact that κ is not a limit of Woodins in Q enables us to omit some of the steps.)
Thus $F = F' | \beta \in K^e$, where F is an ω-complete extender on K^e. Let $\pi'' : K^e \xrightarrow[F]{} W'$. (Hence $\pi' = \pi'' \upharpoonright J_\tau^E$.)
But since $\sigma \upharpoonright \beta = \mathrm{id}$, we have:
$\pi''(A) \cap \beta = \pi'(A) \cap \beta = \sigma\pi'(A) \cap \beta = \pi(A) \cap \beta = A \cap \beta$.
Contr! QED (Lemma 7)